Generalizations of ss-supplemented modules

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We introduce the concept of (strongly) ss-radical supplemented modules. We prove that if a submodule \( N \) of \( M \) is strongly ss-radical supplemented and \( \text{Rad}(M/N) = M/N \), then \( M \) is strongly ss-radical supplemented. For a left good ring \( R \), we show that \( \text{Rad}(R) \subseteq \text{Soc}(R) \) if and only if every left \( R \)-module is ss-radical supplemented. We characterize the rings over which all modules are strongly ss-radical supplemented. We also prove that over a left \( WV \)-ring every supplemented module is ss-supplemented.

Key words and phrases: semisimple module, (strongly) ss-radical supplemented module, \( WV \)-ring.

1 Introduction

Throughout the paper, all rings are associative with identity and all modules are unitary left modules. Let \( R \) be a ring and \( M \) be an \( R \)-module. By \( \text{Rad}(M) \) and \( \text{Soc}(M) \), we will denote the radical of \( M \) and the socle of \( M \), respectively. A submodule \( K \subseteq M \) is called small in \( M \), written \( K \ll M \), if for every submodule \( N \subseteq M \) the equality \( M = N + K \) implies \( N = M \).

In [9], D.X. Zhou and X.R. Zhang introduced the concept of socle of a module \( M \) to that of \( \text{Soc}_s(M) = \sum\{N \ll M | N \text{ is simple}\} \). It is clear that \( \text{Soc}_s(M) \subseteq \text{Rad}(M) \) and \( \text{Soc}_s(M) \subseteq \text{Soc}(M) \). A module \( M \) is called supplemented if every submodule \( N \) of \( M \) has a supplement, i.e. a submodule \( K \) minimal with respect to \( M = N + K \). \( K \) is a supplement of \( N \) in \( M \) if and only if \( M = N + K \) and \( N \cap K \ll K \).

For more properties of supplemented modules we refer to [3]. In [10], H. Zöschinger introduced a notion of modules whose radical has supplements and called them radical supplemented. In the same paper and in [11], he determined the structure of radical supplemented modules. In [2], E. Büyükaşık and E. Türkmen call a module \( M \) strongly radical supplemented if every submodule containing the radical has a supplement in \( M \). In [6], E. Kaynar, E. Türkmen and H. Çalışçı call a submodule \( V \) an ss-supplement of \( U \) in \( M \) if and only if \( M = U + V \) and \( U \cap V \subseteq \text{Soc}_s(V) \).

A module \( M \) is ss-supplemented if every submodule \( U \) of \( M \) has an ss-supplement in \( M \). It is shown in [6, Theorem 3.30] that a ring \( R \) is semiperfect and \( \text{Rad}(R) \subseteq \text{Soc}_s(R) \) if and only if every left \( R \)-module is ss-supplemented. Motivated by these results, we call a module ss-radical supplemented if \( \text{Rad}(M) \) has an ss-supplement in \( M \) and we call a module strongly ss-radical supplemented if every submodule containing the radical has an ss-supplement in \( M \).
Also we obtain the various properties of (strongly) ss-radical supplemented modules. We show that homomorphic images of strongly ss-radical supplemented modules are strongly ss-radical supplemented (Proposition 3) and for a left good ring R, we prove that \( \text{Rad}(R) \subseteq \text{Soc}_R(R) \) if and only if every left R-module is ss-radical supplemented. If \( R \) is a left \( WV \)-ring, then \( R \) is a left good ring and every left \( R \)-module is ss-radical supplemented. We have characterized the rings over which all modules are strongly ss-radical supplemented, by Theorem 3. We study on (strongly) ss-radical supplemented modules over Dedekind domains.

2 ss-Radical supplemented and strongly ss-radical supplemented modules

In this section, we give some properties of the (strongly) ss-radical supplemented modules. In particular, we provide characterizations of some classes of rings.

Recall that a module \( M \) is called radical if \( M \) has no maximal submodules, i.e. \( \text{Rad}(M) = M \).

Proposition 1. Let \( M \) be radical module. Then \( M \) is a strongly ss-radical supplemented module.

Proof. It is clear that \( \text{Rad}(M) \) has the trivial ss-supplement 0 in \( M \). Thus \( M \) is strongly ss-radical supplemented.

Let \( M \) be a module. By \( P(M) \) we denote the sum of all radical submodules of \( M \). Then \( P(M) \) is the largest radical submodule of \( M \).

Corollary 1. \( P(M) \) is strongly ss-radical supplemented for every module \( M \).

Example 1. (1) Consider the \( \mathbb{Z} \)-module \( M = \mathbb{Z} \). Then \( \mathbb{Z} \) is an ss-radical supplemented module because \( \text{Rad}(\mathbb{Z}) = 0 \).

(2) Let \( R \) be a commutative domain and \( K(R) \) be the fractions field of \( R \). It follows from Proposition 1 that \( K(R) \) is a strongly ss-radical supplemented \( R \)-module.

It is well known that every module with small radical is radical supplemented. The following example shows that a module with small radical need to be ss-radical supplemented. Firstly we need the following facts.

Lemma 1. Let \( M \) be a module and \( \text{Rad}(M) \subseteq \text{Soc}(M) \). Then \( M \) is ss-radical supplemented.

Proof. Obviously, \( M = M + \text{Rad}(M) \) and \( M \cap \text{Rad}(M) = \text{Rad}(M) \subseteq \text{Soc}(M) \). Therefore \( \text{Rad}(M) \) is semisimple and so \( M \) is an ss-supplement of \( \text{Rad}(M) \) in \( M \). Hence \( M \) is ss-radical supplemented.

Using Lemma 1, we obtain the following Corollary 2 and Corollary 3.

Corollary 2. Let \( M \) be a module and \( \text{Rad}(M) \ll M \). Then \( M \) is an ss-radical supplemented module if and only if \( \text{Rad}(M) \subseteq \text{Soc}(M) \).

Proof. By Lemma 1 and by [6, Corollary 3.3].

Example 2. Consider the \( \mathbb{Z} \)-module \( M = \mathbb{Z}_p \), where \( p \) is any prime integer. Then \( \text{Rad}(M) = < \overline{p} > \ll M \). Since \( \text{Rad}(M) \) is not semisimple, by Corollary 2, \( M \) is not ss-radical supplemented.
Let $R$ be a ring. $R$ is said to be a left max ring if every non-zero left $R$-module has a maximal submodule.

**Corollary 3.** Let $R$ be a left max ring and $M$ be an $R$-module. Then $M$ is ss-radical supplemented if and only if $\text{Rad}(M) \subseteq \text{Soc}(M)$.

**Proof.** Since $R$ is a left max ring, $M$ has a small radical. By Corollary 2, the proof follows.

Recall that a module $M$ is coatomic if every proper submodule of $M$ is contained in some maximal submodule of $M$.

**Proposition 2.** Let $R$ be a semilocal ring and $M$ be an ss-radical supplemented module. Then, every ss-supplement of $\text{Rad}(M)$ in $M$ is coatomic.

**Proof.** If $M = \text{Rad}(M)$, the proof is clear. Assume that $M \neq \text{Rad}(M)$. Let $V$ be an ss-supplement of $\text{Rad}(M)$ in $M$. Then $\text{Rad}(M) \cap V = \text{Rad}(V)$ is semisimple and so $\text{Rad}(V)$ is coatomic. Since $R$ is semilocal, it follows from [7, Theorem 3.5] that

$$
\frac{M}{\text{Rad}(M)} \cong \frac{V}{\text{Rad}(M) \cap V} = \frac{V}{\text{Rad}(V)}
$$

is semisimple. By [10, Lemma 3], we get that $V$ is coatomic.

**Proposition 3.** Every homomorphic image of a strongly ss-radical supplemented module is a strongly ss-radical supplemented module.

**Proof.** Let $M$ be a strongly ss-radical supplemented module and $L \subseteq N \subseteq M$ with $\text{Rad}(\frac{M}{L}) \subseteq \frac{N}{L}$. Consider the canonical projection $\pi : M \rightarrow \frac{M}{L}$. Then $\pi(\text{Rad}(M)) = \frac{\text{Rad}(M)}{L} \subseteq \frac{\text{Rad}(M)}{L} \subseteq \frac{N}{L}$ and so $\text{Rad}(M) \subseteq N$. By the hypothesis, $N$ has an ss-supplement, say $K$, in $M$. Clearly, we can write $\frac{M}{L} = \frac{N}{L} + \frac{K+L}{L}$. By [8, 19.3 (4)], we obtain that $\pi(N \cap K) = \frac{N \cap K + L}{L} = \frac{N \cap K}{L} \ll \pi(K) = \frac{K+L}{L}$ and $\pi(N \cap K) = \frac{N \cap K + L}{L}$ is semisimple. Hence $M$ is strongly ss-radical supplemented. 

**Corollary 4.** Let $M$ be a strongly ss-radical supplemented module. Then $\frac{M}{\text{Rad}(M)}$ is semisimple.

**Proof.** By Proposition 3, $\frac{M}{\text{Rad}(M)}$ is strongly ss-radical supplemented. Since $\text{Rad}(\frac{M}{\text{Rad}(M)}) = 0$, we get that $\frac{M}{\text{Rad}(M)}$ is semisimple.

**Proposition 4.** Let $M$ be a strongly ss-radical supplemented module and $\text{Rad}(M) \subseteq U$. Then every ss-supplement of $U$ in $M$ is coatomic.

**Proof.** Let $V$ be an ss-supplement of $U$ in $M$. Then, we can write $M = U + V$, $U \cap V \ll V$ and $U \cap V$ is semisimple. Therefore, $U \cap V$ is coatomic. Note that $\frac{M}{\text{Rad}(M)} \cong \frac{M}{U}$, thus, $\frac{M}{U}$ is semisimple by Corollary 4. It follows that $\frac{M}{U} \cong \frac{V}{U \cap V}$ is semisimple. By [10, Lemma 3], we obtain that $V$ is coatomic.

In the following example, we show that a factor module of an ss-radical supplemented module need not be ss-radical supplemented, in general.

**Example 3.** Put $M = \mathbb{Z}/2\mathbb{Z}$. Clearly, $M$ is ss-radical supplemented. Consider the factor module $\frac{\mathbb{Z}}{2\mathbb{Z}}$ of $M$. Then $\text{Rad}(\frac{\mathbb{Z}}{2\mathbb{Z}}) = 2\mathbb{Z} \ll \mathbb{Z}$. Hence, $\frac{\mathbb{Z}}{2\mathbb{Z}}$ is not ss-radical supplemented by Corollary 2.
Proposition 5. Let $M$ be an ss-radical supplemented module and $N \subseteq \text{Rad}(M)$. Then, $\frac{M}{N}$ is ss-radical supplemented.

Proof. Consider the canonical $\pi : M \rightarrow \frac{M}{N}$. Since $N \subseteq \text{Rad}(M)$, $\pi(\text{Rad}(M)) = \frac{\text{Rad}(M) + N}{N} = \text{Rad}(\frac{M}{N})$ by [8, 19.3 (4)]. By the assumption, we can write $M = \text{Rad}(M) + V$, $\text{Rad}(M) \cap V \ll V$ and $\text{Rad}(M) \cap V$ semisimple for some submodule $V$ of $M$. By a similar discussion in the proof of Proposition 3, we deduce that $\frac{V + N}{N}$ is an ss-supplement of $\text{Rad}(\frac{M}{N})$ in $\frac{M}{N}$.

Let $M$ be a module. $M$ is called a good module if

$$f(\text{Rad}(M)) = \text{Rad}(f(M))$$

for any homomorphism $f : M \rightarrow N$.

For a good module $M$, we have the following fact.

Proposition 6. Let $M$ be a good module. If $M$ is ss-radical supplemented, $\frac{M}{N}$ is ss-radical supplemented for every submodule $N$ of $M$.

Proof. Since $M$ is a good module, we can write $\text{Rad}(\frac{M}{N}) = \frac{\text{Rad}(M) + N}{N}$. By similar discussion in the proof of Proposition 3, we get that $\frac{M}{N}$ is ss-radical supplemented.

Proposition 7. Let $M$ be a module and $N$ be a submodule of $M$. Then, if $N$ is strongly ss-radical supplemented and $\text{Rad}(\frac{M}{N}) = \frac{M}{N}$, $M$ is strongly ss-radical supplemented.

Proof. Let $U$ be submodule of $M$ with $\text{Rad}(M) \subseteq U \subseteq M$. Since $\text{Rad}(\frac{M}{N}) = \frac{M}{N}$, $\text{Rad}(M) + N = M$ and so $U + N = M$. Since $\text{Rad}(N) \subseteq \text{Rad}(M) \subseteq U$ and $\text{Rad}(N) \subseteq N$, we can write $\text{Rad}(N) \subseteq (U \cap N)$. Since $N$ is strongly ss-radical supplemented, there exists submodule $V$ of $M$ such that $(U \cap N) + V = N$, $U \cap V \ll V$ and $U \cap V$ semisimple. Hence, we have $M = U + N = U + (U \cap N) + V = U + V$. That is, $M$ is strongly ss-radical supplemented.

Lemma 2. Let $M$ be a module, $M_1$, $N \subseteq M$ and $\text{Rad}(M) \subseteq N$. If $M_1$ is a strongly ss-radical supplemented module and $M_1 + N$ has an ss-supplement in $M$, then $N$ has an ss-supplement in $M$.

Proof. Suppose that $L$ is an ss-supplement of $M_1 + N$ in $M$ and $K$ is an ss-supplement of $(L + N) \cap M_1$ in $M_1$. Then $M = L + K + N$ and $(L + K) \cap N \ll L + K$. Thus $L \cap (K + N)$ is semisimple. $K \cap (L + N) \cap M_1 = K \cap (L + N)$ is semisimple and, by [5, 8.1(5)], $(L + K) \cap N$ is semisimple. Then $K + L$ is an ss-supplement of $N$ in $M$.

Proposition 8. Let $M_1$, $M_2$ be any submodules of a module $M$ such that $M = M_1 + M_2$. Then if $M_1$ and $M_2$ are strongly ss-radical supplemented, $M$ is strongly ss-radical supplemented.

Proof. Suppose that $N \subseteq M$ with $\text{Rad}(M) \subseteq N$. Clearly $M_1 + M_2 + N$ has the trivial ss-supplement 0 in $M$, so by Lemma 2, $M_1 + N$ has an ss-supplement in $M$. Applying the lemma once more, we obtain an ss-supplement for $N$ in $M$.

Corollary 5. Every finite sum of strongly ss-radical supplemented modules is a strongly ss-radical supplemented module.

Proposition 9. Let $M$ be a module with small radical. Then $M$ is strongly ss-radical supplemented if and only if it is ss-supplemented.
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Proof. ($\Longrightarrow$) Let $U$ be a submodule of $M$. Then $\text{Rad}(M) \subseteq \text{Rad}(M) + U$ and $\text{Rad}(M) + U$ has an $ss$-supplement, say $V$, in $M$. So $M = \text{Rad}(M) + U + V$, $[\text{Rad}(M) + U] \cap V \ll V$ and $[\text{Rad}(M) + U] \cap V$ is semisimple. Since $\text{Rad}(M) \ll M$, we have $M = U + V$ and also $U \cap V \subseteq [\text{Rad}(M) + U] \cap V$. Hence, $U$ has an $ss$-supplement $V$ in $M$. Thus, $M$ is an $ss$-supplemented module.

$(\Longleftarrow)$ Clear.

Corollary 6. Let $M$ be a coatomic module. Then $M$ is $ss$-supplemented if and only if $M$ is a strongly $ss$-radical supplemented module.

Theorem 1. Let $M$ be a module with $\text{Rad}(M) \ll M$. The following statements are equivalent.

1. $M$ is $ss$-supplemented.
2. $M$ is supplemented and $M$ is $ss$-radical supplemented.
3. $M$ is strongly radical supplemented and $\text{Rad}(M) \subseteq \text{Soc}(M)$.
4. $M$ is strongly $ss$-radical supplemented.

Proof. (1) $\implies$ (2) Clear.
(2) $\implies$ (3) It follows from Corollary 2.
(3) $\implies$ (4) Suppose that $U \subseteq M$ with $\text{Rad}(M) \subseteq U$. Since $M$ is strongly radical supplemented, we have $V$ supplement with $M = U + V$ and $U \cap V \ll V$. Since $U \cap V \subseteq \text{Rad}(V) \subseteq \text{Rad}(M) \subseteq \text{Soc}(M)$, $M$ is strongly $ss$-radical supplemented.
(4) $\implies$ (1) By Proposition 9.

The following result is a direct consequence of Theorem 1.

Corollary 7. Let $R$ be a left max ring. Then every strongly $ss$-radical supplemented $R$-module is $ss$-supplemented.

Let $M$ be a module. $M$ is called strongly local if it is local and $\text{Rad}(M) \subseteq \text{Soc}(M)$ [6].

Corollary 8. Let $M$ be a local module. Then the following statements are equivalent.

1. $M$ is strongly local.
2. $M$ is $ss$-supplemented.
3. $M$ is $ss$-radical supplemented.

Proof. Since local modules have small radical, the proof follows from Theorem 1 and [6, Proposition 3 (4)].

Now, we shall characterize the rings over which all modules are strongly $ss$-radical supplemented.

Proposition 10. The following statements are equivalent for a ring $R$.

1. Every projective left $R$-module is $ss$-radical supplemented.
2. Every free left $R$-module is $ss$-radical supplemented.
(3) Every finitely generated free left $R$-module is $ss$-radical supplemented.

(4) $\text{Rad}(R) \subseteq \text{Soc}(RR)$.

Proof. (1) $\implies$ (2) and (2) $\implies$ (3) are clear.
(3) $\implies$ (4) By (3), $RR$ is $ss$-radical supplemented. It follows from [6, Theorem 3(30)] that $\text{Rad}(R) \subseteq \text{Soc}(RR)$.

(4) $\implies$ (1) Let $P$ be any projective left $R$-module. Then, by [8, 21.17 (2)], $\text{Rad}(P) = \text{Rad}(R)P \subseteq \text{Soc}(RR)P = \text{Soc}(P)$ and so $\text{Rad}(P)$ is small in $P$. Applying [8, 21.17 (2)], we obtain that $P$ is $ss$-radical supplemented.

Example 4. Given the ring $\mathbb{Z}$. It is well known that $\text{Rad}(\mathbb{Z}) = \text{Soc}(\mathbb{Z} \mathbb{Z}) = 0$. Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \mathbb{Z}_16$. Note that $M$ is not projective. Then, $\text{Rad}(M) \ll M$ and $M$ is not $ss$-radical supplemented by Corollary 2.

A ring $R$ is called a left good ring if $RR$ is a good module.

Theorem 2. Let $R$ be a left good ring. Then $\text{Rad}(R) \subseteq \text{Soc}(RR)$ if and only if every left $R$-module is $ss$-radical supplemented.

Proof. Let $M$ be a left $R$-module and $\text{Rad}(R) \subseteq \text{Soc}(RR)$. Since $R$ is a left good ring, we can write $\text{Rad}(M) = \text{Rad}(R)$, $M \subseteq \text{Soc}(RR)$, $M \subseteq \text{Soc}(M)$ by [7]. Hence, $M$ is $ss$-radical supplemented by Lemma 1. The converse follows from Corollary 2.

Let $R$ be a ring. $R$ is called a left WV-ring if every simple $R$-module is $R/I$-injective, where $R \not\cong R$ and $I$ is any ideal of $R$. Clearly, left WV-rings are a generalization of $V$-rings [4].

Lemma 3. Let $R$ be a left WV-ring. Then $R$ is a left good ring and a left max ring.

Proof. If $R$ is a left $V$-ring, then $R$ is a left good ring and a left max ring. Assume that $R$ is not a left $V$-ring. By [4, Corollary 6 (8)], $\frac{R}{\text{Rad}(R)}$ is a left $V$-ring. It follows from [8, 23 (2)] that $R$ is a left good ring. Let $M \neq 0$. Since $R$ is a left good ring, $\text{Rad}(M) = \text{Rad}(R)$, $M \subseteq \text{Soc}(RR)$, $M \subseteq \text{Soc}(M)$ and so $\text{Rad}(M)$ is semisimple. It means that $\text{Rad}(M) \neq M$. Hence, $R$ is a left max ring.

Note that, in general, a left max ring and a left good ring need not be a left WV-ring. For example, the $R = \mathbb{Z}_8$ is an Artinian ring which is not a left WV-ring.

Corollary 9. Let $R$ be a left WV-ring. Then

(1) every left $R$-module is $ss$-radical supplemented;

(2) every supplemented left $R$-module is $ss$-supplemented.

Proof. (1) By Theorem 2 and Lemma 3.

(2) Let $M$ be a supplemented module. Since $R$ is a left WV-ring, by Lemma 3, $\text{Rad}(M) \subseteq \text{Soc}(M)$. It follows from Theorem 1 that $M$ is $ss$-supplemented.
**Theorem 3.** For a ring $R$, following statements are equivalent.

1. Every left $R$-module is strongly $ss$-radical supplemented.

2. Every finitely generated left $R$-module is strongly $ss$-radical supplemented.

3. $R$ is strongly $ss$-radical supplemented.

4. $R$ is $ss$-supplemented.

5. $R$ is semilocal and $\text{Rad}(R) \subseteq \text{Soc}(R)$.

**Proof.** (1) $\implies$ (2), (2) $\implies$ (3) and (3) $\implies$ (4) are clear.

(4) $\implies$ (5) and (5) $\implies$ (1) by [6, Theorem 3 (30)].

It is well known that, over an Artinian ring, every left $R$-module is supplemented and so every module is strongly radical supplemented. However, any module over an Artinian ring need not be strongly $ss$-radical supplemented. For example, consider the ring $\mathbb{Z}_8$. Then $R$ is not strongly $ss$-radical supplemented.

Unless stated otherwise, here in after, we assume that every ring is a Dedekind domain which is not field.

For a module $M$, $P(M)$ indicates the sum of all radical submodules of $M$. If $P(M) = 0$, then $M$ is called reduced. Note that $P(M)$ is the largest radical submodule of $M$. Let $R$ be a Dedekind domain and let $M$ be an $R$-module. Since $R$ is a Dedekind domain, $P(M)$ is the divisible part of $M$. By [1, Lemma 4 (4)], $P(M)$ is (divisible) injective and so there exists a submodule $N$ of $M$ such that $M = P(M) \oplus N$. Here $N$ is called the reduced part of $M$.

**Proposition 11.** Let $R$ be a Dedekind domain and let $M$ be an $R$-module. Then $M$ is a strongly $ss$-radical supplemented module if and only if $N$ is strongly $ss$-radical supplemented, where $N$ is reduced part of $M$.

**Proof.** $N$ is a strongly $ss$-radical supplemented module as a homomorphic image of $M$ by Proposition 3. The converse is by Proposition 8.

**Lemma 4.** Let $R$ be a Dedekind domain and $M$ be a torsion-free $R$-module. Then, the following statements are equivalent.

1. $M$ is strongly $ss$-radical supplemented.

2. $M$ is injective.

**Proof.** (1) $\implies$ (2) Let $U$ be a maximal submodule of $M$. Then, $U$ has an $ss$-supplement, say $V$, in $M$. By [8, 41.1 (3)], $V$ is local and so $V \subseteq T(M) = 0$. Therefore $V = 0$. Hence, $M$ has no maximal submodules, i.e. $M = \text{Rad}(M)$. By [1, Lemma 4 (4)], $M$ is injective.

(2) $\implies$ (1) By [1, Lemma 4 (4)] $\text{Rad}(M) = M$ and so, by Proposition 1, $M$ is strongly $ss$-radical supplemented.

**Corollary 10.** Let $R$ be a Dedekind domain and $M$ be a strongly $ss$-radical supplemented module. Then, $\frac{M}{T(M)}$ is injective.
Proposition 12. Let $R$ be a Dedekind domain, $M$ be an $R$-module and $T(M)$ is strongly $ss$-radical supplemented. Then, $M$ is strongly $ss$-radical supplemented if and only if $\frac{M}{T(M)}$ is injective.

Proof. $(\implies)$ Since $M$ is strongly $ss$-radical supplemented it follows from Proposition 3 that $\frac{M}{T(M)}$ is strongly $ss$-radical supplemented. Hence, $\frac{M}{T(M)}$ is injective.

$(\impliedby)$ Since $\frac{M}{T(M)}$ is injective, we can write $\text{Rad} \left( \frac{M}{T(M)} \right) = \frac{M}{T(M)}$ by [1, Lemma 4 (4)]. Therefore, this can be proved by taking $N = T(M)$ in the Proposition 7 and hypothesis.

References


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