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# NILPOTENT LIE ALGEBRAS OF DERIVATIONS WITH THE CENTER OF SMALL CORANK

Let $\mathbb{K}$ be a field of characteristic zero, $A$ be an integral domain over $\mathbb{K}$ with the field of fractions $R = Frac(A)$, and $Der_{\mathbb{K}}A$ be the Lie algebra of all $\mathbb{K}$-derivations on $A$. Let $W(A) := RDer_{\mathbb{K}}A$ and $L$ be a nilpotent subalgebra of rank $n$ over $R$ of the Lie algebra $W(A)$. We prove that if the center $Z = Z(L)$ is of rank $\geq n-2$ over $R$ and $F = F(L)$ is the field of constants for $L$ in $R$, then the Lie algebra $FL$ is contained in a locally nilpotent subalgebra of $W(A)$ of rank $n$ over $R$ with a natural basis over the field $R$. It is also proved that the Lie algebra $FL$ can be isomorphically embedded (as an abstract Lie algebra) into the triangular Lie algebra $u_n(F)$, which was studied early by other authors.

*Key words and phrases:* derivation, vector field, Lie algebra, nilpotent algebra, integral domain.

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## INTRODUCTION

Let $\mathbb{K}$ be a field of characteristic zero, $A$ be an integral domain over $\mathbb{K}$, and $R = \mathrm{Frac}(A)$ be its field of fractions. Recall that a $\mathbb{K}$-derivation $D$ on $A$ is a $\mathbb{K}$-linear operator on the vector space $A$ satisfying the Leibniz rule $D(ab) = D(a)b + aD(b)$ for any $a, b \in A$. The set $\mathrm{Der}_{\mathbb{K}} A$ of all $\mathbb{K}$-derivations on $A$ is a Lie algebra over $\mathbb{K}$ with the Lie bracket $[D_1, D_2] = D_1D_2 - D_2D_1$. The Lie algebra $\mathrm{Der}_{\mathbb{K}} A$ can be isomorphically embedded into the Lie algebra $\mathrm{Der}_{\mathbb{K}} R$ (any derivation $D$ on $A$ can be uniquely extended on $R$ by the rule $D(a/b) = (D(a)b - aD(b))/b^2$, $a, b \in A$). We denote by $W(A)$ the subalgebra $R\,\mathrm{Der}_{\mathbb{K}} A$ of the Lie algebra $\mathrm{Der}_{\mathbb{K}} R$ (note that $W(A)$ and $\mathrm{Der}_{\mathbb{K}} R$ are Lie algebras over the field $\mathbb{K}$ but not over $R$). Nevertheless, $W(A)$ and $\mathrm{Der}_{\mathbb{K}} R$ are vector spaces over the field $R$, so one can define the rank $\mathrm{rk}_R L$ for any subalgebra L of the Lie algebra $W(A)$ by the rule $\mathrm{rk}_R L = \dim_R RL$. Every subalgebra $L$ of the Lie algebra $W(A)$ determines its field of constants in $R$ by

$$F = F(L) := \{r \in R \mid D(r) = 0 \ \text{ for all } D \in L\}.$$

The product $FL = \{\sum \alpha_i D_i \mid \alpha_i \in F,\ D_i \in L\}$ is a Lie algebra over the field $F$, this Lie algebra often has simpler structure than $L$ itself (note that such an extension of the ground field preserves the main properties of $L$ from the viewpoint of Lie theory).

We study nilpotent subalgebras $L \subseteq W(A)$ of rank $n \geq 3$ over $R$ with the center $Z = Z(L)$ of rank $\geq n-2$ over $R$, i.e. with the center of corank $\leq 2$ over $R$. We prove that $FL$ is contained

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in a locally nilpotent subalgebra of $W(A)$ with a natural basis over $R$, similar to the standard basis of the triangular Lie algebra $U_n(F)$ (Theorem 1). As a consequence, we get an isomorphic embedding (as Lie algebras) of the Lie algebra $FL$ over $F$ into the triangular Lie algebra $u_n(F)$ over $F$ (Theorem 2). These results generalize main results of the papers [8] and [9]. Note that the problem of classifying finite dimensional Lie algebras from Theorem 1 up to isomorphism is wild (i.e., it contains the hopeless problem of classifying pairs of square matrices up to similarity, see [3]). Triangular Lie algebras were studied in [1] and [2], they are locally nilpotent but not nilpotent.

We use standard notations. The ground field $\mathbb{K}$ is arbitrary of characteristic zero. If $F$ is a subfield of a field $R$ and $r_1, \dots, r_k \in R$, then $F\langle r_1, \dots, r_k\rangle$ is the set of all linear combinations of $r_i$ with coefficients in $F$, it is a subspace in the $F$-space $R$, for an infinite set $\{r_1, \dots, r_k, \dots\}$ we use the notation $F\left\langle \{r_i\}_{i=1}^{\infty}\right\rangle$. The triangular subalgebra $u_n(\mathbb{K})$ of the Lie algebra $W_n(\mathbb{K}) := \operatorname{Der}_{\mathbb{K}} \mathbb{K}[x_1, \dots, x_n]$ consists of all the derivations on $\mathbb{K}[x_1, \dots, x_n]$ of the form

$$D = f_1(x_2, \dots, x_n)\frac{\partial}{\partial x_1} + \cdots + f_{n-1}(x_n)\frac{\partial}{\partial x_{n-1}} + f_n\frac{\partial}{\partial x_1},$$

where $f_i \in \mathbb{K}[x_{i+1}, \dots, x_n]$, $f_n \in \mathbb{K}$. If $D \in W(A)$, then $\operatorname{Ker} D$ denotes the field of constants for $D$ in $R$, i.e., $\operatorname{Ker} D = \{r \in R \mid D(r) = 0\}$.

## 1 MAIN PROPERTIES OF NILPOTENT SUBALGEBRAS OF W(A)

We often use the next relations for derivations which are well known (see, for example [7]). Let $D_1, D_2 \in W(A)$ and $a, b \in R$. Then

1) $[aD_1, bD_2] = ab[D_1, D_2] + aD_1(b)D_2 - bD_2(a)D_1$;

2) if $a, b \in \operatorname{Ker} D_1 \cap \operatorname{Ker} D_2$, then $[aD_1, bD_2] = ab[D_1, D_2]$.

The next two lemmas contain some results about derivations and Lie algebras of derivations.

**Lemma 1** ([6], Lemma 2). *Let $L$ be a subalgebra of the Lie algebra $\operatorname{Der}_{\mathbb{K}} R$ and $F$ the field of constants for $L$ in $R$. Then $FL$ is a Lie algebra over $F$, and if $L$ is abelian, nilpotent or solvable, then so is $FL$, respectively.*

**Lemma 2** ([6], Proposition 1). *Let $L$ be a nilpotent subalgebra of the Lie algebra $W(A)$ with $\operatorname{rk}_R L < \infty$ and $F = F(L)$ the field of constants for $L$ in $R$. Then*

*1) $FL$ is finite dimensional over $F$;*

*2) if $\operatorname{rk}_R L = 1$, then $L$ is abelian and $\dim_F FL = 1$;*

*3) if $\operatorname{rk}_R L = 2$, then $FL$ is either abelian with $\dim_F FL = 2$ or $FL$ is of the form*

$$FL = F\left\langle D_2, D_1, aD_1, \dots, \frac{a^k}{k!}D_1\right\rangle,$$

*for some $D_1, D_2 \in FL$ and $a \in R$ such that $[D_1, D_2] = 0$, $D_2(a) = 1$, $D_1(a) = 0$.*

**Lemma 3.** *Let $L$ be a nilpotent subalgebra of the Lie algebra $W(A)$ of rank $n$ over $R$ with the center $Z = Z(L)$ of rank $k$ over $R$. Then $I := RZ \cap L$ is an abelian ideal of $L$ with $\mathrm{rk}_R\, I = k$.*

*Proof.* By Lemma 4 from [6], $I$ is an ideal of the Lie algebra $L$. Let us show that $I$ is abelian. Let us choose an arbitrary basis $D_1, \dots, D_k$ of the center $Z$ over $R$ ( i.e., a maximal by inclusion linearly independent over $R$ subset of $Z$). One can easy to see that $D_1, \dots, D_k$ is a basis of the ideal $I$ as well, so we can write for each element $D \in I$

$$D = a_1 D_1 + \cdots + a_k D_k$$

for some $a_1, \dots, a_k \in R$. Since $D_j \in Z,\ j = 1, \dots, k$, it holds

$$[D_j, D] = [D_j, \sum_{i=1}^{k} a_i D_i] = \sum_{i=1}^{k} D_j(a_i) D_i = 0 \tag{1}$$

for $j = 1, \dots, k$. The derivations $D_1, \dots, D_n$ are linearly independent over the field $R$, hence we obtain from (1) that $D_j(a_i) = 0,\ i, j = 1, \dots, k$. Therefore we have for each element $\overline{D} = b_1 D_1 + \dots b_k D_k$ of the ideal $I$ the next equalities

$$[D, \overline{D}] = [\sum_{i=1}^{k} a_i D_i, \sum_{j=1}^{k} b_j D_j] = \sum_{i,j=1}^{k} a_i b_j [D_i, D_j] = 0,$$

since $D_i(b_j) = D_j(a_i) = 0$ as mentioned above. The latter means that $I$ is an abelian ideal. Besides, obviously $\mathrm{rk}_R\, I = k$. □

**Lemma 4.** *Let $L$ be a nilpotent subalgebra of the Lie algebra $W(A)$, $Z = Z(L)$ the center of $L$, $I := RZ \cap L$ and $F$ the field of constants for $L$ in $R$. If for some $D \in L$ it holds $[D, FI] \subseteq FI$, $[D, FI] \neq 0$, then there exist a basis $D_1, \dots, D_m$ of the ideal $FI$ of the Lie algebra $FL$ over $R$ and $a \in R$ such that $D(a) = 1,\ D_i(a) = 0,\ i = 1, \dots, m$. Besides, each element $\overline{D} \in FI$ is of the form $\overline{D} = f_1(a) D_1 + \cdots + f_m(a) D_m$ for some polynomials $f_i \in F_1[t]$, where $F_1$ is the field of constants for the subalgebra $L_1 = FI + FD$ in $R$.*

*Proof.* By Lemma 3, the intersection $I = RZ \cap L$ is an abelian ideal of the Lie algebra $L$ and therefore $FI$ is an abelian ideal of the Lie algebra $FL$. Choose a basis $D_1, \dots, D_m$ of $FI$ over the field $R$ in such a way that $D_1, \dots, D_m \in Z$. Then $FZ$ is the center of the Lie algebra $FL$. Now take any basis $T_1, \dots, T_s$ of the $F$-space $FI$ (note that the Lie algebra $FL$ is finite dimensional over the field $F$ by [6]). Every basis element $T_i$ can be written in the form $T_i = \sum_{j=1}^{m} r_{ij} D_j$, $i = 1, \dots, s$, for some $r_{ij} \in R$. Denote by $B$ the subring $B = F[r_{ij}, i = 1, \dots, s, j = 1, \dots, m]$ of the field $R$ generated by $F$ and the elements $r_{ij}$. Since the linear operator $\mathrm{ad}\, D$ is nilpotent on the $F$-space $FI$ the derivation $D$ is locally nilpotent on the ring $B$. Indeed,

$$[D, T_i] = [D, \sum_{j=1}^{m} r_{ij} D_j] = \sum_{j=1}^{m} D(r_{ij}) D_j$$

and therefore

$$(\mathrm{ad}\, D)^{k_i}(T_i) = \sum_{j=1}^{m} D^{k_i}(r_{ij}) D_j = 0$$

for some natural $k_i$, $i = 1, \ldots, s$. Denoting $\overline{k} = \max_{1 \le t \le s} k_t$, we get $D^{\overline{k}}(r_{ij}) = 0$ and therefore $D$ is locally nilpotent on $B$. One can easily show that there exists an element $p \in B$ (a preslice) such that $D(p) \in \operatorname{Ker} D, D(p) \neq 0$. Then denoting $a := p/D(p)$, we have $D(a) = 1$ (such an element $a$ is called a slice for $D$). The ring $B$ is contained in the localization $B[c^{-1}]$, where $c := D(p)$ and the derivation $D$ is locally nilpotent on $B[c^{-1}]$. Note that $B[c^{-1}] \subseteq F_1$, where $F_1$ is the field of constants for $L_1 = FI + FD$ in $R$. Besides, by Principle 11 from [4] it holds $B[c^{-1}] = B_0[a]$, where $B_0$ is the kernel of $D$ in $B[c^{-1}]$. This completes the proof because $B \subseteq B[c^{-1}]$ and every element $\overline{D}$ of $FI$ is of the form $\overline{D} = b_1 D_1 + \ldots b_m D_m, b_i \in B$. □

**Lemma 5.** *Let $L$ be a nilpotent subalgebra of the Lie algebra $W(A)$, $Z = Z(L)$ the center of $L$, $F$ the field of constants of $L$ in $R$ and $I = RZ \cap L$. Let $\operatorname{rk}_R Z = n-2$. Then the following statements for the Lie algebra $FL/FI$ hold*

1) *if $FL/FI$ is abelian, then $\dim_F FL/FI = 2$;*

2) *if $FL/FI$ is nonabelian, then there exist elements $D_{n-1}, D_n \in FL$, $b \in R$ such that*

$$FL/FI = F\left\langle D_{n-1} + FI,\, bD_{n-1} + FI,\, \ldots,\, \frac{b^k}{k!} D_{n-1} + FI,\, D_n + FI \right\rangle$$

*with $k \ge 1$, $D_n(b) = 1$, $D_{n-1}(b) = 0$, $D(b) = 0$ for all $D \in FI$.*

*Proof.* Let us choose a basis $D_1, \ldots, D_{n-2}$ of the center $Z$ over the field $R$ and any central ideal $FD_{n-1} + FI$ of the quotient algebra $FL/FI$. Denote the intersection $R(I + \mathbb{K}D_{n-1}) \cap L$ by $I_1$. Then it is easy to see that $FI_1$ is an ideal of the Lie algebra $FL$ of rank $n-1$ over $R$ and the Lie algebra $FL/FI_1$ is of dimension 1 over $F$ (by Lemma 5 from [6]). Let us choose an arbitrary element $D_n \in FL \setminus FI_1$. Then $D_1, \ldots, D_n$ is a basis of the Lie algebra $FL$ over the field $R$.

Case 1. The quotient algebra $FL/FI$ is abelian. Let us show that

$$FL/FI = F\langle D_{n-1} + FI,\, D_n + FI\rangle.$$

Indeed, let us take any elements $S_1 + FI,\ S_2 + FI$ of $FL/FI$ and write

$$S_1 = \sum_{i=1}^{n} r_i D_i, \quad S_2 = \sum_{i=1}^{n} s_i D_i, \quad r_i,\ s_i \in R,\ i, j = 1, \ldots, n.$$

From the equalities $[D_i, S_1] = [D_i, S_2] = 0$, $i = 1, \ldots, n-2$ (recall that $D_i \in Z(L)$, $i = 1, \ldots, n-2$) it follows that

$$D_i(r_j) = D_i(s_j) = 0,\ i = 1 \ldots, n-2,\ j = 1, \ldots, n. \tag{2}$$

Since $[FL, FI] \subseteq FI$ we have $[D_i, S_1], [D_i, S_2] \in FI$ for $i = n-1, n$. Taking into account the equalities (2) we derive that

$$D_i(s_j) = D_i(r_j) = 0,\ i = n-1,\ n,\ j = n-1,\ n.$$

Therefore it holds $s_i, r_i \in F$ for $i = n-1, n$ and the elements $D_{n-1} + FI$, $D_n + FI$ form a basis for the abelian Lie algebra $FL/FI$ over the field $F$.

Case 2. $FL/FI$ is nonabelian. Then $\dim_F FL/FI \ge 3$ because the Lie algebra $FL/FI$ is nilpotent. Let us show that the ideal $FI_1/FI$ of the Lie algebra $FL/FI$ is abelian (recall that

$I_1 = R(I + \mathbb{K}D_{n-1}) \cap L$). Since $D_{n-1} + FI$ lies in the center of the quotient algebra $FL/FI$ we have for any element $rD_{n-1} + FI$ of the ideal $FI_1/FI$ the following equality

$$[D_{n-1} + FI, rD_{n-1} + FI] = FI.$$

Hence $D_{n-1}(r)D_{n-1} + FI = FI$. The last equality implies $D_{n-1}(r) = 0$. But then for any elements $rD_{n-1} + FI$, $sD_{n-1} + FI$ of $FI_1/FI$ we get

$$\begin{aligned}[rD_{n-1} + FI, sD_{n-1} + FI] &= [rD_{n-1}, sD_{n-1} + FI] \\ &= (D_{n-1}(s)r - sD_{n-1}(r))D_{n-1} + FI = FI.\end{aligned}$$

The latter means that $FI_1/FI$ is an abelian ideal of $FL/FI$.

Further, the nilpotent linear operator $\operatorname{ad} D_n$ acts on the linear space $FI_1/FI$ with $\operatorname{Ker}(\operatorname{ad} D_n) = FD_{n-1} + FI$. Indeed, let $\operatorname{ad} D_n(rD_{n-1} + FI) = FI$. Then $[D_n, rD_{n-1}] \in FI$ and therefore $D_n(r)D_{n-1} \in FI$. This relation implies $D_n(r) = 0$ and taking into account the equalities $D_i(r) = 0$, $i = 1, \dots, n-1$, we get that $r \in F$ and $\operatorname{Ker}(\operatorname{ad} D_n) = FD_{n-1} + FI$. It follows from this relation that the linear operator $\operatorname{ad} D_n$ on $FI/FI_1$ has only one Jordan chain and the Jordan basis can be chosen with the first element $D_{n-1} + FI$. Since $\dim FI_1/FI \geq 2$ (recall that $\dim_F FL/FI \geq 3$) the chain is of length $\geq 2$. Let us take the second element of the Jordan chain in the form $bD_{n-1} + FI$, $b \in R$. Then $\operatorname{ad} D_n(bD_{n-1} + FI) = D_{n-1} + FI$ and hence $D_n(b) = 1$. The inclusion $[D_{n-1}, bD_{n-1}] \in FI$ implies the equality $D_{n-1}(b) = 0$, and analogously one can obtain $D_i(b) = 0$, $i = 1, \dots, n-2$.

If $\dim FI_1/FI \geq 3$ and $cD_{n-1} + FI$ is the third element of the Jordan chain of $\operatorname{ad} D_n$, then repeating the above considerations we get $D_n(c) = b$. Then the element $\alpha = \frac{b^2}{2!} - c \in R$ satisfies the relations $D_{n-1}(\alpha) = D_n(\alpha) = 0$ and $D_i(\alpha) = 0$, $i = 1, \dots, n-2$, since $D_i(b) = D_i(c) = 0$. Therefore, $\alpha = \frac{b^2}{2!} - c \in F$ and $c = \frac{b^2}{2!} + \alpha$. Since $\alpha D_{n-1} + FI \in \operatorname{Ker}(\operatorname{ad} D_n)$, we can take the third element of the Jordan chain in the form $\frac{b^2}{2!}D_{n-1} + FI$. Repeating the consideration one can build the needed basis of the Lie algebra $FL/FI$. □

**Lemma 6.** *Let $L$ be a nilpotent subalgebra of $W(A)$ with the center $Z = Z(L)$ of $\operatorname{rk}_R Z = n-2$, $F$ the field of constants for $L$ in $R$ and $I = RZ \cap L$. If $S, T$ are elements of $L$ such that $[S, T] \in I$, the rank of the subalgebra $L_1$ spanned by $I, S, T$ equals $n$ and $C_{FL}(FI) = FI$, then there exist elements $a, b \in R$ such that $S(a) = 1$, $T(a) = 0$, $S(b) = 0$, $T(b) = 1$ and $D(a) = D(b) = 0$ for each $D \in I$. Besides, every element $D \in FI$ can be written in the form $D = f_1(a,b)D_1 + \dots + f_{n-2}(a,b)D_{n-2}$ with some polynomials $f_i(u,v) \in F[u,v]$.*

*Proof.* Let us choose a basis $D_1, \dots, D_{n-2}$ of $Z$ over $R$. By the lemma conditions, one can easily see that $D_1, \dots, D_{n-2}, S, T$ is a basis of $L$ over $R$. The ideal $FI$ of the Lie algebra $FL$ is abelian by Lemma 3 and $\operatorname{ad} S, \operatorname{ad} T$ are commuting linear operators on the vector space $FI$ (over $F$). Take a basis $T_1, \dots, T_s$ of $FI$ over $F$ (recall that $\dim_F FL < \infty$ by Theorem 1 from [6]) and write $T_i = \sum_{j=1}^{n-2} r_{ij}D_j$ for some $r_{ij} \in R, i = 1, \dots, s, j = 1, \dots, n-2$. Denote by

$$B = F[r_{ij}, i = 1, \dots, s, j = 1, \dots, n-2],$$

the subring of $R$ generated by $F$ and all the coefficients $r_{ij}$. Then $B$ is invariant under the derivations $S$ and $T$, these derivations are locally nilpotent on $B$ and linearly independent over $R$ (by the condition $C_{FL}(FI) = FI$ of the lemma). By Lemma 4, there exists an element $a \in B[c^{-1}]$ such that

$$S(a) = 1, \quad D_i(a) = 0, \quad i = 1, \dots, n-2,$$

(here $c = S(p)$ for a preslice $p$ for $S$ in $B$). Since $c \in \operatorname{Ker} S$ and $[S, T] = 0$ one can assume without loss of generality that $T(c) \in \operatorname{Ker} T$. But then $T$ is a locally nilpotent derivation on the subring $B[c^{-1}]$. Repeating these considerations we can find an element $b \in B[c^{-1}][d^{-1}]$ with $T(b) = 1$ (here $d$ is a preslice for the derivation $T$ in $B[c^{-1}]$). Denote $B_1 = B[c^{-1}, d^{-1}]$, the subring of $R$ generated by $B, c^{-1}, d^{-1}$. Then using standard facts about locally nilpotent derivations (see, for example Principle 11 in [4]) one can show that $B_1 = B_0[a, b]$, where $B_0 = \operatorname{Ker} S \cap \operatorname{Ker} T$. Therefore every element $h$ of $B_1$ can be written in the form $h = f(a, b)$ with $f(u, v) \in F[u, v]$. Note that

$$F = \operatorname{Ker} T \cap \operatorname{Ker} S \cap_{i=1}^{n-2} \operatorname{Ker} D_i.$$

It follows from this representation of elements of $B_1$ that every element of the ideal $FI$ can be written in the form

$$D = f_1(a, b)D_1 + \cdots + f_{n-2}(a, b)D_{n-2}$$

with some polynomials $f_i(u, v) \in F[u, v]$. □

## 2 THE MAIN RESULTS

**Theorem 1.** *Let $L$ be a nilpotent subalgebra of rank $n \geq 3$ over $R$ from the Lie algebra $W(A)$, $Z = Z(L)$ the center of $L$ with $\operatorname{rk}_R Z \geq n-2$, $F$ the field of constants of $L$ in $R$. Then one of the following statements holds:*

1) $\dim_F FL = n$ *and $FL$ is either abelian or is a direct sum of a nonabelian nilpotent Lie algebra of dimension* 3 *and an abelian Lie algebra;*

2) $\dim_F FL \geq n+1$ *and $FL$ lies in one of the locally nilpotent subalgebras $L_1, L_2$ of $W(A)$ of rank $n$ over $R$, which have a basis $D_1, \dots, D_n$ over $R$ satisfying the relations $[D_i, D_j] = 0$, $i, j = 1, \dots, n$, and are one of the form*

$$L_1 = F\left\langle \left\{ \frac{b^i}{i!} D_1 \right\}_{i=0}^{\infty}, \dots, \left\{ \frac{b^i}{i!} D_{n-1} \right\}_{i=0}^{\infty}, D_n \right\rangle$$

*for some $b \in R$ such that $D_i(b) = 0$, $i = 1, \dots, n-1$, and $D_n(b) = 1$,*

$$L_2 = F\left\langle \left\{ \frac{a^i b^j}{i!j!} D_1 \right\}_{i,j=0}^{\infty}, \dots, \left\{ \frac{a^i b^j}{i!j!} D_{n-2} \right\}_{i,j=0}^{\infty}, \left\{ \frac{b^i}{i!} D_{n-1} \right\}_{i=0}^{\infty}, D_n \right\rangle$$

*for some $a, b \in R$ such that $D_{n-1}(a) = 1$, $D_n(a) = 0$, $D_{n-1}(b) = 0$, $D_n(b) = 1$, $D_i(a) = D_i(b) = 0$, $i = 1, \dots, n-2$.*

*Proof.* By Lemma 3, $I = RZ \cap L$ is an abelian ideal of $L$ and therefore $FI$ is an abelian ideal of the Lie algebra $FL$ (here the Lie algebra $FL$ is considered over the field $F$). Let $\dim_F FL = n$. It is obvious that $\dim_F M = \operatorname{rk}_R M$ for any subalgebra $M$ of the Lie algebra $FL$, in particular $\dim_F FZ \geq n-2$ because of conditions of the theorem. We may restrict ourselves only on nonabelian algebras and assume $\dim_F FZ = n-2$ (in case $\dim_F FZ \geq n-1$ the Lie algebra $FL$ is abelian). Since $FL$ is nilpotent of nilpotency class 2, one can easily show that $FL$ is a direct sum of a nonabelian Lie algebra of dimension 3 and an abelian algebra and satisfies the condition 1) of the theorem. So, we may assume further that $\dim_F FL \geq n+1$.

Case 1. $\mathrm{rk}_R Z = n-1$. Then $FI$ is of codimension 1 in $FL$ by Lemma 5 from [6]. Therefore $\dim_F FI \geq n$ because of $\dim_F FL \geq n+1$ and $\dim_F FL/FI = 1$. We obtain the strong inclusion $FZ \subsetneqq FI$ because of $\dim_F FZ = n-1$. Take a basis $D_1, \ldots, D_{n-1}$ of $Z$ over $R$ and an element $D_n \in FL \setminus FI$. Then $D_1, \ldots, D_n$ is a basis for $FL$ over $R$ and $[D_n, FI] \neq 0$. Using Lemma 4 one can easily show that $FL$ is contained in a subalgebra of type $L_1$ from $W(A)$.

Case 2. $\mathrm{rk}_R Z = n-2$ and $\dim_F FI = n-2$. Then $FI = FZ$, $\dim_F FL/FI \geq 3$ and therefore by Lemma 5 the quotient algebra $FL/FI$ is of the form

$$FL/FI = F\left\langle \left\{ \frac{b^i}{i!} D_{n-1} + FI \right\}_{i=0}^{k}, D_n + FI \right\rangle$$

for some $k \geq 1$, $b \in R$ such that $D_n(b) = 1$, $D_{n-1}(b) = 0$ and $D(b) = 0$ for each $D \in FI$.

The $F$-space

$$J = F\left\langle \left\{ \frac{b^i}{i!} D_1 \right\}_{i=0}^{\infty}, \ldots, \left\{ \frac{b^i}{i!} D_{n-1} \right\}_{i=0}^{\infty} \right\rangle$$

is an abelian subalgebra of $W(A)$ and $[FL, J] \subseteq J$. Therefore the sum

$$J + F\left\langle \left\{ \frac{b^i}{i!} D_{n-1} \right\}_{i=0}^{\infty}, D_n \right\rangle$$

is a subalgebra of the Lie algebra $W(A)$. If $[D_n, D_{n-1}] \neq 0$, then taking into account the relation $[D_n, D_{n-1}] \in FI$ one can write

$$[D_n, D_{n-1}] = \alpha_1 D_1 + \cdots + \alpha_{n-2} D_{n-2}$$

for some $\alpha_i \in F$ (recall that $FI = FZ$). Consider the element of $W(A)$ of the form

$$\widetilde{D}_{n-1} = D_{n-1} - \alpha_1 b D_1 - \cdots - \alpha_{n-2} b D_{n-2}.$$

Since $[D_n, \widetilde{D}_{n-1}] = 0$, $\widetilde{D}_{n-1}(b) = 0$, one can replace the element $D_{n-1}$ with the element $\widetilde{D}_{n-1}$ and assume without loss of generality that $[D_n, D_{n-1}] = 0$. As a result we get the Lie algebra of the type $L_1$ from the statement of the theorem.

Case 3. $\mathrm{rk}_R Z = n-2$ and $\dim_F FI > n-2$. First, suppose $C_{FL}(FI) = FI$. Then by Lemma 6 there are a basis $D_1, \ldots, D_{n-2}$ of the ideal $FI$ over $R$ and elements $a, b \in R$ such that

$$D_{n-1}(a) = 1,\ D_n(a) = 0,\ D_{n-1}(b) = 0,\ D_n(b) = 1$$

and

$$D_i(a) = D_i(b) = 0,\ i = 1, \ldots, n-2,$$

and each element $D \in FI$ can be written in the form

$$D = f_1(a,b) D_1 + \cdots + f_{n-2}(a,b) D_{n-2}$$

for some polynomials $f_i(u,v) \in F[u,v]$.

Consider the $F$-subspace

$$J = F[a,b]D_1 + \cdots + F[a,b]D_{n-2}$$

of the Lie algebra $W(A)$. It is easy to see that $J$ is an abelian subalgebra of $W(A)$ and $[FL, J] \subseteq J$. If $[D_n, D_{n-1}] = 0$, then it is obvious that the subalgebra $FL + J$ is of type $L_2$ of the theorem and $FL \subset L_1$. Let $[D_n, D_{n-1}] \neq 0$. Since $[D_n, D_{n-1}] \in FI$, it follows

$$[D_n, D_{n-1}] = h_1(a,b)D_1 + \cdots + h_{n-2}D_{n-2}$$

for some polynomials $h_i(u,v) \in F[u,v]$. Then the subalgebra $J$ has such an element

$$T = u_1(a,b)D_1 + \ldots u_{n-2}(a,b)D_{n-2}$$

that $D_n(u_i(a,b)) = h_i(a,b), i = 1, \ldots, n-2$ (recall that $D_n(a) = 0,\ D_n(b) = 1$), and hence the element $\widetilde{D}_{n-1} = D_{n-1} - T$ satisfies the equality $[D_n, T] = 0$. Replacing $D_{n-1}$ with $\widetilde{D}_{n-1}$ we get the needed basis of the Lie algebra $FL + J$ and see that $FL$ can be embedded into the Lie $L_2$ of $W(A)$. So in case of $C_{FL}(FI) = FI$ the Lie algebra $FL$ can be isomorphically embedded into the Lie algebra of type $L_2$ from the statement of the theorem.

Further, suppose $C_{FL}(FI) \neq FI$. Since $C_{FL}(FI) \supseteq FI$ one can easily show that $D_{n-1} \in C_{FL}(FI) \setminus FI$ (note that $FL/FI$ has the unique minimal ideal $FD_{n-1} + FI$). Then $[D_{n-1}, FI] = 0$, and therefore $[D_n, FI] \neq 0$. Therefore by Lemma 4 there is an element $c \in R$ such that

$$D_n(c) = 1,\ D_{n-1}(c) = 0,\ D_i(c) = 0,\ i = 1, \ldots, n-2.$$

Moreover, each element of $FI$ is of the form $g_1(c)D_1 + \cdots + g_{n-2}(c)D_{n-2}$ for some polynomials $g_i(u) \in F[u]$. By Lemma 5, the quotient algebra $FL/FI$ is of the form

$$FL/FI = F\left\langle \left\{ \frac{b^i}{i!}D_{n-1} + FI \right\}_{i=0}^{k}, D_n + FI \right\rangle$$

for some $b \in R, k \geq 1$ such that $D_n(b) = 1,\ D_{n-1}(b) = 0$. But then

$$D_{n-1}(b-c) = 0,\ D_n(b-c) = 0,\ D_i(b-c) = 0,$$

and hence $b - c = \alpha$ for some $\alpha \in F$. Without loss of generality we can assume $b = c$. The locally nilpotent subalgebra

$$L_1 = F\left\langle \left\{ \frac{a^i b^j}{i!j!}D_1 \right\}_{i,j=0}^{\infty}, \ldots, \left\{ \frac{a^i b^j}{i!j!}D_{n-2} \right\}_{i,j=0}^{\infty}, \left\{ \frac{b^i}{i!}D_{n-1} \right\}_{i=0}^{\infty}, D_n \right\rangle$$

of the Lie algebra $W(A)$ contains $FL$ and satisfies the conditions for the Lie algebra of type $L_2$ from the statement of the theorem, possibly except the condition $[D_n, D_{n-1}] = 0$. If $[D_n, D_{n-1}] \neq 0$, then from the inclusion $[D_n, D_{n-1}] \in FI$ it follows that

$$[D_n, D_{n-1}] = f_1(b)D_1 + \cdots + f_{n-2}(b)D_{n-2}$$

for some polynomials $f_i(u) \in F[u]$.

One can easily show that there is such an element

$$\overline{D} = h_1(b)D_1 + \cdots + h_{n-2}(b)D_{n-2} \in L_1,$$

that $[D_n, \overline{D}] = [D_n, D_{n-1}]$ (one can take antiderivations $h_i$ for polynomials $f_i$, $i = 1, \ldots, n-2$). Replacing $D_{n-1}$ with $D_{n-1} - \overline{D}$ we get the needed basis over $R$ of the Lie algebra $L_2$. $\square$

**Remark 1.** *Any Lie algebra of dimension $n$ over $F$ can be realized as a Lie algebra of rank $n$ over $R$ by Theorem 2 from [5]. So the Lie algebra of type 1) from Theorem 1 can be chosen in any way possible.*

As a corollary we get the next statement about embedding of Lie algebras of derivations.

**Theorem 2.** *Let $L$ be a nilpotent subalgebra of rank $n$ over $R$ of the Lie algebra $W(A)$, $Z = Z(L)$ be the center of $L$ and $F$ be the field of constants of $L$ in $R$. If $\mathrm{rk}_R Z \geq n-2$, then the Lie algebra $FL$ can be isomorphically embedded (as an abstract Lie algebra) into the triangular Lie algebra $u_n(F)$.*

*Proof.* First, suppose $\dim_F FL = n$. If $FL$ is abelian, then $FL$ is isomorphically embeddable into the Lie algebra $u_n(F)$ because the subalgebra $F\left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\rangle$ of $u_n(F)$ is abelian of dimension $n$ over $F$. So one can assume that $FL$ is nonabelian. Then by Theorem 1, $FL = M_1 \oplus M_2$, where $M_1$ is an abelian Lie algebra of dimension $n-3$ over $F$ and $M_2$ is nilpotent nonabelian with $\dim_F M_2 = 3$. The subalgebra $H_2 = F\left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3} \right\rangle$ of the Lie algebra $u_n(F)$ is obviously isomorphic to $M_2$. The abelian subalgebra $H_1 = F\left\langle \frac{\partial}{\partial x_4}, \dots, \frac{\partial}{\partial x_n} \right\rangle, n \geq 4$, is isomorphic to the Lie algebra $M_1$. So $FL \simeq H_1 \oplus H_2$ is isomorphic to a subalgebra of $u_n(F)$. Note that $H_1 \oplus H_2$ is of rank $n$ over the field $\mathbb{K}(x_1, \dots, x_n)$ of rational functions in $n$ variables.

Next, let $\dim_F FL > n$. By Theorem 1, the Lie algebra $FL$ lies in one of the subalgebras of types $L_1$ or $L_2$. Therefore it is sufficient to show that the subalgebras $L_1, L_2$ of $W(A)$ from Theorem 1 can be isomorphically embedded into the Lie algebra $u_n(F)$. In case $L_1$, we define a mapping $\varphi$ on the basis $D_1, \dots, D_n, \left\{\frac{b^i}{i!}D_i\right\}_{i=1}^{\infty}$ of $L_1$ over $R$ by the rule $\varphi(D_i) = \frac{\partial}{\partial x_i}, i = 1, \dots, n$, $\varphi(\frac{b^i}{i!}D_i) = \frac{x_n^i}{i!}\frac{\partial}{\partial x_i}, i = 1, \dots, n-1$, and then extend it on $L_1$ by linearity. One can easily see that the mapping $\varphi$ is an isomorphic embedding of the Lie algebra $L_1$ into $u_n(F)$. Analogously, on $L_2$ we define a mapping $\psi : L_2 \to u_n(F)$ by the rule

$$\psi(D_i) = \frac{\partial}{\partial x_i}, \quad i = 1, \dots, n, \qquad \psi(\frac{a^i b^j}{i!j!}D_k) = \frac{x_{n-1}^i x_n^j}{i!j!}\frac{\partial}{\partial x_k}, \quad k = 1, \dots, n-2$$

$$\psi(\frac{b^i}{i!}D_{n-1}) = \frac{x_n^i}{i!}\frac{\partial}{\partial x_{n-1}}, \quad i \geq 1, j \geq 1,$$

and further by linearity. Then $\psi$ is an isomorphic embedding of the Lie algebra $L_2$ into the Lie algebra $u_n(F)$. □

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Нехай $\mathbb{K}$ — поле характеристики нуль, $A$ — область цілісності над $\mathbb{K}$ з полем часток $R = Frac(A)$, і $Der_{\mathbb{K}}A$ — алгебра Лі $\mathbb{K}$-диференціювань $A$. Нехай $W(A) := RDer_{\mathbb{K}}A$ і $L$ — нільпотентна підалгебра рангу $n$ над $R$ Лі алгебри $W(A)$. Ми показуємо, що якщо центр $Z = Z(L)$ має ранг $\geq n-2$ над $R$ і $F = F(L)$ — поле констант алгебри Лі $L$ в $R$, то алгебра Лі $FL$ міститься в локально нільпотентній підалгебрі рангу $n$ над $R$ з природнім базисом над полем $R$. Також доводиться, що Лі алгебра $FL$ може бути ізоморфно вкладена (як абстрактна Лі алгебра) в трикутну алгебру Лі $u_n(F)$, що була досліджена раніше іншими авторами.

*Ключові слова і фрази:* диференціювання, векторне поле, алгебра Лі, нільпотентна алгебра, область цілісності.