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LEGENDRIAN NORMALLY FLAT SUBMANIFOLS OF $\mathcal{S}$-SPACE FORMS

In the present study, we consider a Legendrian normally flat submanifold M of $(2n+s)$-dimensional $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$ of constant φ-sectional curvature c. We have shown that if M is pseudo-parallel then M is semi-parallel or totally geodesic.

We also prove that if M is Ricci generalized pseudo-parallel, then either it is minimal or $L=\frac{1}{n-1}$, when $c \neq -3s$.

Key words and phrases: $\mathcal{S}$-space form, Legendrian submanifold, normally flat submanifold, pseudo-parallel submanifold, Ricci generalized pseudo-parallel submanifold.

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INTRODUCTION

An n-dimensional submanifold M in an m-dimensional Riemannian manifold $\widetilde{M}$ is *pseudo-parallel* [1,2], if its second fundamental form σ satisfies the following condition

$$\widetilde{R}\,.\,\sigma = L\,Q(g,\sigma), \tag{1}$$

where $\widetilde{R}$ is the curvature operator with respect to the Van der Waerden-Bortolotti connection $\widetilde{\nabla}$ of $\widetilde{M}$, L is some smooth function on M and $Q(g,\sigma)$ is a $(0,4)$ tensor on M determined by $Q(g,\sigma)(Z,W;X,Y) = ((X\wedge_g Y)\,.\,\sigma)(Z,W)$. Recall that the $(0,k+2)$-tensor $Q(B,T)$ associated with any $(0,k)$-tensor field T, $k\geq 1$, and $(0,2)$-tensor field B, is defined by

$$\begin{aligned} Q(B,T)(X_1,X_2,\ldots,X_k;X,Y) &= ((X\wedge_B Y)\,.\,T)(X_1,X_2,\ldots,X_k) \\ &= -T((X\wedge_B Y)X_1,X_2,\ldots,X_k) - \ldots - T(X_1,X_2,\ldots,X_{k-1},(X\wedge_B Y)X_k), \end{aligned} \tag{2}$$

where $X\wedge_B Y$ is defined by

$$(X\wedge_B Y)Z = B(Y,Z)X - B(X,Z)Y. \tag{3}$$

In particular, if $L=0$, M is called a *semi-parallel* submanifold. Pseudo-parallel submanifolds were introduced in [1,2] as naturel extension of semi-parallel submanifolds and as the extrinsic analogues of pseudo-symmetric Riemannian manifolds in the sense of Deszcz [7], which generalize semi-symmetric Riemannian manifolds. On the other hand, Murathan et al. [11] defined submanifolds satisfying the condition

$$\widetilde{R}\,.\,\sigma = L\,Q(S,\sigma), \tag{4}$$

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where S is the Ricci tensor of M. The kind of submanifolds are called *Ricci generalized pseudo-parallel*. Recently, many authors studied pseudo-parallel and Ricci generalized pseudo-parallel submanifolds on various spaces, where the ambient manifold $\widetilde{M}$ has constant sectional curvature, we refer for example to [2, 5, 10–12, 14]. An integral submanifold of maximal dimension M^n of an $\mathcal{S}$-manifold $\widetilde{M}^{2n+s}$ is called *Legendrian* and it plays an important role in contact geometry. The study of Legendrian submanifolds of Sasakian manifolds from the Riemannian geometry point of view was initiated in 1970's. Legendrian submanifolds like their analogues in symplectic geometry, i.e. Lagrangian submanifolds. In [12], authors showed that a pseudo-parallel integral minimal submanifold M^n of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$ is totally geodesic if $Ln - \frac{1}{4}(n(c+3s)+c-s) \geq 0$.

In this work, we mainly prove that if a Legendrian normally flat submanifold M of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$ is pseudo-parallel (resp. Ricci generalized pseudo-parallel) then it is semi-parallel or totally geodesic (resp. minimal or $L = \frac{1}{n-1}$).

1 PRELIMINARIES

We remember some necessary useful notions and results for our next considerations. Let $\widetilde{M}^n$ be an n-dimensional Riemannian manifold and M^m an m-dimensional submanifold of $\widetilde{M}^n$. Let g be the metric tensor field on $\widetilde{M}^n$ as well as the metric induced on M^m. We denote by $\widetilde{\nabla}$ the covariant differentiation in $\widetilde{M}^n$ and by ∇ the covariant differentiation in M^m. Let $T\widetilde{M}$ (resp. TM) be the Lie algebra of vector fields on $\widetilde{M}^n$ (resp. on M^m) and $T^{\perp}M$ the set of all vector fields normal to M^m. The Gauss-Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \quad \widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V,$$

$X, Y \in TM$, $V \in T^{\perp}M$, where $\nabla^{\perp}$ is the connection in the normal bundle, σ is the second fundamental form of M^m and A_V is the Weingarten endomorphism associated with V. A_V and σ are related by $g(A_V X, Y) = g(\sigma(X,Y), V) = g(X, A_V Y)$.

The submanifold M^m is said to be *totally geodesic* in $\widetilde{M}^n$ if its second fundamental form is identically zero and it is said to be *minimal* if $H \equiv 0$, where H is the mean curvature vector defined by $H = \frac{1}{m} trace(\sigma)$ [13].

We denote by $\widetilde{R}$ and R the curvature tensors associated with $\widetilde{\nabla}$, ∇ and $\nabla^{\perp}$ respectively.

The basic equations of Gauss and Ricci are

$$g(\widetilde{R}(X,Y)Z,W) = g(R(X,Y)Z,W) + g(\sigma(X,Z),\sigma(Y,W)) - g(\sigma(X,W),\sigma(Y,Z)),$$

$$g(\widetilde{R}(X,Y)N,V) = g(R^{\perp}(X,Y)N,V) - g([A_N, A_V]X,Y),$$

respectively, $X, Y, Z, W \in TM$, $N, V \in T^{\perp}M$.

The covariant derivative $\widetilde{\nabla}\sigma$ of the second fundamental form σ is given by

$$\widetilde{\nabla}_X\sigma(Y,Z) = \nabla^{\perp}_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z).$$

The operators $\widetilde{R}(X,Y)$ from the curvature of $\widetilde{\nabla}$ and $X \wedge Y$ can be extended as derivations of tensor fields in the usual way, so

$$(\widetilde{R}(X,Y).\sigma)(Z,W) = R^{\perp}(X,Y)(\sigma(Z,W)) - \sigma(R(X,Y)Z,W) - \sigma(Z,R(X,Y)W). \tag{5}$$

Putting $B = g$, $T = \sigma$ in (2) and (3), we get

$$\begin{aligned} Q(g,\sigma)(Z,W;X,Y) &= ((X \wedge Y) \,.\, \sigma)(Z,W) = -\sigma((X \wedge Y)Z,W) - \sigma(Z,(X \wedge Y)W) \\ &= -g(Y,Z)\sigma(X,W) + g(X,Z)\sigma(Y,W) - g(Y,W)\sigma(Z,X) + g(X,W)\sigma(Z,Y). \end{aligned} \tag{6}$$

Let $\widetilde{M}^{2n+s}$ be a $(2n+s)$-dimensional Riemannian manifold endowed with an φ-structure (that is a tensor field of type $(1,1)$ and rank $2n$ satisfying $\varphi^3 + \varphi = 0$). If moreover there exist on $\widetilde{M}^{2n+s}$ global vector fields $\xi_1, \dots, \xi_s$ (called structure vector fields), and their duals 1-forms $\eta_1, \dots, \eta_s$ such that for all $X, Y \in T\widetilde{M}$ and $\alpha, \beta \in \{1, \dots, s\}$ (see [8])

$$\eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}, \;\; \varphi\xi_\alpha = 0, \;\; \eta_\alpha(\varphi X) = 0, \;\; \varphi^2 X = -X + \textstyle\sum_{\alpha=1}^{s} \eta_\alpha(X)\xi_\alpha \,, \tag{7}$$

then there exists on $\widetilde{M}$ a Riemannian metric g satisfying

$$g(X,Y) = g(\varphi X, \varphi Y) + \textstyle\sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y), \tag{8}$$

and

$$\eta_\alpha(X) = g(X,\xi_\alpha), \;\; g(\varphi X, Y) = -g(X, \varphi Y), \tag{9}$$

for all $\alpha \in \{1, \dots, s\}$, $\widetilde{M}$ is then said to be a metric φ-manifold. The φ-structure is normal if $N_\varphi + 2\sum_{\alpha=1}^{s} \xi_\alpha \otimes d\eta_\alpha = 0$, where N_φ is the Nijenhuis torsion of φ.

Let Φ be the fundamental 2-form on M defined for all vector fields X, Y on $\widetilde{M}$ by $\Phi(X,Y) = g(X, \varphi Y)$. A normal metric φ-structure with closed fundamental 2-form will be called K-structure and $\widetilde{M}^{2n+s}$ called K-manifold. Finally, if $d\eta_1 = \ldots = d\eta_s = \Phi$, the K-structure is called $\mathcal{S}$-structure and $\widetilde{M}$ is called *$\mathcal{S}$-manifold*.

The Riemannian connection $\widetilde{\nabla}$ of an $\mathcal{S}$-manifold satisfies [3]

$$\widetilde{\nabla}_X \xi_\alpha = -\varphi X, \; \alpha \in \{1, \dots, s\},$$

$$(\widetilde{\nabla}_X \varphi)Y = \textstyle\sum_{\alpha=1}^{s} (g(\varphi X, \varphi Y)\xi_\alpha + \eta_\alpha(Y)\varphi^2 X), \quad X, Y \in T\widetilde{M},$$

where $\widetilde{\nabla}$ is the Levi-Civita connection of g.

A plane section π is called an *φ-section* if it is determined by a unit vector X, normal to the structure vector fields and φX. The sectional curvature of π is called an *φ-sectional curvature*. An $\mathcal{S}$-manifold is said to be an *$\mathcal{S}$-space form* if it has constant φ-sectional curvature c and then, it is denoted by $\widetilde{M}^{2n+s}(c)$ $(n > 1)$ and its curvature tensor has the form [9]

$$\begin{aligned} \widetilde{R}(X,Y)Z = &\frac{c+3s}{4}\Big\{g(\varphi X, \varphi Z)\varphi^2 Y - g(\varphi Y, \varphi Z)\varphi^2 X\Big\} \\ &+ \frac{c-s}{4}\Big\{g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y + 2g(X, \varphi Y)\varphi Z\Big\} \\ &+ \sum_{\alpha,\beta=1}^{s} \Big\{\eta_\alpha(X)\eta_\beta(Z)\varphi^2 Y - \eta_\alpha(Y)\eta_\beta(Z)\varphi^2 X \\ &\qquad + g(\varphi Y, \varphi Z)\eta_\alpha(X)\xi_\beta - g(\varphi X, \varphi Z)\eta_\alpha(Y)\xi_\beta\Big\}, \end{aligned} \tag{10}$$

for all $X, Y, Z \in T\widetilde{M}$.

When $s = 1$, an $\mathcal{S}$-space form $\widetilde{M}(c)$ reduces to a Sasakian space form $\widetilde{M}(c)$ and $s = 0$ becomes a complex space form.

2 PSEUDO-PARALLEL LEGENDRIAN SUBMANIFOLDS OF AN $\mathcal{S}$-SPACE FORM

Let M^n be an n-dimensional submanifold of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$. If $\eta_\alpha(X) = 0$, $\alpha \in \{1, \ldots, s\}$, for every tangent vector X to M, then we say M is a *Legendrian submanifold*. Recall that a submanifold M of $\widetilde{M}$ is an anti-invariant submanifold if $\varphi(TM) \subseteq T^{\perp}M$. So, a Legendrian submanifold is identical with an anti-invariant submanifold normal to the structure vector fields $\xi_1, \ldots, \xi_s$. Actually, a Legendrian submanifold is special an integral submanifold. Therefore, from (8) and (9) we obtain

$$g(\varphi X, \varphi Y) = g(X, Y), \ \eta_\alpha(X) = g(X, \xi_\alpha) = 0,$$

for any $X, Y \in TM$ and $\alpha \in \{1, \ldots, s\}$. Then we have the following known Lemma (see [4]).

Lemma 1. *Let M^n be a Legendrian submanifold of an $\mathcal{S}$-manifold, then*

$$A_{\xi_\alpha} = 0,$$

$$A_{\varphi X}Y = A_{\varphi Y}X, \tag{11}$$

for all $\alpha \in \{1, \ldots, s\}$ and $X, Y \in TM$.

The previous Lemma implies immediately the following result.

Lemma 2. *For a Legendrian submanifold M^n of an $\mathcal{S}$-manifold $\widetilde{M}^{2n+s}$, the following equations*

$$g(\sigma(X, Y), \varphi Z) = g(\sigma(X, Z), \varphi Y), \tag{12}$$

$$A_{\varphi X}Y = -\varphi\sigma(X, Y) = A_{\varphi Y}X \tag{13}$$

hold for all $X, Y, Z \in TM$.

Moreover, from (7) and (13) we obtain

$$\varphi A_{\varphi X}Y = \sigma(X, Y) = \varphi A_{\varphi Y}X. \tag{14}$$

Using (14), (9) and the Gauss equation, we have

$$\widetilde{R}(X, Y) = R(X, Y) - [A_{\varphi X}, A_{\varphi Y}]. \tag{15}$$

We recall that the submanifold M is said to have *flat normal connection* (or trivial normal connection) if $R^{\perp} = 0$. If M has normal connection flat then we call it to be *normally flat*.

Then, making use of (14), (5) and (6), if M is normally flat, the pseudo-parallelity condition (1) turns into

$$\begin{aligned} -A_{\varphi W}R(X, Y)Z - A_{\varphi Z}R(X, Y)W = L\big\{ &- g(Y, Z)A_{\varphi X}W + g(X, Z)A_{\varphi Y}W \\ &- g(Y, W)A_{\varphi X}Z + g(X, W)A_{\varphi Y}Z\big\}. \end{aligned} \tag{16}$$

So, a Legendrian normally flat submanifold M^n of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$ is pseudo-parallel if and only if the equation (16) holds.

In particular, if $L = 0$ in (16) the M is said to be semi-parallel.

As a parallel submanifold, $\widetilde{\nabla}\sigma = 0$ (in particular, totally geodesic submanifold $\sigma = 0$) is semi-parallel it is obvious that also is a pseudo-parallel submanifold.

The following two propositions are the analogous results to [5, Prop. 3.1, Prop. 3.2] in case of pseudo-parallel Legendrian submanifold of an $\mathcal{S}$-space form, respectively.

Proposition 1. *Let M^n be a pseudo-parallel Legendrian submanifold of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$. If there is another smooth function L' satisfying* (1)*, then $L = L'$ at least on $M - K$, where $K = \{p \in M/\sigma_p = 0\}$.*

Proof. If L and L' are two functions that satisfy (1), we get $(L - L')\, Q(g,\sigma) = 0$. Choose an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM, $p \in M$. We have

$$\begin{aligned}(L - L')\, Q(g,\sigma)(e_k, e_l; e_i, e_j) &= (L - L')[(e_i \wedge e_j).\sigma](e_k, e_l)\\ &= (L - L')\big\{ - g(e_j, e_k)\sigma(e_i, e_l) + g(e_i, e_k)\sigma(e_j, e_l)\\ &\qquad - g(e_j, e_l)\sigma(e_k, e_i) + g(e_i, e_l)\sigma(e_k, e_j)\big\}\\ &= (L - L')\big\{ - \delta_{jk}\sigma(e_i, e_l) + \delta_{ik}\sigma(e_j, e_l)\\ &\qquad - \delta_{jl}\sigma(e_k, e_i) + \delta_{il}\sigma(e_k, e_j)\big\} = 0.\end{aligned}$$

For $i = k \neq j = l$ we get

$$(L - L')\{\sigma(e_j, e_j) - \sigma(e_i, e_i)\} = 0.$$

For $i = k = l \neq j$ we get

$$(L - L')\sigma(e_i, e_j) = 0.$$

If $L(p) \neq L'(p)$, $p \in M$, then

$$\sigma(e_i, e_j) = 0, \quad \sigma(e_i, e_i) = \sigma(e_j, e_j), \qquad \forall i, j \in \{1, \dots, n\}.$$

Moreover, since $i \neq j$ and from (12)

$$g(\sigma(e_i, e_i), \varphi e_j) = g(\sigma(e_i, e_j), \varphi e_i) = 0,$$

$$\begin{aligned} g(\sigma(e_i, e_i), \varphi e_i) &= g(\sigma(e_j, e_j), \varphi e_i) = g(\sigma(e_i, e_j), \varphi e_j) = 0,\\ g(\sigma(e_i, e_i), \xi_\alpha) &= g(\varphi A_{\varphi e_i} e_i, \xi_\alpha) = 0, \qquad \forall \alpha \in \{1, \dots, s\}.\end{aligned}$$

So, we obtain $g(\sigma(e_i, e_i), N) = 0\, \forall i \in \{1, \dots, n\}, \forall N \in T^\perp M$ and since $\{\varphi e_1, \dots, \varphi e_n, \xi_1, \dots, \xi_s\}$ is a basis of $T^\perp M$ for a Legendrian submanifold M, then $\sigma = 0$. Consequently

$$\{p \in M, L(p) \neq L'(p)\} \subseteq K.$$

This proves the proposition. □

Proposition 2. *Let M^n be a pseudo-parallel Legendrian normally flat submanifold of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$, then for any vector fields $X, Y \in TM$ we have*

$$R(X, Y)\varphi H = L\{g(\varphi H, X)Y - g(\varphi H, Y)X\},$$

where H is a mean curvature vector.

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of TM and Z unit vector field of T_pM for $p \in M$. $\forall U \in TM$, (16) can be rewritten as

$$\begin{aligned} g(R(X, Y)Z, A_{\varphi W}U) + g(R(X, Y)W, A_{\varphi Z}U) &= L\big\{g(Y, Z)g(A_{\varphi X}W, U)\\ &- g(X, Z)g(A_{\varphi Y}W, U) + g(Y, W)g(A_{\varphi X}Z, U) - g(X, W)g(A_{\varphi Y}Z, U)\big\}.\end{aligned} \tag{17}$$

If we put $W = U = e_i$ in (17), we obtain

$$g(R(X,Y)Z, A_{\varphi e_i}e_i) + g(R(X,Y)e_i, A_{\varphi Z}e_i) = L\{g(Y,Z)g(A_{\varphi X}e_i, e_i) - g(X,Z)g(A_{\varphi Y}e_i, e_i) + g(Y,e_i)g(A_{\varphi X}Z, e_i) - g(X,e_i)g(A_{\varphi Y}Z, e_i)\}.$$

Assuming that $\{\lambda_1, \dots, \lambda_n\}$ are the eigenvalues of $A_{\varphi Z}$ corresponding to frame $\{e_1, \dots, e_n\}$. Using (11) in the above equation, we have

$$\begin{aligned} -g(R(X,Y)A_{\varphi e_i}e_i, Z) + \lambda_i g(R(X,Y)e_i, e_i) &= L\{g(Y,Z)g(A_{\varphi e_i}e_i, X) - g(X,Z)g(A_{\varphi e_i}e_i, Y) \\ &\quad + g(Y,e_i)g(A_{\varphi Z}e_i, X) - g(X,e_i)g(A_{\varphi Z}e_i, Y)\} \\ &= L\{g(Y,Z)g(A_{\varphi e_i}e_i, X) - g(X,Z)g(A_{\varphi e_i}e_i, Y) \\ &\quad + \lambda_i g(Y,e_i)g(e_i, X) - \lambda_i g(X,e_i)g(e_i, Y)\}. \end{aligned}$$

So that

$$-g(R(X,Y)A_{\varphi e_i}e_i, Z) = L\{g(Y,Z)g(A_{\varphi e_i}e_i, X) - g(X,Z)g(A_{\varphi e_i}e_i, Y)\}.$$

From (13), we get

$$g(R(X,Y)\varphi H, Z) = -\frac{1}{n}\sum_{i=1}^{n} g(R(X,Y)A_{\varphi e_i}e_i, Z) = L\{g(Y,Z)g(\varphi H, X) - g(X,Z)g(\varphi H, Y)\}.$$

□

3 MAIN RESULTS

Theorem 1. *Let M^n be a Legendrian normally flat submanifold of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$ with $c \leq s$, then M^n is pseudo-parallel if and only if it is semi-parallel or totally geodesic.*

Proof. Since M^n is a Legendrian submanifold and from (10) we have

$$\widetilde{R}(X,Y)Z = \frac{c+3s}{4}\Big\{g(Y,Z)X - g(X,Z)Y\Big\}, \tag{18}$$

for any $X, Y, Z \in TM$, so that

$$\widetilde{R}(X,Y)\varphi H = \frac{c+3s}{4}\Big\{g(Y,\varphi H)X - g(X,\varphi H)Y\Big\},$$

where H is the mean curvature vector. As $R^{\perp} = 0$ and from (18), the Ricci equation reduces to $[A_{\varphi X}, A_{\varphi Y}] = 0$, so from (15) we get $\widetilde{R}(X,Y)\varphi H = R(X,Y)\varphi H$, thus

$$R(X,Y)\varphi H = \frac{c+3s}{4}\Big\{g(Y,\varphi H)X - g(X,\varphi H)Y\Big\}.$$

Using the above equation and Proposition 2, we obtain

$$\Big(\frac{c+3s}{4} + L\Big)\Big\{g(Y,\varphi H)X - g(X,\varphi H)Y\Big\} = 0, \tag{19}$$

this implies that $L = -\frac{c+3s}{4}$ or $H = 0$.

When $L = -\frac{c+3s}{4}$, if $c = -3s$, i.e. $L = 0$, that is, M is semi-parallel. If $c \neq -3s$, so $L \neq 0$, then from (16), (10) and (11) we have

$$-g(Y,Z)A_{\varphi X}W + g(X,Z)A_{\varphi Y}W - g(Y,W)A_{\varphi X}Z + g(X,W)A_{\varphi Y}Z = 0. \tag{20}$$

Thus by using (20) and Proposition 1, we have $\sigma = 0$, i.e. M is totally geodesic.

Now, assuming that $L \neq -\frac{c+3s}{4}$, then from (19), $H = 0$. By substituting (18) into (16) we obtain

$$\Big(L - \frac{c+3s}{4}\Big)\Big\{ - g(Y,Z)A_{\varphi X}W + g(X,Z)A_{\varphi Y}W - g(Y,W)A_{\varphi X}Z + g(X,W)A_{\varphi Y}Z\Big\} = 0.$$

Putting $X = W = e_i$ and summing over $i = 1,\dots,n$, as $H = 0$ we get $L = \frac{c+3s}{4}$ or $A_{\varphi Y}Z = 0$ (i.e. M is totally geodesic), for all $Y, Z \in TM$.

On the other hand, if we suppose that $L = \frac{c+3s}{4}$. Notice that in [12] the authors gave a necessary condition for a minimal pseudo-parallel integral submanifold M^n (of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$) to be totally geodesic is $Ln - \frac{1}{4}[n(c+3s) + c - s] \geq 0$. Hence, in this case M is totally geodesic. Conversely, if M is semi-parallel or totally geodesic obviously it is trivial pseudo-parallel. $\square$

From (19), we easily prove the following result.

Corollary 1. *Let M^n $(n > 1)$ be a Legendrian normally flat submanifold of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$, with $c \neq -3s$. If M^n is semi-parallel then it is minimal.*

In [12], the authors have shown that for a minimal Legendrian submanifold M^n of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$, if it is semi-parallel and satisfies $n(c+3s) + c - s \leq 0$, then it is totally geodesic. Therefore, by Corollary 1 we have the following assertion.

Corollary 2. *Let M^n $(n > 1)$ be a Legendrian normally flat submanifold of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$, with $c < -3s$. If M^n is semi-parallel then it is totally geodesic.*

Theorem 2 ([4]). *Let M^m $(m \leq n)$ be a minimal anti-invariant submanifold of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$ normal to the structure vector fields. Then the following assertions are equivalent.*

1. *M^m is totally geodesic.*
2. *M^m is of constant curvature $k = \frac{c+3s}{4}$.*
3. *The Ricci tensor $S = \frac{1}{4}(m-1)(c+3s)g$.*
4. *The scalar curvature $\rho = \frac{1}{4}m(m-1)(c+3s)$.*

By the hypothesis of flat normal connection, M^n is of constant curvature $k = \frac{c+3s}{4}$, in view of Corollary 1 we get

Corollary 3. *Let M^n be a Legendrian normally flat submanifold of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$ with $c \neq -3s$. If M^n is semi-parallel, then the following statements are equivalent.*

1. *M^n is totally geodesic.*
2. *The Ricci tensor $S = \frac{1}{4}(n-1)(c+3s)g$.*
3. *The scalar curvature $\rho = \frac{1}{4}n(n-1)(c+3s)$.*

It is well known that the equation of Ricci shows that the triviality of the normal connection of M into space form $\widetilde{M}^{n+d}(c)$ (and more generally, for submanifolds in a locally conformally flat space) is equivalent to the fact that all second fundamental tensors are mutually commute, or that all second fundamental tensors are mutually diagonalizable (see [6]).

So, for any $p \in M$ there exists a local orthogonal frame $\{e_i\}$ of M^n such that all the second fundamental form tensors are mutually diagonalizable, then

$$A_N(e_i) = \lambda_i^N e_i$$

for any unit normal vector field N and λ_i^N are the principle curvatures of M with respect to N.

Next, we assume that M^n is a Legendrian normally flat submanifold of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$, with $c \neq -3s$. In this case, from (10) and (15) we have

$$R(X,Y)Z = \widetilde{R}(X,Y)Z = \frac{c+3s}{4}\Big\{g(Y,Z)X - g(X,Z)Y\Big\} \tag{21}$$

for any vector $X, Y, Z \in TM$. For an orthonormal frame $\{e_1, \dots, e_n\}$ of M, the Ricci tensor S of M is defined by $S(X,Y) = \sum_{i=1}^n g(R(e_i, X)Y, e_i)$. So, from (21) we have

$$S(X,Y) = \frac{c+3s}{4}(n-1)g(X,Y). \tag{22}$$

Putting $B = S$, $T = \sigma$ in (2) and (3), we get

$$\begin{aligned} Q(S,\sigma)(Z,W;X,Y) = &- S(Y,Z)\sigma(X,W) + S(X,Z)\sigma(Y,W) \\ &- S(Y,W)\sigma(Z,X) + S(X,W)\sigma(Z,Y). \end{aligned} \tag{23}$$

From (14), (5), (11) and (23), the condition (4) turns into

$$\begin{aligned} -A_{\varphi W}R(X,Y)Z - A_{\varphi Z}R(X,Y)W = L\{ &- S(Y,Z)A_{\varphi X}W + S(X,Z)A_{\varphi Y}W \\ &- S(Y,W)A_{\varphi X}Z + S(X,W)A_{\varphi Y}Z\}. \end{aligned} \tag{24}$$

So, a Legendrian normally flat submanifold M^n of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$ is Ricci generalized pseudo-parallel if and only if the equation (24) holds.

Theorem 3. *Let $\widetilde{M}^{2n+s}(c)$, $c \neq -3s$, be an $\mathcal{S}$-space form of constant φ-sectional curvature c and M^n be a Legendrian normally flat submanifold of $\widetilde{M}^{2n+s}(c)$. If M^n is Ricci generalized pseudo-parallel, then either M^n is minimal or $L = \frac{1}{n-1}$.*

Proof. Let M be a Ricci generalized pseudo-parallel, since M is a Legendrian normally flat submanifold, we choose an orthonormal basis of $T_p^\perp M$ of the form $\{e_{n+1} = \varphi e_1, \dots, e_{2n} = \varphi e_n, e_{2n+1} = \xi_1, \dots, e_{2n+s} = \xi_s\}$ and for any $i, j \in \{1, \dots, n\}$, $\alpha \in \{1, \dots, s\}$ denote λ_i^{n+j} by the principle curvatures with respect to the normal vector field φe_j, i.e.

$$A_{\varphi e_j}(e_i) = \lambda_i^{n+j} e_i\,. \tag{25}$$

In this case the mean curvature vector can be written as

$$H^{n+j} = \frac{1}{n}\sum_{i=1}^n \lambda_i^{n+j}\,.$$

In view of (24), setting $X = e_i, Y = e_j, Z = e_k, W = e_l$ we obtain

$$-A_{\varphi e_l} R(e_i, e_j)e_k - A_{\varphi e_k} R(e_i, e_j)e_l = L\big\{ - S(e_j, e_k)A_{\varphi e_i} e_l + S(e_i, e_k)A_{\varphi e_{l_j}} e_l - S(e_j, e_l)A_{\varphi e_i} e_k + S(e_i, e_l)A_{\varphi e_j} e_k\big\}. \tag{26}$$

Substituting (22), (25) into (26) and for any $e_m \in TM$, we get

$$-\lambda_m^{n+l} R_{ijkm} - \lambda_m^{n+k} R_{ijlm} = \frac{c+3s}{4}(n-1)L\big\{ - \lambda_l^{n+i}\delta_{jk}\delta_{lm} + \lambda_l^{n+j}\delta_{ik}\delta_{lm} - \lambda_k^{n+i}\delta_{jl}\delta_{km} + \lambda_k^{n+j}\delta_{il}\delta_{km}\big\}, \tag{27}$$

where $g(e_i, e_j) = \delta_{ij}$ and $1 \le i, j, k, l, m \le n$. Since,

$$R_{ijkm} = \frac{c+3s}{4}\Big\{\delta_{jk}\delta_{im} - \delta_{ik}\delta_{jm}\Big\}, \qquad R_{ijlm} = \frac{c+3s}{4}\Big\{\delta_{jl}\delta_{im} - \delta_{il}\delta_{jm}\Big\}, \tag{28}$$

by the use of (28), equation (27) turns into

$$-\lambda_m^{n+l}(\delta_{jk}\delta_{im} - \delta_{ik}\delta_{jm}) - \lambda_m^{n+k}(\delta_{jl}\delta_{im} - \delta_{il}\delta_{jm}) = (n-1)L\big\{ - \lambda_l^{n+i}\delta_{jk}\delta_{lm} + \lambda_l^{n+j}\delta_{ik}\delta_{lm} - \lambda_k^{n+i}\delta_{jl}\delta_{km} + \lambda_k^{n+j}\delta_{il}\delta_{km}\big\}.$$

Hence, if we put $k = i$, $m = j$, we get

$$-\lambda_j^{n+l}(\delta_{ij} - \delta_{ii}\delta_{jj}) - \lambda_j^{n+j}\delta_{ij}(\delta_{jl}\delta_{ij} - \delta_{il}\delta_{jj}) = (n-1)L\big\{ - \lambda_l^{n+l}\delta_{il}\delta_{ij}\delta_{jl} + \lambda_l^{n+l}\delta_{jl}\delta_{ii} - \lambda_i^{n+i}\delta_{jl}\delta_{ij} + \lambda_i^{n+i}\delta_{ij}\delta_{il}\big\}, \tag{29}$$

because it follows from (11) that

$$\lambda_i^{n+j} = g(A_{\varphi e_j} e_i, e_i) = g(A_{\varphi e_i} e_i, e_j) = \lambda_i^{n+i}\delta_{ij}\,.$$

Summing over $i = 1, \dots, n$ and $j = 1, \dots, n$ in (29) respectively, we have

$$H^{n+l} = \frac{n-1}{n} L\lambda_l^{n+l}. \tag{30}$$

On the other hand, by substituting (21) and (22) in (24), we obtain

$$[(n-1)L - 1]\{-g(Y, Z)A_{\varphi X} W + g(X, Z)A_{\varphi Y} W - g(Y, W)A_{\varphi X} Z + g(X, W)A_{\varphi Y} Z\} = 0. \tag{31}$$

By setting $X = e_i, Y = e_j, Z = e_k, W = e_l$ and substituting (25) into (31), for any $e_m \in TM$ we get

$$[(n-1)L - 1]\{\lambda_l^{n+j}\delta_{ik}\delta_{lm} - \lambda_l^{n+i}\delta_{jk}\delta_{lm} + \lambda_k^{n+j}\delta_{il}\delta_{km} - \lambda_k^{n+i}\delta_{jl}\delta_{km}\} = 0.$$

In the same way, we put $k = i$, $m = j$ in the above equation

$$[(n-1)L - 1]\{\lambda_l^{n+l}\delta_{ii}\delta_{jl} - \lambda_l^{n+l}\delta_{il}\delta_{ij}\delta_{jl} + \lambda_i^{n+i}\delta_{ij}\delta_{il} - \lambda_i^{n+i}\delta_{jl}\delta_{ij}\} = 0. \tag{32}$$

Furthermore, by summing over $i = 1, \dots, n$ and $j = 1, \dots, n$ in (32), we obtain

$$[(n-1)L - 1](n-1)\lambda_l^{n+l} = 0.$$

As $n > 1$ we have

$$[(n-1)L - 1]\lambda_l^{n+l} = 0. \tag{33}$$

Comparing (30) and (33), we deduce that if $L = 0$ then $H^{n+l} = 0$ for any $1 \le l \le n$, i.e. M is minimal. If $L \neq 0$, then $\frac{[(n-1)L-1]n}{(n-1)L} H^{n+l} = 0$, which implies $H^{n+l} = 0$ or $L = \frac{1}{n-1}$. $\square$

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У поданому дослідженні ми розглядаємо лежандровий нормально плоский підмноговид M $(2n+s)$-вимірної $\mathcal{S}$-просторової форми $\widetilde{M}^{2n+s}(c)$ сталої φ-секційної кривизни c. Ми показали, що якщо M є псевдопаралельним, то M є напівпаралельним або тотально геодезичним.

Ми також довели, що якщо M є узагальнено псевдопаралельним підмноговидом Річчі, то або M є мінімальним, або $L = \frac{1}{n-1}$ при $c \neq -3s$.

Ключові слова і фрази: $\mathcal{S}$-просторова форма, лежандровий підмноговид, нормально плоский підмноговид, псевдопаралельний підмноговид, узагальнено псевдопаралельний підмноговид Річчі.