APPLICATIONS ON OPERATIONS ON WEAKLY COMPACT GENERALIZED TOPOLOGICAL SPACES

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In this paper, we have introduced the notion of operations on a generalized topological space \((X, \mu)\) to investigate the notion of \(\gamma_\mu\)-compact subsets of a generalized topological space and to study some of its properties. It is also shown that, under some conditions, \(\gamma_\mu\)-compactness of a space is equivalent to some other weak forms of compactness. Characterizations of such sets are given. We have then introduced the concept of \(\gamma_\mu\)-\(T_2\) spaces to study some properties of \(\gamma_\mu\)-compact spaces. This operation enables us to unify different results due to S. Kasahara.

Key words and phrases: operation, \(\gamma_\mu\)-open set, \(\gamma_\mu\)-compact space.

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INTRODUCTION

The notion of an operation on a topological space was introduced by S. Kasahara [6] in 1979. After then D.S. Janković [5] introduced the concept of \(\alpha\)-closed sets and investigated some properties of functions with \(\alpha\)-closed graphs. The notion of \(\gamma\)-open sets was studied by H. Ogata [8] to investigate some new separation axioms. Recently, the notion of operations on the family of all semi-open sets and pre-open sets is investigated in [7, 12].

In this paper, our aim is to study the concept of \(\gamma_\mu\)-compact subsets that are defined via operations, where an operation is defined on a collection of generalized open sets instead of a topology. The notion of generalized open sets was introduced by Á. Császár. We recall some notions defined in [2]. Let \(X\) be a non-empty set and \(\mathcal{P}(X)\) be the power set of \(X\). We call a class \(\mu \subseteq \exp X\) a generalized topology (briefly, GT) if \(\emptyset \in \mu\) and any union of elements of \(\mu\) belongs to \(\mu\). A set \(X\) with a GT \(\mu\) on it is said to be a generalized topological space (briefly, GTS) and is denoted by \((X, \mu)\). A GT \(\mu\) on \(X\) is said to be strong if \(X \in \mu\). A GTS \((X, \mu)\) is said to be quasi topological space [4] if it is closed under finite intersection.

For a GTS \((X, \mu)\), the elements of \(\mu\) are called \(\mu\)-open sets and the complement of \(\mu\)-open sets are called \(\mu\)-closed sets. For \(A \subseteq X\), we denote by \(c_\mu(A)\) the intersection of all \(\mu\)-closed sets containing \(A\), i.e. the smallest \(\mu\)-closed set containing \(A\); and by \(i_\mu(A)\) the union of all \(\mu\)-open sets contained in \(A\), i.e., the largest \(\mu\)-open set contained in \(A\) (see [1, 2]).

It is easy to observe that \(i_\mu\) and \(c_\mu\) are idempotent and monotonic, where \(\gamma: \mathcal{P}(X) \to \mathcal{P}(X)\) is said to be idempotent iff for each \(A \subseteq X\), \(\gamma(\gamma(A)) = \gamma(A)\), and monotonic iff \(\gamma(A) \subseteq \gamma(B)\) whenever \(A \subseteq B \subseteq X\). It is also well known [2, 3] that let \(\mu\) be a GT on \(X\) and \(A \subseteq X\),
$x \in X$, then $x \in c_\mu(A)$ if and only if $M \cap A \neq \emptyset$ for every $M \in \mu$ containing $x$ and that $c_\mu(X \setminus A) = X \setminus i_\mu(A)$. We note that $x \in i_\mu(A)$ if and only if there exists some $\mu$-open set $U$ containing $x$ such that $U \subseteq A$. A subset $A$ of $X$ is $\mu$-open (resp. $\mu$-closed) if and only if $A = i_\mu(A)$ (resp. $A = c_\mu(A)$).

1. **$\gamma_\mu$-Open Sets**

**Definition 1** ([9]). Let $(X, \mu)$ be a GTS. Let $\gamma_\mu : \mu \to \mathcal{P}(X)$ be a function from $\mu$ to $\mathcal{P}(X)$ such that $U \subseteq \gamma_\mu(U)$ for each $U \in \mu$. The function $\gamma_\mu$ is called an operation on $\mu$ and the image $\gamma_\mu(U)$ will be denoted by $U^{\gamma_\mu}$.

**Definition 2** ([9]). Let $(X, \mu)$ be a GTS and $\gamma_\mu$ be an operation on $\mu$. A subset $A$ of $X$ is called $\gamma_\mu$-open if for each $x \in A$ there exists $U \in \mu$ such that $x \in U \subseteq U^{\gamma_\mu} \subseteq A$. The family of all $\gamma_\mu$-open sets of $(X, \mu)$ is denoted by $\gamma_\mu$-O(X). We assume that $\emptyset$ is a $\gamma_\mu$-open set.

**Theorem 1.** Let $(X, \mu)$ be a strong GTS and $\gamma_\mu$ be an operation on $\mu$. For $\gamma_\mu$-O(X), the following properties hold:

(i) $\emptyset, X \in \gamma_\mu$-O(X);

(ii) $\gamma_\mu$-O(X) is closed under arbitrary union and hence $\gamma_\mu$-O(X) is a GT on $X$;

(iii) $\gamma_\mu$-O(X) $\subseteq \mu$.

**Proof.** (i) We assumed that $\emptyset \in \gamma_\mu$-O(X). For each $x \in X$, there exists $X \in \mu$ such that $x \in X^{\gamma_\mu} \subseteq X$. Thus $X \in \gamma_\mu$-O(X).

(ii) Let $\{A_\alpha : \alpha \in \Lambda\}$ be a family of $\gamma_\mu$-open sets in $(X, \mu)$ and $x \in \cup\{A_\alpha : \alpha \in \Lambda\}$. Then there exists an $\alpha_0 \in \Lambda$ such that $x \in A_{\alpha_0}$. Since $A_{\alpha_0}$ is a $\gamma_\mu$-open set, there exists a $\mu$-open set $U$ such that $x \in U \subseteq U^{\gamma_\mu} \subseteq A_{\alpha_0} \subseteq \cup\{A_\alpha : \alpha \in \Lambda\}$. Thus $\cup\{A_\alpha : \alpha \in \Lambda\}$ is a $\gamma_\mu$-open set.

(iii) Let $A \in \gamma_\mu$-O(X). Then for each $x \in A$, there exists a $\mu$-open set $U$ such that $x \in U \subseteq U^{\gamma_\mu} \subseteq A$. Hence $A = \cup\{U : x \in A\}$. Hence $A$ is $\mu$-open. Thus $\gamma_\mu$-O(X) $\subseteq \mu$. \[\square\]

**Remark 1.** It follows that $\gamma_\mu$-O(X) is a GT on X. But it is not closed under finite intersection, i.e. the intersection of two $\gamma_\mu$-open sets is not always $\gamma_\mu$-open. It follows from the example below.

**Example 1.** Let $X = \{1, 2, 3\}$ and $\mu = \{\emptyset, \{1\}, \{1, 2\}, \{2, 3\}, X\}$. Then $(X, \mu)$ is a GTS. Now $\gamma_\mu : \mu \rightarrow \mathcal{P}(X)$ defined by

$$\gamma_\mu(A) = \begin{cases} A, & \text{if } 1 \in A, \\ \{2, 3\}, & \text{otherwise}, \end{cases}$$

is an operation. It can be easily checked that $\{1, 2\}$ and $\{2, 3\}$ are two $\gamma_\mu$-open sets but their intersection $\{2\}$ is not so.

**Definition 3.** A GTS $(X, \mu)$ is said to be $\gamma_\mu$-regular if for each $x \in X$ and each $U \in \mu$ containing $x$, there exists $V \in \mu$ such that $x \in V \subseteq V^{\gamma_\mu} \subseteq U$.

**Theorem 2.** For a strong GTS $(X, \mu)$, the following properties are equivalent:

(i) $\mu = \gamma_\mu$-O(X);

(ii) $(X, \mu)$ is $\gamma_\mu$-regular;

(iii) for each $x \in X$ and each $U \in \mu$ containing $x$, there exists $W \in \gamma_\mu$-O(X) such that $x \in W \subseteq W^{\gamma_\mu} \subseteq U$. 

Proof. (i) $\Leftrightarrow$ (ii) This follows immediately from Definitions 2 and 3.

(ii) $\Rightarrow$ (iii) For each $x \in X$ and $U \in \mu$ containing $x$, by (ii) there exists $W \in \mu$ such that $x \in W \subseteq W^\gamma \subseteq U$. Now by (i), $\mu = \gamma_\mu$-O(X) and hence $W$ is a $\gamma_\mu$-open set such that $x \in W \subseteq W^\gamma \subseteq U$.

(iii) $\Rightarrow$ (i) By Theorem 1, $\gamma_\mu$-O(X) $\subseteq \mu$. Let $U \in \mu$. Then for any $x \in U$, by (iii) there exists $W_x \in \gamma_\mu$-O(X) such that $x \in W_x \subseteq U$. Thus by Theorem 1, we have $U = \cup\{W_x : x \in U\} \in \gamma_\mu$-O(X).

Definition 4. An operation $\gamma_\mu$ on a GTS $(X, \mu)$ is said to be regular if for each $x \in X$ and each $U, V \in \mu$ containing $x$, there exists $W \in \mu$, such that $x \in W \subseteq W^\gamma \subseteq U^\gamma \cap V^\gamma$.

Theorem 3. Let $\gamma_\mu : \mu \to \mathcal{P}(X)$ be a regular operation on $\mu$. Then $A \cap B$ is a $\gamma_\mu$-open set for any $\gamma_\mu$-open sets $A$ and $B$. If $\mu$ is in addition strong, then $\gamma_\mu$ is a topology on $X$.

Proof. Let $A$ and $B$ be two $\gamma_\mu$-open sets in a GTS $(X, \mu)$. We shall show that $A \cap B$ is also a $\gamma_\mu$-open set. Let $x \in A \cap B$. Then there exist two $\mu$-open sets $U$ and $V$ containing $x$ such that $U^\gamma \subseteq A$ and $V^\gamma \subseteq B$. Since $\gamma_\mu : \mu \to \mathcal{P}(X)$ is a regular operation, there exists a $\mu$-open set $W$ containing $x$ such that $W^\gamma \subseteq U^\gamma \cap V^\gamma \subseteq A \cap B$. Thus $A \cap B$ is $\gamma_\mu$-open. The rest follows from Theorem 1.

Definition 5. Let $(X, \mu)$ be a GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation.

(a) It follows from Theorem 1 (ii) that $\gamma_\mu$ is a GT on $X$. The $\gamma_\mu$-closure [9] of a set $A$ in $X$ is denoted by $c_\gamma(A)$ and is defined as $c_\gamma(A) = \cap\{F : F$ is in addition strong, then $\gamma_\mu$ is a topology on $X$.

Theorem 3. Let $\gamma_\mu : \mu \to \mathcal{P}(X)$ be a regular operation on $\mu$. Then $A \cap B$ is a $\gamma_\mu$-open set for any $\gamma_\mu$-open sets $A$ and $B$. If $\mu$ is in addition strong, then $\gamma_\mu$ is a topology on $X$.

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Definition 6. A subset $A$ of a strong GTS $(X, \mu)$ is said to be $\mu$-compact [11] (weakly $\mu$-compact [10]) if every cover $\{U_\alpha : \alpha \in \Lambda\}$ of $A$ by $\mu$-open subsets of $X$ has a finite subset $\Lambda_0$ of $\Lambda$ such that $A \subseteq \cup\{U_\alpha : \alpha \in \Lambda_0\}$ (resp. $A \subseteq \cup\{c_\mu(U_\alpha) : \alpha \in \Lambda_0\}$).

If $A = X$, then the $\mu$-compact (resp. $\mu$-closed) subset $A$ is known as a $\mu$-compact space (resp. $\mu$-closed) space.

Definition 7. Let $(X, \mu)$ be a strong GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation on $\mu$. A subset $A$ of $(X, \mu)$ is said to be $\gamma_\mu$-compact if every cover $\{U_\alpha : \alpha \in \Lambda\}$ of $A$ by $\mu$-open subsets of $X$, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $A \subseteq \cup\{U_\alpha^\gamma : \alpha \in \Lambda_0\}$. If $A = X$, then the $\gamma_\mu$-compact subset $A$ is called a $\gamma_\mu$-compact space.

Remark 3. If $(X, \mu)$ be a strong GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation on $\mu$. If $\gamma_\mu$ is an identity (resp. $\mu$-closure) operation, then the notion of a $\gamma_\mu$-compact space coincides with that of a $\mu$-compact (weakly $\mu$-compact) space.
Theorem 4. Let \((X, \mu)\) be a strong GTS and \(\gamma : \mu \rightarrow \mathcal{P}(X)\) be a regular operation on \(\mu\). Then the following are equivalent:

(i) \((X, \mu)\) is \(\mu\)-compact;
(ii) \((X, \mu)\) is \(\gamma\)-compact;
(iii) \((X, \gamma \circ \Lambda(X))\) is \(\mu\)-compact;
(iv) \((X, \gamma \circ O(X))\) is \(\gamma\)-compact.

The implications (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iii) \(\Rightarrow\) (iv) hold without the assumption of \(\gamma\)-regularity on \((X, \mu)\).

Proof. (i) \(\Rightarrow\) (ii) Let \((X, \mu)\) be a \(\mu\)-compact space. For any cover \(\{U_i : i \in \Lambda\}\) of \(\mu\)-open subsets of \(X\), there exists a finite subset \(\Lambda_0\) of \(\Lambda\) such that \(X = \bigcup U_i \subseteq \bigcup \{U_i^{\gamma} : i \in \Lambda_0\}\). Therefore \((X, \mu)\) is \(\gamma\)-compact.

(ii) \(\Rightarrow\) (iii) Let \((X, \mu)\) be a \(\gamma\)-compact space and \(\{U_i : i \in \Lambda\}\) be a cover of \(X\) by \(\gamma\)-open subsets of \(X\). For each \(x \in X\), there exists \(a(x) \in \Lambda\) such that \(x \in U_{a(x)}\). Since \(U_{a(x)}\) is \(\gamma\)-open, there exists \(V_{a(x)} \subseteq \mu\) such that \(x \in V_{a(x)} \subseteq \bigcup U_{a(x)}\). Then \(\{V_{a(x)} : x \in X\}\) is a \(\gamma\)-open cover of \(X\) by \(\mu\)-open subsets of \(X\). Since \((X, \mu)\) is \(\gamma\)-compact, there exists a finite subset \(\{x_1, x_2, \ldots, x_n\}\) of \(X\) such that \(\bigcup \{V_{a(x_i)}^{\gamma} : i = 1, 2, \ldots, n\} = X\) and hence \(X = \bigcup \{U_{a(x_i)} : i = 1, 2, \ldots, n\}\). Thus, \((X, \gamma \circ \Lambda(X))\) is \(\mu\)-compact.

(iii) \(\Rightarrow\) (iv) This follows from the fact that \(\gamma \circ \Lambda(X) \subseteq \mu\).

(iv) \(\Rightarrow\) (i) Let \((X, \mu)\) be \(\gamma\)-regular and \((X, \gamma \circ \Lambda(X))\) be \(\gamma\)-compact. Then by Theorem 2, \(\mu = \gamma \circ \Lambda(X)\) and \((X, \mu)\) is \(\gamma\)-compact. Let \(\{U_i : i \in \Lambda\}\) be a \(\mu\)-open cover of \(X\). Then for each \(x \in X\), there exists \(a(x) \in \Lambda\) such that \(x \in U_{a(x)}\). Since \((X, \mu)\) is \(\gamma\)-regular, there exists \(V_{a(x)} \subseteq \mu\) such that \(x \in V_{a(x)} \subseteq \bigcup U_{a(x)}\). Since \(\{V_{a(x)} : x \in X\}\) is a \(\mu\)-open cover of \(X\) and \((X, \mu)\) is \(\gamma\)-compact, there exists a finite subset \(\{x_1, x_2, \ldots, x_n\}\) of \(X\) such that \(\bigcup \{V_{a(x)}^{\gamma} : i = 1, 2, \ldots, n\} = X\). Thus, \(\bigcup \{U_{a(x_i)} : i = 1, 2, \ldots, n\} = X\). Hence, \((X, \mu)\) is \(\mu\)-compact.

Remark 4. Let \((X, \mu)\) be a \(\mu\)-compact space. If \(\gamma : \mu \rightarrow \mathcal{P}(X)\) is an operation on \(\mu\) then \((X, \mu)\) is \(\gamma\)-compact.

Example 2. Let \(\mathbb{N}\) be the set of natural numbers. Let \(\mu = \mathcal{P}(\mathbb{N})\) (= the power set of \(\mathbb{N}\)). Now \(\gamma : \mathcal{P}(\mathbb{N}) \rightarrow \mathcal{P}(\mathbb{N})\) defined by

\[
\gamma(A) = \begin{cases} 
\mathbb{N}, & \text{if } A \neq \emptyset, \\
\emptyset, & \text{otherwise}.
\end{cases}
\]

Then \((\mathbb{N}, \mu)\) is a \(\gamma\)-compact space which is not \(\mu\)-compact.

Definition 8. Let \((X, \mu)\) be a GTS and \(\gamma : \mu \rightarrow \mathcal{P}(X)\) be an operation on \(\mu\). A filterbase \(\mathcal{F}\) on \(X\) is said to be

(a) \(\gamma\)-converge to a point \(x \in X\) if for each \(\mu\)-open set \(U\) containing \(x\), there exists \(F \in \mathcal{F}\) such that \(F \subseteq U^{\gamma}\);

(b) \(\gamma\)-accumulate at \(x \in X\) if for each \(F \in \mathcal{F}\) and each \(\mu\)-open set \(U\) containing \(x\), \(F \cap U^{\gamma} \neq \emptyset\).
Theorem 5. Let $(X, \mu)$ be a strong GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be a $\mu$-regular operation on $\mu$. If a filterbase $\mathcal{F}$ on $X$ $\gamma_\mu$-accumulates at $x \in X$, then there exists a filterbase $\mathcal{G}$ on $X$ such that $\mathcal{F} \subseteq \mathcal{G}$ and $\mathcal{G}$ $\gamma_\mu$-converges to $x$.

Proof. Let $\mathcal{F}$ be the filterbase which $\gamma_\mu$-accumulates at $x$. Hence for each $\mu$-open set $U$ containing $x$ and each $A \in \mathcal{F}$, $A \cap U^\mu \neq \emptyset$. Hence $x \in \gamma_\mu(A)$ for each $A \in \mathcal{F}$. Let $\mathcal{H} = \{A \cap U^\mu : U$ is a $\mu$-open set containing $x$ and $A \in \mathcal{F}\}$. Suppose that $H_1, H_2 \in \mathcal{H}$. Then $H_1 \cap H_2 = (A_1 \cap U^\mu_1) \cap (A_2 \cap U^\mu_2) = (A_1 \cap A_2) \cap (U^\mu_1 \cap U^\mu_2)$ for every $A_1, A_2 \in \mathcal{F}$ and every $\mu$-open set $U_1, U_2$ in $X$. Since $\gamma_\mu$ is $\mu$-regular, there exists a $\mu$-open set $U_3$ of $X$ containing $x$ such that $U^\mu_3 \subseteq U^\mu_1 \cap U^\mu_2$. Since $\mathcal{F}$ is a filterbase, there exists $A_3 \in \mathcal{F}$ such that $A_3 \subseteq A_1 \cap A_2$. Hence $A_3 \cap U^\mu_3 \subseteq H_1 \cap H_2$. Thus $\mathcal{H}$ is a filterbase. Now set $\mathcal{G} = \{B : \exists C \subseteq B\}$. Then $\mathcal{G}$ is a filter generated by $\mathcal{H}$. Now for each $\mu$-open set $U$ containing $x$ and each $A \in \mathcal{F}$, $U^\mu \supseteq A \cap U^\mu \in \mathcal{G}$, where $A \cap U^\mu \in \mathcal{H}$. So $\mathcal{G}$ $\gamma_\mu$-converges to $x$. Also for each $A \in \mathcal{F}$, $A = \bigcap U^\mu \cap A \in \mathcal{H}$. So $A \in \mathcal{G}$. Hence, $\mathcal{F} \subseteq \mathcal{G}$. □

Corollary 1. Let $(X, \mu)$ be a strong GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be a $\mu$-monotonic operation on $\mu$. If a maximal filter in $X$ that $\gamma_\mu$-accumulates at a point $x \in X$, then it also $\gamma_\mu$-converges to $x$.

Proof. Let $\mathcal{F}$ be a maximal filterbase which $\gamma_\mu$-accumulates at some point $x_0$ of $X$. If $\mathcal{F}$ does not $\gamma_\mu$-converge to $x_0$, then there exists $U_0 \in \mu$ containing $x_0$ such that $F \cap U^\mu_0 \neq \emptyset$ and $F \cap (X \setminus U^\mu_0) \neq \emptyset$ for every $F \in \mathcal{F}$. Then $\mathcal{F} \cup \{F \cap U^\mu_0 : F \in \mathcal{F}\}$ is a filterbase on $X$ which strictly contains $\mathcal{F}$. This is a contradiction to the maximality of $\mathcal{F}$. □

Theorem 6. If a GTS $(X, \mu)$ is $\gamma_\mu$-compact, for some operation $\mu$ such that$(X, \mu)$ is $\gamma_\mu$-regular, then $(X, \mu)$ is $\mu$-compact.

Proof. Let $U = \{U_\alpha : \alpha \in \Lambda\}$ be a $\mu$-open cover of $X$. For any $x \in X$, there exists $\alpha \in \Lambda$ such that $x \in U_\alpha$. Since $X$ is $\gamma_\mu$-regular, there exists $V_\alpha \in \mu$ such that $x \in V_\alpha \subseteq V^\mu_\alpha \subseteq U_\alpha$. Since $(X, \mu)$ is $\gamma_\mu$-compact and $\{V_\alpha : \alpha \in \Lambda\}$ is a $\mu$-open cover of $X$, there is a finite subset $\Lambda_0$ of $\Lambda$ such that $X = \cup\{V^\mu_\alpha : \alpha \in \Lambda_0\} \subseteq \cup\{U_\alpha : \alpha \in \Lambda_0\}$. □

Theorem 7. Let $(X, \mu)$ be a strong GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation on $\mu$. Then the following are equivalent:

(i) $(X, \mu)$ is $\gamma_\mu$-compact;

(ii) each maximal filter $\mathcal{F}$ on $X$ $\gamma_\mu$-converges to some point of $X$;

(iii) each filterbase in $X$ $\gamma_\mu$-accumulates at some point of $X$.

Proof. (i) $\Rightarrow$ (ii) Let $(X, \mu)$ be $\gamma_\mu$-compact and $\mathcal{F}_0$ be a maximal filter on $X$. Suppose that $\mathcal{F}_0$ does not $\gamma_\mu$-converge to any point of $X$. Then by Corollary 1, $\mathcal{F}_0$ does not $\gamma_\mu$-accumulate at any point of $X$. Then for each $x \in X$, there exist $F_x \in \mathcal{F}_0$ and $U_x \in \mu$ containing $x$ such that $F_x \cap U^\mu_x = \emptyset$. Then the family $\{U_x : x \in X\}$ is a cover of $X$ by $\mu$-open subsets of $X$. Thus by (i), there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of $X$ such that $X = \cup\{U^\mu_{x_i} : i = 1, 2, \ldots, n\}$. Since $\mathcal{F}_0$ is a filterbase on $X$, there exists $F_0 \in \mathcal{F}_0$ such that $F_0 \subseteq \cap\{F_{x_i} : i = 1, 2, \ldots, n\}$. Then $F_0 = F_0 \cap \cup\{U^\mu_{x_i} : i = 1, 2, \ldots, n\} = \cup\{F_0 \cap U^\mu_{x_i} : i = 1, 2, \ldots, n\} \subseteq \cup\{F_{x_i} \cap U^\mu_{x_i} : i = 1, 2, \ldots, n\} = \emptyset$. This is a contradiction to the fact that $F_0 \in \mathcal{F}_0$. Thus $\mathcal{F}_0$ $\gamma_\mu$-converges to some point of $X$. □
Theorem 10. Let $\mathcal{F}$ be a filterbase on $X$. Then there exists a maximal filterbase $\mathcal{F}_0$ such that $\mathcal{F} \subseteq \mathcal{F}_0$. By (ii), $\mathcal{F}_0$ $\gamma$-converges to some point $x_0 \in X$. For any $\mu$-open set $U$ containing $x_0$, there exists $F_0 \in \mathcal{F}_0$ such that $F_0 \subseteq U^{\gamma}$. For any $F \in \mathcal{F}$, $F \in \mathcal{F}_0$ and $\emptyset \neq F \cap F_0 \subseteq F \cap U^{\gamma}$. Thus each filterbase $\gamma$-accumulates at some point of $X$.

Remark 5. Finite union of $\gamma$-compact subsets of $X$ is also $\gamma$-compact.

**Theorem 8.** Let $A$ be any subset of a strong GTS $(X, \mu)$ such that $A$ and $X \setminus A$ are both $\gamma$-compact subsets of $X$, then $(X, \mu)$ is $\gamma$-compact.

**Proof.** Let $\{V_\alpha : \alpha \in \Lambda\}$ be a $\mu$-open cover of $X$. Then $\{V_\alpha : \alpha \in \Lambda\}$ is a $\mu$-open cover of $A$ and $X \setminus A$ also. Thus there exist finite subsets $\Lambda_1$ and $\Lambda_2$ of $\Lambda$ such that $A \subseteq \{V_{\alpha_1}^{\mu} : \alpha_1 \in \Lambda_1\}$ and $X \setminus A \subseteq \{V_{\alpha_2}^{\mu} : \alpha_2 \in \Lambda_2\}$. Thus $X = A \cup (X \setminus A) \subseteq \{V_{\alpha_1}^{\mu} : \alpha_1 \in \Lambda_1 \cup \Lambda_2\}$. This completes the proof.

**Definition 9.** Let $(X, \mu)$ be a GTS and $\gamma : \mu \to \mathcal{P}(X)$ be an operation. Then $(X, \mu)$ is said to be $\gamma$-T$_2$ if for any two distinct points $x$ and $y$ in $X$, there exist $\mu$-open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $U^{\gamma} \cap V^{\gamma} = \emptyset$.

We observe that every $\gamma$-T$_2$ space is $\mu$-T$_2$ space. But the converse is false as shown by Example 2.

**Theorem 9.** Let $(X, \mu)$ be QTS $\gamma$-regular and $\gamma : \mu \to \mathcal{P}(X)$ be a $\mu$-monotonic operation. If $(X, \mu)$ is $\gamma$-T$_2$ and $K \subseteq X$ is $\gamma$-compact, then $K$ is a $\gamma$-closed set.

**Proof.** It is sufficient to show that $X \setminus K$ is a $\gamma$-open set. Let $x \in X \setminus K$. For each $y \in K$, there exist $\mu$-open sets $U_y$ and $V_y$ such that $x \in U_y$, $y \in V_y$ and $U_y^{\gamma} \cap V_y^{\gamma} = \emptyset$. Thus we can construct a cover $U = \{V_y : y \in K\}$ of $K$ by $\mu$-open sets of $X$. Since $K$ is $\gamma$-compact, there exists a finite collection $\{V_{y_1}, V_{y_2}, \ldots, V_{y_n}\}$ of $U$ such that $K \subseteq \bigcup_{i=1}^{n} V_{y_i}^{\gamma}$. Let $U = \bigcap_{i=1}^{n} U_{y_i}$. Then $U$ is a $\mu$-open set containing $x$ such that $U^{\gamma} \subseteq X \setminus K$. Then by $\gamma$-regulartility of $X$, there exists a $\mu$-open set $W$ containing $x$ such that $x \in W \subseteq \bigcup_{i=1}^{n} V_{y_i}^{\gamma} \subseteq U \subseteq U^{\gamma}$. Thus $W \subseteq U^{\gamma} \subseteq X \setminus K$. Hence, $X \setminus K$ is $\gamma$-open.

We call an operation $\gamma : \mu \to \mathcal{P}(X)$ on a GTS $(X, \mu)$ to be additive if for any $A, B \in \mu$, $(A \cup B)^{\gamma} = A^{\gamma} \cup B^{\gamma}$.

**Theorem 10.** Let $(X, \mu)$ be QTS and $\gamma : \mu \to \mathcal{P}(X)$ be an additive, $\mu$-monotonic operation on $\mu$. If $Y \subseteq X$ is $\gamma$-compact, $x \in X \setminus Y$ and $(X, \mu)$ is $\gamma$-T$_2$, then there exist $\mu$-open sets $U$ and $V$ with $x \in U$, $Y \subseteq V^{\gamma}$ and $U^{\gamma} \cap V^{\gamma} = \emptyset$. 

(ii) $\Rightarrow$ (iii) Let $\mathcal{F}$ be a filterbase on $X$. Then there exists a maximal filterbase $\mathcal{F}_0$ such that $\mathcal{F} \subseteq \mathcal{F}_0$. By (ii), $\mathcal{F}_0$ $\gamma$-converges to some point $x_0 \in X$. For any $\mu$-open set $U$ containing $x_0$, there exists $F_0 \in \mathcal{F}_0$ such that $F_0 \subseteq U^{\gamma}$. For any $F \in \mathcal{F}$, $F \in \mathcal{F}_0$ and $\emptyset \neq F \cap F_0 \subseteq F \cap U^{\gamma}$. Thus each filterbase $\gamma$-accumulates at some point of $X$.
Proof. For each $y \in Y$, let $V_y$ and $U_y$ be $\mu$-open sets such that $V_y \cap U_y = \emptyset$, with $y \in V_y$ and $x \in U_y$. The collection $\mathcal{V} = \{V_y : y \in Y\}$ is then a cover of $Y$ by $\mu$-open sets. Now since $Y$ is $\gamma_\mu$-compact, there exists a finite subcollection $\{V_{y_1}, V_{y_2}, \ldots, V_{y_n}\}$ of $\mathcal{V}$ such that $Y \subseteq \bigcup_{i=1}^{n} V_{y_i}$.

Let $U = \bigcap_{i=1}^{n} U_{y_i}$ and $V = \bigcup_{i=1}^{n} V_{y_i}$. Since $U \subseteq U_{y_i}$ for every $i = 1, 2, \ldots, n$ and $\gamma_\mu$ is monotonic, $U_{y_i} \cap V_{y_i} = \emptyset$ for $i = 1, 2, \ldots, n$. Thus, $U_{y_i} \cap V_{y_i} = \emptyset$ (as $\gamma_\mu$ is an additive operation on $\mu$). Thus, $Y \subseteq V_{y_i}$ and $x \in U$.

\[\square\]

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