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Generalizations of ss-supplemented modules

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We introduce the concept of (strongly) ss-radical supplemented modules. We prove that if a submodule N of M is strongly ss-radical supplemented and $Rad(M/N) = M/N$, then M is strongly ss-radical supplemented. For a left good ring R, we show that $Rad(R) \subseteq Soc({}_RR)$ if and only if every left R-module is ss-radical supplemented. We characterize the rings over which all modules are strongly ss-radical supplemented. We also prove that over a left WV-ring every supplemented module is ss-supplemented.

Key words and phrases: semisimple module, (strongly) ss-radical supplemented module, WV-ring.

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1 Introduction

Throughout the paper, all rings are associative with identity and all modules are unitary left modules. Let R be a ring and M be an R-module. By $Rad(M)$ and $Soc(M)$, we will denote the *radical* of M and the *socle* of M, respectively. A submodule $K \subseteq M$ is called *small* in M, written $K \ll M$, if for every submodule $N \subseteq M$ the equality $M = N + K$ implies $N = M$. In [9], D.X. Zhou and X.R. Zhang introduced the concept of socle of a module M to that of $Soc_s(M)$ by considering the class of all simple submodules M that are small in M in place of the class of all simple submodules of M, that is $Soc_s(M) = \sum\{N \ll M \,|\, N \textit{ is simple}\}$. It is clear that $Soc_s(M) \subseteq Rad(M)$ and $Soc_s(M) \subseteq \mathrm{Soc}(M)$. A module M is called *supplemented* if every submodule N of M has a supplement, i.e. a submodule K minimal with respect to $M = N + K$. K is a supplement of N in M if and only if $M = N + K$ and $N \cap K \ll K$. For more properties of supplemented modules we refer to [3]. In [10], H. Zöschinger introduced a notion of modules whose radical has supplements and called them *radical supplemented*. In the same paper and in [11], he determined the structure of radical supplemented modules. In [2], E. Büyükaşık and E. Türkmen call a module M *strongly radical supplemented* if every submodule containing the radical has a supplement in M. In [6], E. Kaynar, E. Türkmen and H. Çalışıcı call a submodule V an ss-supplement of U in M if and only if $M = U + V$ and $U \cap V \subseteq \mathrm{Soc}_s(V)$. A module M is *ss-supplemented* if every submodule U of M has an ss-supplement V in M. It is shown in [6, Theorem 3.30] that a ring R is semiperfect and $Rad(R) \subseteq Soc({}_RR)$ if and only if every left R-module is ss-supplemented. Motivated by these results, we call a module *ss-radical supplemented* if $Rad(M)$ has an ss-supplement in M and we call a module *strongly ss-radical supplemented* if every submodule containing the radical has an ss-supplement in M.

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Also we obtain the various properties of (strongly) ss-radical supplemented modules. We show that homomorphic images of strongly ss-radical supplemented modules are strongly ss-radical supplemented (Proposition 3) and for a left good ring R, we prove that $Rad(R) \subseteq \mathrm{Soc}({}_RR)$ if and only if every left R-module is ss-radical supplemented. If R is a left WV-ring, then R is a left good ring and every left R-module is ss-radical supplemented. We have characterized the rings over which all modules are strongly ss-radical supplemented, by Theorem 3. We study on (strongly) ss-radical supplemented modules over Dedekind domains.

2 *ss*-Radical supplemented and strongly *ss*-radical supplemented modules

In this section, we give some properties of the (strongly) ss-radical supplemented modules. In particular, we provide characterizations of some classes of rings.

Recall that a module M is called *radical* if M has no maximal submodules, i.e. $Rad(M) = M$.

Proposition 1. *Let M be radical module. Then M is a strongly ss-radical supplemented module.*

Proof. It is clear that $Rad(M)$ has the trivial ss-supplement 0 in M. Thus M is strongly ss-radical supplemented. □

Let M be a module. By $P(M)$ we denote the sum of all radical submodules of M. Then $P(M)$ is the largest radical submodule of M.

Corollary 1. *$P(M)$ is strongly ss-radical supplemented for every module M.*

Example 1. *(1) Consider the $\mathbb{Z}$-module $M = \mathbb{Z}$. Then $\mathbb{Z}$ is an ss-radical supplemented module because $Rad(\mathbb{Z}) = 0$.*

(2) Let R be a commutative domain and $K(R)$ be the fractions field of R. It follows from Proposition 1 that $K(R)$ is a strongly ss-radical supplemented R-module.

It is well known that every module with small radical is radical supplemented. The following example shows that a module with small radical need to be ss-radical supplemented. Firstly we need the following facts.

Lemma 1. *Let M be a module and $Rad(M) \subseteq Soc(M)$. Then M is ss-radical supplemented.*

Proof. Obviously, $M = M + Rad(M)$ and $M \cap Rad(M) = Rad(M) \subseteq Soc(M)$. Therefore $Rad(M)$ is semisimple and so M is an ss-supplement of $Rad(M)$ in M. Hence M is ss-radical supplemented. □

Using Lemma 1, we obtain the following Corollary 2 and Corollary 3.

Corollary 2. *Let M be a module and $Rad(M) \ll M$. Then M is an ss-radical supplemented module if and only if $Rad(M) \subseteq Soc(M)$.*

Proof. By Lemma 1 and by [6, Corollary 3.3]. □

Example 2. *Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^3}$, where p is any prime integer. Then $Rad(M) =< \overline{p} >\ll M$. Since $Rad(M)$ is not semisimple, by Corollary 2, M is not ss-radical supplemented.*

Let R be a ring. R is said to be a *left max ring* if every non-zero left R-module has a maximal submodule.

Corollary 3. *Let R be a left max ring and M be an R-module. Then M is ss-radical supplemented if and only if $Rad(M) \subseteq Soc(M)$.*

Proof. Since R is a left max ring, M has a small radical. By Corollary 2, the proof follows. □

Recall that a module M is *coatomic* if every proper submodule of M is contained in some maximal submodule of M.

Proposition 2. *Let R be a semilocal ring and M be an ss-radical supplemented module. Then, every ss-supplement of $Rad(M)$ in M is coatomic.*

Proof. If $M = Rad(M)$, the proof is clear. Assume that $M \neq Rad(M)$. Let V be an ss-supplement of $Rad(M)$ in M. Then $Rad(M) \cap V = Rad(V)$ is semisimple and so $Rad(V)$ is coatomic. Since R is semilocal, it follows from [7, Theorem 3.5] that

$$\frac{M}{Rad(M)} \cong \frac{V}{Rad(M) \cap V} = \frac{V}{Rad(V)}$$

is semisimple. By [10, Lemma 3], we get that V is coatomic. □

Proposition 3. *Every homomorphic image of a strongly ss-radical supplemented module is a strongly ss-radical supplemented module.*

Proof. Let M be a strongly ss-radical supplemented module and $L \subseteq N \subseteq M$ with $Rad(\frac{M}{L}) \subseteq \frac{N}{L}$. Consider the canonical projection $\pi : M \longrightarrow \frac{M}{L}$. Then $\pi(Rad(M)) = \frac{Rad(M)+L}{L} \subseteq Rad(\frac{M}{L}) \subseteq \frac{N}{L}$ and so $Rad(M) \subseteq N$. By the hypothesis, N has an ss-supplement, say K, in M. Clearly, we can write $\frac{M}{L} = \frac{N}{L} + \frac{(K+L)}{L}$. By [8, 19.3 (4)], we obtain that $\pi(N \cap K) = \frac{N \cap K+L}{L} = \frac{N}{L} \cap \frac{K+L}{L} \ll \pi(K) = \frac{K+L}{L}$ and $\pi(N \cap K) = \frac{N}{L} \cap \frac{K+L}{L}$ is semisimple. Hence M is strongly ss-radical supplemented. □

Corollary 4. *Let M be a strongly ss-radical supplemented module. Then $\frac{M}{Rad(M)}$ is semisimple.*

Proof. By Proposition 3, $\frac{M}{Rad(M)}$ is strongly ss-radical supplemented. Since $Rad(\frac{M}{Rad(M)}) = 0$, we get that $\frac{M}{Rad(M)}$ semisimple. □

Proposition 4. *Let M be a strongly ss-radical supplemented module and $Rad(M) \subseteq U$. Then every ss-supplement of U in M is coatomic.*

Proof. Let V be an ss-supplement of U in M. Then, we can write $M = U + V$, $U \cap V \ll V$ and $U \cap V$ is semisimple. Therefore, $U \cap V$ is coatomic. Note that $\frac{\frac{M}{Rad(M)}}{\frac{U}{Rad(M)}} \cong \frac{M}{U}$ thus, $\frac{M}{U}$ is semisimple by Corollary 4. It follows that $\frac{M}{U} \cong \frac{V}{U \cap V}$ is semisimple. By [10, Lemma 3], we obtain that V is coatomic. □

In the following example, we show that a factor module of an ss-radical supplemented module need not be ss-radical supplemented, in general.

Example 3. *Put $M = {}_{\mathbb{Z}}\mathbb{Z}$. Clearly, M is ss-radical supplemented. Consider the factor module $\frac{\mathbb{Z}}{8\mathbb{Z}}$ of M. Then $Rad\left(\frac{\mathbb{Z}}{8\mathbb{Z}}\right) = \frac{2\mathbb{Z}}{8\mathbb{Z}} \ll \frac{\mathbb{Z}}{8\mathbb{Z}}$. Hence, $\frac{\mathbb{Z}}{8\mathbb{Z}}$ is not ss-radical supplemented by Corollary 2.*

Proposition 5. *Let M be an ss-radical supplemented module and $N \subseteq Rad(M)$. Then, $\frac{M}{N}$ is ss-radical supplemented.*

Proof. Consider the canonical $\pi : M \longrightarrow \frac{M}{L}$. Since $N \subseteq Rad(M)$, $\pi(Rad(M)) = \frac{Rad(M)+N}{N} = Rad(\pi(M)) = Rad\left(\frac{M}{N}\right)$ by [8, 19.3 (4)]. By the assumption, we can write $M = Rad(M) + V$, $Rad(M) \cap V \ll V$ and $Rad(M) \cap V$ semisimple for some submodule V of M. By a similar discussion in the proof of Proposition 3, we deduce that $\frac{V+N}{N}$ is an ss-supplement of $Rad\left(\frac{M}{N}\right)$ in $\frac{M}{N}$. □

Let M be a module. M is called a *good module* if

$$f(Rad(M)) = Rad(f(M)) \text{ for any homomorphism } f : M \longrightarrow N.$$

For a good module M, we have the following fact.

Proposition 6. *Let M be a good module. If M is ss-radical supplemented, $\frac{M}{N}$ is ss-radical supplemented for every submodule N of M.*

Proof. Since M is a good module, we can write $Rad\left(\frac{M}{N}\right) = \frac{Rad(M)+N}{N}$. By similar discussion in the proof of Proposition 3, we get that $\frac{M}{N}$ is ss-radical supplemented. □

Proposition 7. *Let M be a module and N be a submodule of M. Then, if N is strongly ss-radical supplemented and $Rad\left(\frac{M}{N}\right) = \frac{M}{N}$, M is strongly ss-radical supplemented.*

Proof. Let U be submodule of M with $Rad(M) \subseteq U \leq M$. Since $Rad\left(\frac{M}{N}\right) = \frac{M}{N}$, $Rad(M) + N = M$ and so $U + N = M$. Since $Rad(N) \subseteq Rad(M) \subseteq U$ and $Rad(N) \subseteq N$, we can write $Rad(N) \subseteq (U \cap N)$. Since N is strongly ss-radical supplemented, there exists submodule V of M such that $(U \cap N) + V = N$, $U \cap V \ll V$ and $U \cap V$ semisimple. Hence, we have $M = U + N = U + (U \cap N) + V = U + V$. That is, M is strongly ss-radical supplemented. □

Lemma 2. *Let M be a module, M_1, $N \leq M$ and $Rad(M) \subseteq N$. If M_1 is a strongly ss-radical supplemented module and $M_1 + N$ has an ss-supplement in M, then N has an ss-supplement in M.*

Proof. Suppose that L is an ss-supplement of $M_1 + N$ in M and K is an ss-supplement of $(L + N) \cap M_1$ in M_1. Then $M = L + K + N$ and $(L + K) \cap N \ll L + K$. Thus $L \cap (K + N)$ is semisimple. $K \cap [(L + N) \cap M_1] = K \cap (L + N)$ is semisimple and, by [5, 8.1(5)], $(L + K) \cap N$ is semisimple. Then $K + L$ is an ss-supplement of N in M. □

Proposition 8. *Let M_1, M_2 be any submodules of a module M such that $M = M_1 + M_2$. Then if M_1 and M_2 are strongly ss-radical supplemented, M is strongly ss-radical supplemented.*

Proof. Suppose that $N \subseteq M$ with $Rad(M) \subseteq N$. Clearly $M_1 + M_2 + N$ has the trivial ss-supplement 0 in M, so by Lemma 2, $M_1 + N$ has an ss-supplement in M. Applying the lemma once more, we obtain an ss-supplement for N in M. □

Corollary 5. *Every finite sum of strongly ss-radical supplemented modules is a strongly ss-radical supplemented module.*

Proposition 9. *Let M be a module with small radical. Then M is strongly ss-radical supplemented if and only if it is ss-supplemented.*

Proof. ($\Longrightarrow$) Let U be a submodule of M. Then $Rad(M) \subseteq Rad(M) + U$ and $Rad(M) + U$ has an ss-supplement, say V, in M. So $M = Rad(M) + U + V$, $[Rad(M) + U] \cap V \ll V$ and $[Rad(M) + U] \cap V$ is semisimple. Since $Rad(M) \ll M$, we have $M = U + V$ and also $U \cap V \subseteq [Rad(M) + U] \cap V$. Hence, U has an ss-supplement V in M. Thus, M is an ss-supplemented module.

($\Longleftarrow$) Clear. □

Corollary 6. *Let M be a coatomic module. Then M is ss-supplemented if and only if M is a strongly ss-radical supplemented module.*

Theorem 1. *Let M be a module with $Rad(M) \ll M$. The following statements are equivalent.*

(1) M is ss-supplemented.

(2) M is supplemented and M is ss-radical supplemented.

(3) M is strongly radical supplemented and $Rad(M) \subseteq Soc(M)$.

(4) M is strongly ss-radical supplemented.

Proof. (1) $\Longrightarrow$ (2) Clear.

(2) $\Longrightarrow$ (3) It follows from Corollary 2.

(3) $\Longrightarrow$ (4) Suppose that $U \subseteq M$ with $Rad(M) \subseteq U$. Since M is strongly radical supplemented, we have V supplement with $M = U + V$ and $U \cap V \ll V$. Since $U \cap V \subseteq Rad(V) \subseteq Rad(M) \subseteq Soc(M)$, M is strongly ss-radical supplemented.

(4) $\Longrightarrow$ (1) By Proposition 9. □

The following result is a direct consequence of Theorem 1.

Corollary 7. *Let R be a left max ring. Then every strongly ss-radical supplemented R-module is ss-supplemented.*

Let M be a module. M is called *strongly local* if it is local and $Rad(M) \subseteq Soc(M)$ [6].

Corollary 8. *Let M be a local module. Then the following statements are equivalent.*

(1) M is strongly local.

(2) M is ss-supplemented.

(3) M is ss-radical supplemented.

Proof. Since local modules have small radical, the proof follows from Theorem 1 and [6, Proposition 3 (4)]. □

Now, we shall characterize the rings over which all modules are strongly ss-radical supplemented.

Proposition 10. *The following statements are equivalent for a ring R.*

(1) Every projective left R-module is ss-radical supplemented.

(2) Every free left R-module is ss-radical supplemented.

(3) Every finitely generated free left R-module is ss-radical supplemented.

(4) $Rad(R) \subseteq Soc({}_RR)$.

Proof. (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3) are clear.

(3) $\Longrightarrow$ (4) By (3), ${}_RR$ is ss-radical supplemented. It follows from [6, Theorem 3(30)] that $Rad(R) \subseteq Soc({}_RR)$.

(4) $\Longrightarrow$ (1) Let P be any projective left R-module. Then, by [8, 21.17 (2)], $Rad(P) = Rad(R)P \subseteq Soc({}_RR)P = Soc(P)$ and so $Rad(P)$ is small in P. Applying [8, 21.17 (2)], we obtain that P is ss-radical supplemented. □

Example 4. *Given the ring $\mathbb{Z}$. It is well known that $Rad(\mathbb{Z}) = Soc({}_\mathbb{Z}\mathbb{Z}) = 0$. Consider the $\mathbb{Z}$-module $M =_\mathbb{Z} \mathbb{Z}_{16}$. Note that M is not projective. Then, $Rad(M) \ll M$ and M is not ss-radical supplemented by Corollary 2.*

A ring R is called *a left good ring* if ${}_RR$ is a good module.

Theorem 2. *Let R be a left good ring. Then $Rad(R) \subseteq Soc({}_RR)$ if and only if every left R-module is ss-radical supplemented.*

Proof. Let M be a left R-module and $Rad(R) \subseteq Soc({}_RR)$. Since R is a left good ring, we can write $Rad(M) = Rad(R)$, $M \subseteq Soc({}_RR)$, $M \subseteq Soc(M)$ by [7]. Hence, M is ss-radical supplemented by Lemma 1. The converse follows from Corollary 2. □

Let R be a ring. R is called a *left WV-ring* if every simple R-module is $\frac{R}{I}$-injective, where $\frac{R}{I} \ncong R$ and I is any ideal of R. Clearly, left WV-rings are a generalization of V-rings [4].

Lemma 3. *Let R be a left WV-ring. Then R is a left good ring and a left max ring.*

Proof. If R is a left V-ring, then R is a left good ring and a left max ring. Assume that R is not a left V-ring. By [4, Corollary 6 (8)], $\frac{R}{Rad(R)}$ is a left V-ring. It follows from [8, 23 (2)] that R is a left good ring. Let $M \neq 0$. Since R is a left good ring, $Rad(M) = Rad(R)$, $M \subseteq Soc({}_RR)$, $M \subseteq Soc(M)$ and so $Rad(M)$ is semisimple. It means that $Rad(M) \neq M$. Hence, R is a left max ring. □

Note that, in general, a left max ring and a left good ring need not be a left WV-ring. For example, the $R = \mathbb{Z}_8$ is an Artinian ring which is not a left WV-ring.

Corollary 9. *Let R be a left WV-ring. Then*

(1) every left R-module is ss-radical supplemented;

(2) every supplemented left R-module is ss-supplemented.

Proof. (1) By Theorem 2 and Lemma 3.

(2) Let M be a supplemented module. Since R is a left WV-ring, by Lemma 3, $Rad(M) \subseteq Soc(M)$. It follows from Theorem 1 that M is ss-supplemented. □

Theorem 3. *For a ring R, following statements are equivalent.*

(1) Every left R-module is strongly ss-radical supplemented.

(2) Every finitely generated left R-module is strongly ss-radical supplemented.

(3) ${}_RR$ is strongly ss-radical supplemented.

(4) ${}_RR$ is ss-supplemented.

(5) R is semilocal and $Rad(R) \subseteq Soc({}_RR)$.

Proof. $(1) \Longrightarrow (2)$, $(2) \Longrightarrow (3)$ and $(3) \Longrightarrow (4)$ are clear.
$(4) \Longrightarrow (5)$ and $(5) \Longrightarrow (1)$ by [6, Theorem 3 (30)]. □

It is well known that, over an Artinian ring, every left R-module is supplemented and so every module is strongly radical supplemented. However, any module over an Artinian ring need not be strongly ss-radical supplemented. For example, consider the ring $\mathbb{Z}_8$. Then ${}_RR$ is not strongly ss-radical supplemented.

Unless stated otherwise, here in after, we assume that every ring is a Dedekind domain which is not field.

For a module M, $P(M)$ indicates the sum of all radical submodules of M. If $P(M) = 0$, then M is called *reduced*. Note that $P(M)$ is the largest radical submodule of M. Let R be a Dedekind domain and let M be an R-module. Since R is a Dedekind domain, $P(M)$ is the divisible part of M. By [1, Lemma 4 (4)], $P(M)$ is (divisible) injective and so there exists a submodule N of M such that $M = P(M) \bigoplus N$. Here N is called the *reduced part* of M.

Proposition 11. *Let R be a Dedekind domain and let M be an R-module. Then M is a strongly ss-radical supplemented module if and only if N is strongly ss-radical supplemented, where N is reduced part of M.*

Proof. N is a strongly ss-radical supplemented module as a homomorphic image of M by Proposition 3. The converse is by Proposition 8. □

Lemma 4. *Let R be a Dedekind domain and M be a torsion-free R-module. Then, the following statements are equivalent.*

(1) M is strongly ss-radical supplemented.

(2) M is injective.

Proof. $(1) \Longrightarrow (2)$ Let U be a maximal submodule of M. Then, U has an ss-supplement, say V, in M. By [8, 41.1 (3)], V is local and so $V \subseteq T(M) = 0$. Therefore $V = 0$. Hence, M has no maximal submodules, i.e. $M = Rad(M)$. By [1, Lemma 4 (4)], M is injective.

$(2) \Longrightarrow (1)$ By [1, Lemma 4 (4)] $Rad(M) = M$ and so, by Proposition 1, M is strongly ss-radical supplemented. □

Corollary 10. *Let R be a Dedekind domain and M be a strongly ss-radical supplemented module. Then, $\frac{M}{T(M)}$ is injective.*

Proposition 12. *Let R be a Dedekind domain, M be an R-module and $T(M)$ is strongly ss-radical supplemented. Then, M is strongly ss-radical supplemented if and only if $\frac{M}{T(M)}$ is injective.*

Proof. $(\Longrightarrow)$ Since M is strongly ss-radical supplemented it follows from Proposition 3 that $\frac{M}{T(M)}$ is strongly ss-radical supplemented. Hence, $\frac{M}{T(M)}$ is injective.

$(\Longleftarrow)$ Since $\frac{M}{T(M)}$ is injective, we can write $Rad\left(\frac{M}{T(M)}\right) = \frac{M}{T(M)}$ by [1, Lemma 4 (4)]. Therefore, this can be proved by taking $N = T(M)$ in the Proposition 7 and hypothesis. □

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Ми вводимо поняття (строго) ss-радикально доповненого модуля. Ми доводимо, що коли підмодуль N модуля M є строго ss-радикально доповненим і $Rad(M/N) = M/N$, то M є строго ss-радикально доповненим. Для доброго лівого кільця R ми показуємо, що $Rad(R) \subseteq Soc({}_RR)$ тоді і тільки тоді, коли кожен лівий R-модуль є ss-радикально доповненим. Ми характеризуємо кільця, над якими всі модулі є строго ss-радикально доповненими. Також ми доводимо, що над лівим WV-кільцем кожен доповнений модуль є ss-доповненим.

Ключові слова і фрази: напівпростий модуль, (строго) ss-радикально доповнений модуль, WV-кільце.