A Kenmotsu metric as a conformal $\eta$-Einstein soliton

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The object of the present paper is to study some properties of Kenmotsu manifold whose metric is conformal $\eta$-Einstein soliton. We have studied certain properties of Kenmotsu manifold admitting conformal $\eta$-Einstein soliton. We have also constructed a 3-dimensional Kenmotsu manifold satisfying conformal $\eta$-Einstein soliton.

Key words and phrases: Einstein soliton, $\eta$-Einstein soliton, conformal $\eta$-Einstein soliton, $\eta$-Einstein manifold, Kenmotsu manifold.

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Introduction

The notion of Einstein soliton was introduced by G. Catino and L. Mazzieri [3] in 2016, which generates self-similar solutions to Einstein flow

$$\frac{dg}{dt} = -2 \left( S - \frac{r}{2} g \right),$$

where $S$ is Ricci tensor, $g$ is Riemannian metric and $r$ is the scalar curvature.

The equation of the $\eta$-Einstein soliton [2] is given by,

$$\mathcal{L}_\xi g + 2S + (2\lambda - r)g + 2\mu \eta \otimes \eta = 0,$$

where $\mathcal{L}_\xi$ is the Lie derivative along the vector field $\xi$, $S$ is the Ricci tensor, $r$ is the scalar curvature of the Riemannian metric $g$, and $\lambda$ and $\mu$ are real constants. For $\mu = 0$, the data $(g, \xi, \lambda)$ is called Einstein soliton.

In 2018, M.D. Siddiqi [6] introduced the notion of conformal $\eta$-Ricci soliton [8] as

$$\mathcal{L}_\xi g + 2S + \left[ 2\lambda - \left( p + \frac{2}{n} \right) \right] g + 2\mu \eta \otimes \eta = 0,$$

where $\mathcal{L}_\xi$ is the Lie derivative along the vector field $\xi$, $S$ is the Ricci tensor, $\lambda$, $\mu$ are constants, $p$ is a scalar non-dynamical field (time dependent scalar field) and $n$ is the dimension of manifold. For $\mu = 0$, conformal $\eta$-Ricci soliton becomes conformal Ricci soliton [7].

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In [9], S. Roy, S. Dey and A. Bhattacharyya have defined conformal Einstein soliton, which can be written as

$$\mathcal{L}_V g + 2S + \left[ 2\lambda - r + \left( p + \frac{2}{n} \right) \right] g = 0,$$

where $\mathcal{L}_V$ is the Lie derivative along the vector field $V$, $S$ is the Ricci tensor, $r$ is the scalar curvature of the Riemannian metric $g$, $\lambda$ is a real constant, $p$ is a scalar non-dynamical field (time dependent scalar field) and $n$ is the dimension of manifold.

So we introduce the notion of conformal $\eta$-Einstein soliton as follows.

**Definition 1.** A Riemannian manifold $(M, g)$ of dimension $n$ is said to admit conformal $\eta$-Einstein soliton if

$$\mathcal{L}_\xi g + 2S + \left[ 2\lambda - r + \left( p + \frac{2}{n} \right) \right] g + 2\mu \eta \otimes \eta = 0,$$

where $\mathcal{L}_\xi$ is the Lie derivative along the vector field $\xi$, $\lambda$, $\mu$ are real constants and $S$, $r$, $p$, $n$ are same as defined in (1).

In the present paper, we study conformal $\eta$-Einstein soliton on Kenmotsu manifold. The paper is organized as follows.

After introduction, section 2 is devoted for preliminaries on $(2n+1)$-dimensional Kenmotsu manifold. In section 3, we have studied conformal $\eta$-Einstein soliton on Kenmotsu manifold. Here we proved that if a $(2n+1)$-dimensional Kenmotsu manifold admits conformal $\eta$-Einstein soliton then the manifold becomes $\eta$-Einstein. We have also characterized the nature of the manifold if the manifold is Ricci symmetric and the Ricci tensor is $\eta$-recurrent. Also we have discussed the condition, when the manifold has cyclic Ricci tensor. Then we have obtained the conditions in a $(2n+1)$-dimensional Kenmotsu manifold admitting conformal $\eta$-Einstein soliton, when a vector field $V$ is pointwise co-linear with $\xi$ and a $(0,2)$-tensor field $h$ is parallel with respect to the Levi-Civita connection associated to $g$. We have also examined the nature of a Ricci-recurrent Kenmotsu manifold admitting conformal $\eta$-Einstein soliton.

In last section, we have given an example of a 3-dimensional Kenmotsu manifold satisfying conformal $\eta$-Einstein soliton.

## 1 Preliminaries

Let $M$ be a $(2n+1)$-dimensional connected almost contact metric manifold with an almost contact metric structure $(\phi, \xi, \eta, g)$, where $\phi$ is a $(1,1)$-tensor field, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is the compatible Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi^\xi = 0,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$g(X, \phi Y) = -g(\phi X, Y),$$

$$g(X, \xi) = \eta(X),$$

for all vector fields $X, Y \in \chi(M)$.

The fundamental 2-form $\Phi$ on an almost contact metric manifold $M^{2n+1}$ is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields $X$ and $Y$ on $M^{2n+1}$. In an almost contact metric manifold, we have $\eta \wedge \Phi^n \neq 0$. 


When $\Phi = d\eta$, an almost contact metric manifold becomes contact metric manifold. An almost contact metric manifold satisfying $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$ is said to be an almost Kenmotsu manifold [4].

An almost contact metric manifold is said to be a Kenmotsu manifold [5] if

$$\nabla_X \phi Y = -g(X, \phi Y)\xi - \eta(Y)\phi X,$$

$$\nabla_X \xi = X - \eta(X)\xi,$$ (6)

where $\nabla$ denotes the Riemannian connection of $g$.

In a Kenmotsu manifold the following relations hold [1]:

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X),$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X,$$ (7)

$$R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X,$$

where $R$ is the Riemannian curvature tensor,

$$S(X, \xi) = -2n\eta(X),$$ (8)

$$S(\phi X, \phi Y) = S(X, Y) + 2n\eta(X)\eta(Y),$$

$$\nabla_X \eta = g(X, Y) - \eta(X)\eta(Y),$$ (9)

for all vector fields $X, Y, Z \in \chi(M)$.

Now we know,

$$(\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi),$$ (10)

for all vector fields $X, Y \in \chi(M)$. Then using (6) and (10), we get,

$$(\xi g)(X, Y) = 2[g(X, Y) - \eta(X)\eta(Y)].$$ (11)

### 2 Conformal $\eta$-Einstein soliton on Kenmotsu manifold

Let $M$ be a $(2n+1)$-dimensional Kenmotsu manifold. Consider the conformal $\eta$-Einstein soliton (2) on $M$ as

$$(\xi g)(X, Y) + 2S(X, Y) + \left[2\lambda - r + \left(p + \frac{2}{2n+1}\right)\right]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0$$

for all vector fields $X, Y \in \chi(M)$.

**Theorem 1.** If the metric of a $(2n+1)$-dimensional Kenmotsu manifold is a conformal $\eta$-Einstein soliton, then the manifold becomes $\eta$-Einstein and the scalar curvature is

$$\left(p + \frac{2}{2n+1}\right) - 4n + 2\lambda + 2\mu.$$
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Theorem 3. If the metric of a $(2n+1)$-dimensional Kenmotsu manifold is a conformal $\eta$-Einstein soliton and the Ricci tensor $S$ is $\eta$-recurrant, then the scalar curvature is

$$2\lambda + 2\mu + \left( p + \frac{2}{2n+1} \right).$$

Proof. If the Ricci tensor $S$ is $\eta$-recurrant, then we have $\nabla S = \eta \otimes S$, which implies

$$(\nabla_X S)(Y, Z) = \eta(X)S(Y, Z)$$

for all vector fields $X, Y, Z$ on $M$. Using (14), the above equation reduces to

$$-(\mu - 1)[\eta(Z)(\nabla_X \eta)Y + \eta(Y)(\nabla_X \eta)Z] = \eta(X)S(Y, Z).$$

Taking $Z = \xi$, $\xi$ in the above equation and using (9), (12), we get

$$\left[ \lambda + \mu - \frac{r}{2} + \frac{p + \frac{2}{2n+1}}{2} \right] \eta(X) = 0,$$

which implies

$$r = 2\lambda + 2\mu + \left( p + \frac{2}{2n+1} \right).$$

This completes the proof.
**Theorem 4.** Let the metric of a \((2n+1)\)-dimensional Kenmotsu manifold \(M\) is a conformal \(\eta\)-Einstein soliton. Then \(M\) has cyclic Ricci tensor if \(\mu = 1\).

**Proof.** Similarly from (14), we get
\[
(\nabla_Y S)(Z, X) = -(\mu - 1)[\eta(X)(\nabla_Y \eta)Z + \eta(Z)(\nabla_X \eta)Y],
\tag{15}
\]
and
\[
(\nabla_Z S)(X, Y) = -(\mu - 1)[\eta(Y)(\nabla_Z \eta)X + \eta(X)(\nabla_Z \eta)Y]
\tag{16}
\]
for all vector fields \(X, Y, Z\) on \(M\).

Then adding (14), (15), (16) and using (9), (4), we obtain
\[
(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = -2(\mu - 1)[\eta(X)g(Y, Z)
+ \eta(Y)g(Z, X) + \eta(Z)g(X, Y)].
\tag{17}
\]
Now, as the manifold \(M\) has cyclic Ricci tensor, i.e
\[
(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0,
\]
then from (17), we have
\[
(\mu - 1)[\eta(X)g(Y, Z) + \eta(Y)g(Z, X) + \eta(Z)g(X, Y)] = 0.
\]
Taking \(X = \xi\) in the above equation and using (3), we get \(\mu = 1\).

Again, if we take \(\mu = 1\) in (17), we obtain \((\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0\), i.e the manifold \(M\) has cyclic Ricci tensor and this completes the proof. \(\square\)

**Corollary 1.** If a \((2n+1)\)-dimensional Kenmotsu manifold \(M\) has a cyclic Ricci tensor and the metric is a conformal \(\eta\)-Einstein soliton, then the scalar curvature is \((p + \frac{2}{2n+1}) - 4n + 2\lambda + 2\).

**Proof.** If \(\mu = 1\), then from (13) we obtain \(r = (p + \frac{2}{2n+1}) - 4n + 2\lambda + 2\). \(\square\)

**Theorem 5.** Let \(M\) be a \((2n+1)\)-dimensional Kenmotsu manifold admitting a conformal \(\eta\)-Einstein soliton \((g, V)\), \(V\) being a vector field on \(M\). If \(V\) is pointwise co-linear with \(\xi\), a vector field on \(M\), then \(V\) is a constant multiple of \(\xi\), provided the scalar curvature is
\[
2\lambda + 2\mu + \left( p + \frac{2}{2n+1} \right) - 4n.
\]

**Proof.** A conformal \(\eta\)-Einstein soliton is defined on a \((2n+1)\)-dimensional Kenmotsu manifold \(M\) as
\[
\mathcal{L}_V g + 2S + \left[ 2\lambda - r + \left( p + \frac{2}{2n+1} \right) \right] g + 2\mu \eta \otimes \eta = 0,
\tag{18}
\]
where \(\mathcal{L}_V\) is the Lie derivative along the vector field \(V\), \(S\) is the Ricci tensor, \(r\) is the scalar curvature of the Riemannian metric \(g\), \(\lambda, \mu\) are real constants, \(p\) is a scalar non-dynamical field (time dependent scalar field).

Since, \(V\) is pointwise co-linear with \(\xi\), let \(V = b\xi\), where \(b\) is a function on \(M\).

Then (18) becomes
\[
(\mathcal{L}_{b\xi} g)(X, Y) + 2S(X, Y) + \left[ 2\lambda - r + \left( p + \frac{2}{2n+1} \right) \right] g(X, Y) + 2\mu \eta(X)\eta(Y) = 0.
\]
for all vector fields $X$, $Y$ on $M$. Applying the property of Lie derivative and Levi-Civita connection, we have

$$\begin{align*}
bg(\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X) + 2S(X, Y) \\
+ \left[2\lambda - r + \left(p + \frac{2}{2n+1}\right)\right]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0.
\end{align*}$$

Now using (6), we get

$$\begin{align*}
2bg(X, Y) - 2b\eta(X)\eta(Y) + (Xb)\eta(Y) + (Yb)\eta(X) + 2S(X, Y) \\
+ \left[2\lambda - r + \left(p + \frac{2}{2n+1}\right)\right]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0.
\end{align*}$$

Taking $Y = \xi$ in the above equation and using (3), (5), (8), we obtain

$$(Xb) + (\xi b)\eta(X) - 4n\eta(X) + \left[2\lambda - r + \left(p + \frac{2}{2n+1}\right)\right]\eta(X) + 2\mu\eta(X) = 0. \quad (19)$$

Then by putting $X = \xi$, the above equation reduces to

$$\xi b = 2n + \frac{r}{2} - \lambda - \mu - \left(p + \frac{2}{2n+1}\right). \quad (20)$$

Using (20), equation (19) becomes

$$(Xb) + \left[\lambda + \mu + \frac{(p + \frac{2}{2n+1})}{2} - 2n - \frac{r}{2}\right]\eta(X) = 0. \quad (21)$$

Applying exterior differentiation in (21), we have

$$\left[\lambda + \mu + \frac{(p + \frac{2}{2n+1})}{2} - 2n - \frac{r}{2}\right]d\eta = 0. \quad (22)$$

Now we know

$$d\eta(X, Y) = \frac{1}{2}\left[(\nabla_X \eta)Y - (\nabla_Y \eta)X\right]$$

for all vector fields $X$, $Y$ on $M$. Using (9), the above equation becomes $d\eta = 0$. Hence the 1-form $\eta$ is closed.

So from (22), either $r = 2\lambda + 2\mu + (p + \frac{2}{2n+1}) - 4n$ or $r \neq 2\lambda + 2\mu + (p + \frac{2}{2n+1}) - 4n$. If $r = 2\lambda + 2\mu + (p + \frac{2}{2n+1}) - 4n$, (21) reduces to $(Xb) = 0$. This implies that $b$ is constant and this completes the proof.

**Theorem 6.** In a $(2n+1)$-dimensional Kenmotsu manifold assume that a symmetric $(0,2)$-tensor field $h = \xi g + 2S + 2\mu\eta \otimes \eta$ is parallel with respect to the Levi-Civita connection associated to $g$. Then $(g, \xi)$ yields a conformal $\eta$-Einstein soliton.

**Proof.** Note that $h$ is a symmetric tensor field of $(0,2)$-type, which we suppose to be parallel with respect to the Levi-Civita connection $\nabla$, i.e $\nabla h = 0$. Applying the Ricci commutation identity, we have

$$\nabla^2 h(X, Y; Z, W) - \nabla^2 h(X, Y; W, Z) = 0$$
for all vector fields \(X, Y, Z, W\) on \(M\). From the above equation we obtain the relation

\[h(R(X, Y)Z, W) + h(Z, R(X, Y)W) = 0.\]

Replacing \(Z = W = \xi\) in the above equation and using (7), we get

\[\eta(X)h(Y, \xi) - \eta(Y)h(X, \xi) = 0.\]

Replacing \(X = \xi\) and using (3), the above equation reduces to

\[h(Y, \xi) = \eta(Y)h(\xi, \xi)\] (23)

for all vector fields \(Y\) on \(M\). Differentiating the above equation covariantly with respect to \(X\), we get

\[\nabla_X(h(Y, \xi)) = \nabla_X(\eta(Y)h(\xi, \xi)).\]

Now expanding the above equation by using (23), (6), (9) and the property \(\nabla h = 0\), we obtain

\[h(X, Y) = h(\xi, \xi)g(X, Y)\] (24)

for all vector fields \(X, Y\) on \(M\).

Let us take

\[h = \mathcal{L}_\xi g + 2S + 2\mu\eta \otimes \eta.\] (25)

Then from (11), (12), we get

\[h(\xi, \xi) = -2\lambda - \left(p + \frac{2}{2n + 1}\right) + r.\]

Then using (25), equation (24) becomes

\[(\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) + \left[2\lambda - r + \left(p + \frac{2}{2n + 1}\right)\right]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0,

which is the conformal \(\eta\)-Einstein soliton. Hence, we complete the proof. \(\square\)

**Definition 2.** A Kenmotsu manifold is said to be Ricci-recurrent manifold if there exists a non-zero 1-form \(A\) such that

\[(\nabla_WS)(Y, Z) = A(W)S(Y, Z)\] (26)

for any vector fields \(W, Y, Z\) on \(M\).

**Theorem 7.** If the metric of a \((2n+1)\)-dimensional Ricci-recurrent Kenmotsu manifold is a conformal \(\eta\)-Einstein soliton with the 1-form \(A\), then the scalar curvature becomes

\[2\lambda + 2\mu + \left(p + \frac{2}{2n + 1}\right) + 4n(A(\xi) - 1).\]

**Proof.** Replacing \(Z\) by \(\xi\) in the equation (26) and using (8), we get

\[(\nabla_WS)(Y, \xi) = -2nA(W)\eta(Y),\]

which implies that

\[WS(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi) = -2nA(W)\eta(Y).\]
Using (8) and (6), the above equation becomes
\[2n(\nabla_W \eta)Y + 2n\eta(W)\eta(Y) + S(Y, W) = 2nA(W)\eta(Y).\]
Again using (9), the above equation reduces to
\[2n\eta(W, Y) + S(Y, W) = 2nA(W)\eta(Y).\]
Taking \(W = \xi\) in the above equation and using (12), we obtain
\[r = 2\lambda + 2\mu + \left(p + \frac{2}{2n + 1}\right) + 4n(A(\xi) - 1).\]
This completes the proof.

**Example 1.** Here, we consider the three-dimensional manifold \(M = \{(x, y, z) \in \mathbb{R}^3, (x, y, z) \neq (0, 0, 0)\}\), where \((x, y, z)\) are standard coordinates in \(\mathbb{R}^3\). The vector fields
\[e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = -\frac{\partial}{\partial z}\]
are linearly independent at each point of \(M\). Let \(g\) be the Riemannian metric defined by
\[g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.\]
Let \(\eta\) be the 1-form defined by \(\eta(Z) = g(Z, e_3)\) for any vector field \(Z\) in \(\mathbb{R}^3\) and \(\phi\) be the \((1, 1)\)-tensor field defined by \(\phi e_1 = -e_2, \phi e_2 = e_1, \phi e_3 = 0\). Then using the linearity of \(\phi\) and \(g\), we have
\[\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W)\]
for any \(Z, W \in \chi(M)\). Thus for \(e_3 = \xi, (\phi, \xi, \eta, g)\) defines an almost contact metric structure on \(M\).

Let \(\nabla\) be the Levi-Civita connection with respect to the Riemannian metric \(g\). Then we have
\[[e_1, e_2] = 0, [e_1, e_3] = e_1, [e_2, e_3] = e_2.\]
The connection \(\nabla\) of the metric \(g\) is given by
\[2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]),\]
which is known as Koszul’s formula.

Using Koszul’s formula, we can easily calculate
\[
\begin{align*}
\nabla_{e_1} e_1 &= -e_3, & \nabla_{e_1} e_2 &= 0, & \nabla_{e_1} e_3 &= e_1, \\
\nabla_{e_2} e_1 &= 0, & \nabla_{e_2} e_2 &= -e_3, & \nabla_{e_2} e_3 &= e_2, \\
\nabla_{e_3} e_1 &= 0, & \nabla_{e_3} e_2 &= 0, & \nabla_{e_3} e_3 &= 0.
\end{align*}
\]
From the above it follows that the manifold satisfies \(\nabla_X \xi = X - \eta(X)\xi\) for \(\xi = e_3\). Hence the manifold is a Kenmotsu manifold. So, here we have considered \(\mathbb{R}^3\) as an almost contact manifold, which turns out to be a 3-dimensional Kenmotsu manifold.

Also, the Riemannian curvature tensor \(R\) is given by
\[R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.\]
Hence,
\[ R(e_1, e_2)e_2 = -e_1, \quad R(e_1, e_3)e_3 = -e_1, \quad R(e_2, e_1)e_1 = -e_2, \]
\[ R(e_2, e_3)e_3 = -e_2, \quad R(e_3, e_1)e_1 = -e_3, \quad R(e_3, e_2)e_2 = -e_3, \]
\[ R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_1 = 0, \quad R(e_3, e_1)e_2 = 0. \]

Then, the Ricci tensor \( S \) is given by
\[ S(e_1, e_1) = -2, \quad S(e_2, e_2) = -2, \quad S(e_3, e_3) = -2. \]
From (12), we have
\[ S(e_3, e_3) = -\left[ \lambda + \mu - \frac{r}{2} + \left( \frac{p + \frac{2}{3}}{2} \right) \right], \]
which implies that
\[ r = 2\lambda + 2\mu - 4 + \left( \frac{p + \frac{2}{3}}{2} \right). \]

Hence \( \lambda \) and \( \mu \) satisfies equation (13) and so \( g \) defines a conformal \( \eta \)-Einstein soliton on the 3-dimensional Kenmotsu manifold \( M \).

References

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Метою даної роботи є вивчення деяких властивостей многовиду Кенмотсу, метрика якого є конформним \( \eta \)-солітоном Айнштайна. Ми дослідили певні властивості многовиду Кенмотсу, що допускає конформний \( \eta \)-солітон Айнштайна. Також ми збували тривимірний многовид Кенмотсу, що задовольняє конформний \( \eta \)-солітон Айнштайна.

Ключові слова і фрази: солітон Айнштайна, \( \eta \)-солітон Айнштайна, конформний \( \eta \)-солітон Айнштайна, \( \eta \)-многовид Айнштайна, многовид Кенмотсу.