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BOUNDEDNESS OF THE HILBERT TRANSFORM ON BESOV SPACES

MAATOUG A., ALLAOUI S.E.

The Hilbert transform along curves is of a great importance in harmonic analysis. It is known that its boundedness on $L^p(\mathbb{R}^n)$ has been extensively studied by various authors in different contexts and the authors gave positive results for some or all $p, 1 < p < \infty$. Littlewood-Paley theory provides alternate methods for studying singular integrals. The Hilbert transform along curves, the classical example of a singular integral operator, led to the extensive modern theory of Calderón-Zygmund operators, mostly studied on the Lebesgue L^p spaces. In this paper, we will use the Littlewood-Paley theory to prove that the boundedness of the Hilbert transform along curve Γ on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ can be obtained by its L^p-boundedness, where $s \in \mathbb{R}, p, q \in]1, +\infty[$, and $\Gamma(t)$ is an appropriate curve in $\mathbb{R}^n$, also, it is known that the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ are embedded into $L^p(\mathbb{R}^n)$ spaces for $s > 0$ (i.e. $B^s_{p,q}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n), s > 0$). Thus, our result may be viewed as an extension of known results to the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ for general values of s in $\mathbb{R}$.

Key words and phrases: Hilbert transform, Littlewood-Paley decomposition, Besov spaces.

Laboratory of Pure and Applied Mathematics, Laghouat University, 03000, Laghouat, Algeria
E-mail: wardamaatoug@gmail.com (Maatoug A.), shallaoui@yahoo.fr (Allaoui S.E.)

1 INTRODUCTION

Let $\Gamma : \mathbb{R} \to \mathbb{R}^n$, $n \geq 2$, be a continuous curve passing through the origin, i.e. $\Gamma(0) = 0$. We define the Hilbert transform along Γ by the principal-valued integral

$$\mathcal{H}f(x) = \text{p.v.} \int_{-\infty}^{\infty} f(x - \Gamma(t)) \frac{dt}{t}, \quad \forall f \in C_0^\infty(\mathbb{R}^n). \tag{1}$$

It is interesting to determine for which curves Γ, and which indices p, one has the L^p-bound

$$\|\mathcal{H}f\|_p \leqslant c\|f\|_p; \tag{2}$$

for a survey of this problem's history through 1977 see [21]. More recent results can be found in a series of papers, see for example [6–8, 14–18] and [4]. Now, we are interesting to determine for which curves Γ, and which indices p, q, s, we have estimates of the form

$$\|\mathcal{H}f\|_{B^s_{p,q}(\mathbb{R}^n)} \leq c\|f\|_{B^s_{p,q}(\mathbb{R}^n)} \tag{3}$$

for $f \in B^s_{p,q}(\mathbb{R}^n)$, and $c < \infty$ depending only on Γ and p, not f ?

At first, note that a simple calculation shows that

$$\widehat{\mathcal{H}f}(\xi) = m(\xi).\widehat{f},$$

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where the "Fourier multiplier" m is the function

$$m(\xi) = \text{p.v.} \int_{-\infty}^{\infty} e^{-i\xi.\Gamma(t)} \frac{dt}{t}, \quad \xi \in \mathbb{R}^n.$$

Next, it is known that for proving the estimate (2) for $p = 2$, it suffices to show that $m(\xi)$ is a bounded function on $\mathbb{R}^n$ and to use the Van Der Corput Lemma and Plancherel's Theorem (see [30, p.197] and [18]).

When Γ is of finite type, i.e the set $\{\Gamma^{(k)}(0) : k \geq 1\}$ spans $\mathbb{R}^n$, we must consider the local version $\mathcal{H}_{loc}$ of the operator $\mathcal{H}$, where the integral defining $\mathcal{H}$ is restricted to $[-1, 1]$. In [21] it is shown that, in this case, $\mathcal{H}_{loc}$ is bounded on $L^p(\mathbb{R}^n)$ for every $p, 1 < p < \infty$. Thus, what can happen in the case when Γ is not of finite type? We restrict our attention to curves γ satisfying

$$\gamma \in C^2(]0, +\infty[), \text{ convex on } [0, +\infty[\text{ and } \gamma(0) = \gamma'(0) = 0; \tag{4}$$

and γ is either even or odd. The convexity hypothesis means that $[\gamma(c) - \gamma(b)]/(c - b) \geq [\gamma(b) - \gamma(a)]/(b - a)$ for $0 \leq a < b < c$. The following notions naturally arise

Definition 1. *(i) A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ belongs to $\mathcal{C}^1$ if there exists $\lambda, 1 < \lambda < \infty$, such that for each $t > 0$ the inequality $f(\lambda t) \geq 2f(t)$ holds. Such a function f is said to be doubling.*

(ii) A differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ belongs to $\mathcal{C}^2$ if there exists $\varepsilon_0 > 0$ such that for $t > 0$ the inequality $f'(t) \geq \varepsilon_0 f(t)/t$ holds. Such a function f is said to be infinitesimally doubling.

If f is nondecreasing on $]0, +\infty[$, then $f \in C^2$ implies $f \in C^1$.

We will also use the function h defined for $t > 0$ by $h(t) = t\gamma'(t) - \gamma(t)$. Because of γ is convex and $\gamma(0) = 0$ we get the following important inequality

$$t\gamma'(t) \geq \gamma(t) \quad \text{for all } t > 0.$$

In [5], it was proved that if γ is even and satisfies (4) for $p \in]1, +\infty[$, then $\mathcal{H}$ is L^p-bounded if and only if $\gamma' \in \mathcal{C}^1$. This is the case when γ is convex and even. In the odd case, the current situation is less satisfactory. In [15], it is shown that if γ is odd, and satisfies (4), then $\mathcal{H}$ is L^2-bounded if and only if $h \in \mathcal{C}^1$. This means that for each $p \in]1, +\infty[$ a necessary condition for $\mathcal{H}$ to be L^p-bounded is $h \in \mathcal{C}^1$. However, it was demonstrated in [3] that this condition is far from sufficient. It is shown that when γ is odd, satisfies (4), and if $h \in \mathcal{C}^2$, then $\mathcal{H}$ is L^p-bounded for all $p \in]1, \infty[$. In the same case, we have the L^p result for any $p \in]1, +\infty[$ if $\gamma' \in \mathcal{C}^1$ (see [5]).

For the case of polynomial curve Γ in $\mathbb{R}^n$ some of related known results are Theorems 2 and 3 (see below and [2,22]). Indeed, the subject of bounds on Hilbert transforms and singular integrals has a rich history and has been studied by many authors on different spaces such as Lebesgue, Sobolev Spaces, which are special cases of Besov spaces. For different states of the Hilbert transform we refer the reader to [2,11,13,23] and many references therein. In particular, and in connection with our work, we mention the work of U. Luther and M.G. Russo [13]. They have studied the Hilbert transform on a new weighted Besov spaces which touched our topic but did not approach exactly. Our method is also different from [13]. Besov spaces are the natural spaces in which many operators related to functional equations, many papers appeared on Besov spaces and some possible related applications, for example Lagrange interpolation in Besov spaces and Cauchy singular integral equations in Sobolev spaces (see [9, 10, 12]).

On some conditions we confirm that the Hilbert transform preserves the boundedness property on Besov $B^s_{p,q}(\mathbb{R}^n)$ spaces, for all $s \in \mathbb{R}$ and $p, q \in]1, \infty[$. In this paper, we will affirm this.

This paper is organized as follows. After giving some preliminaries and notations that will be needed throughout this paper, we recall the decomposition of Littlewood-Palley, the definition of Besov spaces, and their properties. In addition, we will recall some results concerning the $L^p(\mathbb{R}^n)$-boundedness of the Hilbert transform along curves, we mention the following results of [22] and [2], which would guarantee the $L^p(\mathbb{R}^n)$-boundedness for all $p, 1 < p < \infty$. In Section 3, we prove that the boundedness of the Hilbert transform along curves on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ can be obtained by its $L^p(\mathbb{R}^n)$-boundedness. Finally, we present a conclusion and discuss future research in Section 4. The main result of this paper is the following.

Theorem 1. *Let $s \in \mathbb{R}$, $p \in]1, +\infty[$, $q \in]1, +\infty[$. If $\mathcal{H}$ is bounded on $L^p(\mathbb{R}^n)$, then $\mathcal{H}$ is bounded on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$.*

2 PRELIMINARIES

In this section, we recall the basic definitions and notations that will be needed throughout this paper.

2.1 Notations

In this paper, $\mathbb{N} = \{1, 2, \cdots\}$ denotes the set of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. All considered spaces are defined on the Euclidean space $\mathbb{R}^n$, $x.y = x_1y_1 + \cdots + x_ny_n$ denotes the scalar product in $\mathbb{R}^n$. For each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \partial^\alpha f$ denotes the partial derivative $\dfrac{\partial^{|\alpha|} f}{\partial {x_1}^{\alpha_1} \cdots \partial {x_n}^{\alpha_n}}$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n$. $C_0^\infty(\mathbb{R}^n) = \mathcal{D}(\mathbb{R}^n)$ denotes the space of all infinitely differentiable and compactly supported functions in $\mathbb{R}^n$, $\mathcal{D}'(\mathbb{R}^n)$ is the topological dual of $\mathcal{D}(\mathbb{R}^n)$. $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space of all complex-valued, infinitely differentiable, and rapidly decreasing functions, and $\mathcal{S}'(\mathbb{R}^n)$ its topological dual, the space of tempered distributions. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ the Fourier transform defined on both $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ is given by $(\mathcal{F}f)(\varphi) = f(\mathcal{F}\varphi)$, where

$$(\mathcal{F}\varphi)(\xi) = \widehat{\varphi}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix.\xi} \varphi(x) dx, \qquad \xi \in \mathbb{R}^n, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

The mapping $\mathcal{F}$ is a bijection (in both cases) and its inverse is given by

$$(\mathcal{F}^{-1}\varphi)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix.\xi} \varphi(x) dx, \qquad \xi \in \mathbb{R}^n, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

The convolution $\varphi * \psi$ of two integrable functions φ, ψ is defined via the integral

$$(\varphi * \psi)(x) = \int_{\mathbb{R}^n} \varphi(x - y) \psi(y) dy.$$

The symbol $\hookrightarrow$ denotes that the natural injection is a continuous linear operator. For $1 \leq p \leq \infty$, we denote by p' the conjugate exponent of p such that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. The space $L^p(\mathbb{R}^n)$, $0 < p \leqslant \infty$, denotes the set of the measurable functions f such that

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty$$

with the usual modification if $p = \infty$. By l^q we denote the set of sequences $(a_k)_k$ such that $\|(a_k)\|_{l^q} = (\sum_{k=0}^{\infty} |a_k|^q)^{1/q} < \infty$. For p and q such that $0 < p \leq \infty$, $0 < q \leq \infty$, we put

$$\|\{f_k\}_k\|_{l^q(L^p)} = \left(\sum_{k=0}^{\infty} \|f_k(x)\|_p^q \right)^{\frac{1}{q}} < \infty, \qquad \|\{f_k\}_k\|_{L^p(l^q)} = \left\| \left(\sum_{k=0}^{\infty} |f_k(x)|^q \right)^{\frac{1}{q}} \right\|_p < \infty.$$

2.2 The Littlewood-Paley decomposition

We introduce the concept of a dyadic decomposition of unity, and we define the Besov spaces by using the Littlewood-Paley decomposition of tempered distributions.

Let φ be a smooth function in $\mathcal{S}(\mathbb{R}^n)$ which satisfies the conditions: $\varphi(x) = 1$ if $|x| \leq 1$, $\varphi(x) = 0$ if $|x| \geq \frac{3}{2}$, $0 \leq \varphi(x) \leq 1$.

We define

$$\varphi_0(x) = \varphi(x), \quad \varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x), \quad j = 1, 2, \ldots, \quad x \in \mathbb{R}^n,$$

and the following identity holds

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1, \quad \forall x \in \mathbb{R}^n.$$

The system $\{\varphi_j(x)\}_{j \in \mathbb{N}}$ form a dyadic resolution of unity in $\mathbb{R}^n$.

Putting

$$\Phi_0 = \mathcal{F}^{-1}\varphi \quad \text{and} \quad \Phi_j = \mathcal{F}^{-1}\varphi_j,$$

we obtain the Littlewood-Paley decomposition of f, i.e.

$$f = \sum_{j=0}^{\infty} \Phi_j * f$$

for every $f \in \mathcal{S}'(\mathbb{R}^n)$ (see [29]).

2.3 Besov spaces

Definition 2 ([29]). *Let the real numbers s, p, q such that $s \in \mathbb{R}$, $0 < p \leq \infty$, and $0 < q \leq \infty$. The Besov space $B^s_{p,q}(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} \equiv \left(\sum_{j=0}^{\infty} \left(2^{sj} \|\Phi_j * f\|_p \right)^q \right)^{1/q} < \infty$$

with the usual modification if $q = \infty$.

Note that $B^s_{p,q}(\mathbb{R}^n)$ is a Banach space if $p \geq 1, q \geq 1$, which is independent of the chosen system $\{\varphi_j\}_{j=0}^{j=\infty}$.

Proposition 1. *Let $s \in \mathbb{R}$, $p, q \in [1, +\infty]$. The following chain of continuous embeddings*

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n),$$

holds. Furthermore $B^s_{p,q}(\mathbb{R}^n)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ as a dense subspace if $\max(p,q) < \infty$.

The given above definition represents only one of a large collection of possibilities for introducing Besov spaces. For its basic properties we refer to [1, 19, 20, 24, 25].

Now we give the results of [2] and [22], which guarantee the $L^p(\mathbb{R}^n)$-boundedness of the Hilbert transform for all $p, 1 < p < \infty$.

Theorem 2 ([22])**.** *Let $\Gamma(t) = (P_1(t), \ldots, P_n(t))$, where $P_1, \ldots, P_n$ are real polynomials on $\mathbb{R}$. Then $\mathcal{H}$ is bounded on L^p for all $p, 1 < p < \infty$, with bound independent of the coefficients of $P_1, \ldots, P_n$.*

Theorem 3 ([2])**.** *Suppose that P is a real polynomial and γ is convex on $[0, \infty[$ twice differentiable, either even or odd, $\gamma(0) = 0$, and $\gamma'(0) \geq 0$.*

Let $\Gamma(t) = (t, P(\gamma(t)))$, $p \in]1, \infty[$, and either (1) $P'(0)$ is zero, or (2) $P'(0)$ is nonzero and $\gamma' \in \mathcal{C}^1$, then

$$\|\mathcal{H}f\|_p \leq c\|f\|_p.$$

Moreover the constant c depends only on p, γ and the degree of P.

Remark 1. *(1) By taking $\gamma(t) = t$, we recover a form of Theorem 2, it is shown in [22] that for all $p, 1 < p < \infty$, $L^p(\mathbb{R}^n)$-boundedness of $\mathcal{H}$ is obtained. Also taking $P(s) = s$, it is shown in [5] that if γ is odd, satisfies* (4) *and $\gamma' \in \mathcal{C}^1$, then the L^p-boundedness of $\mathcal{H}$ for all $p \in]1, \infty[$ is obtained.*

(2) Some examples of nonconvex curves were studied in [28], and later these were generalized somewhat through a technical theorem in [27]. Although the class of these curves obtained from theorem in [2].

3 PROOF OF THEOREM 1.

The proof of Theorem 1 needs the following lemma.

Lemma 1 ([6])**.** *For all functions f in $S(\mathbb{R}^n)$ we have*

$$\widehat{\mathcal{H}f}(\xi) = m(\xi).\widehat{f}, \tag{5}$$

where the "Fourier multiplier" m is the function

$$m(\xi) = p.v. \int_{-\infty}^{\infty} e^{-i\xi.\Gamma(t)} \frac{dt}{t}, \quad \xi \in \mathbb{R}^n.$$

Proof. By the Fubini theorem we have

$$\widehat{\mathcal{H}f}(\xi) = \text{p.v.} \int_{-\infty}^{\infty} \frac{1}{t} \left\{ \int_{\mathbb{R}^n} e^{(-i\xi.x)} f(x - \Gamma(t)) dx \right\} dt,$$

By changing the variable $u = x - \Gamma(t)$, we obtain

$$\int_{\mathbb{R}^n} f(u) e^{(-i\xi.(u+\Gamma(t)))} du = \widehat{f}(\xi) e^{(-i\xi.\Gamma(t))}.$$

□

Hence we have the following result. By applying the inverse of the Fourier transform for (5), see for instance [26, p.40], we can define the Hilbert transform as a convolution operator by the formula

$$\mathcal{H}f(x) = K * f(x), \tag{6}$$

where

$$K(\xi) = \mathcal{F}^{-1} m(\xi)$$

(see also [8] and [22, p.25]).

Now, by Theorem 2 or Theorem 3, for any p, q, such that $1 < p, q < \infty$, and any $\{f_j\}_j$ in $l^q(L^p)$, we have

$$\left(\sum_{j=0}^{\infty} \|\mathcal{H}f_j\|_p^q \right)^{1/q} \leq c \left(\sum_{j=0}^{\infty} \|f_j\|_p^q \right)^{1/q}, \tag{7}$$

where c is independent of $\{f_j\}$.

It follows from formula (6) that

$$\begin{aligned} \|\mathcal{H}f\|_{B^s_{p,q}(\mathbb{R}^n)} &:= \left(\sum_{j=0}^{\infty} (2^{sj} \|\Phi_j * \mathcal{H}f\|_p)^q \right)^{1/q} \\ &= \left(\sum_{j=0}^{\infty} 2^{jsq} \|\Phi_j * \mathcal{H}f\|_p^q \right)^{1/q} = \left(\sum_{j=0}^{\infty} 2^{jsq} \|\Phi_j * K * f\|_p^q \right)^{1/q}. \end{aligned}$$

By the convolution properties and formula (6), we have

$$\|\mathcal{H}f\|_{B^s_{p,q}(\mathbb{R}^n)} = \left(\sum_{j=0}^{\infty} \|K * (2^{js}\Phi_j * f)\|_p^q \right)^{1/q} = \left(\sum_{j=0}^{\infty} \|\mathcal{H}(2^{js}\Phi_j * f)\|_p^q \right)^{1/q}.$$

Then, by formula (7), there exists a constant c such that

$$\|\mathcal{H}f\|_{B^s_{p,q}(\mathbb{R}^n)} \leq c \left(\sum_{j=0}^{\infty} \|2^{js}\Phi_j * f\|_p^q \right)^{1/q}.$$

Simple calculations show that

$$\|\mathcal{H}f\|_{B^s_{p,q}(\mathbb{R}^n)} \leq c \left(\sum_{j=0}^{\infty} (2^{sj} \|\Phi_j * f\|_p)^q \right)^{1/q} = c \left(\sum_{j=0}^{\infty} 2^{jsq} \|\Phi_j * f\|_p^q \right)^{1/q} = c \|f\|_{B^s_{p,q}(\mathbb{R}^n)}.$$

4 CONCLUSION

In this work, using the Littlewood-Paley decomposition we have proved that the boundedness of the Hilbert transform along curves on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ can be obtained by its $L^p(\mathbb{R}^n)$-boundedness, where $s \in \mathbb{R}$, and $p, q \in]1, +\infty[$. In future work, we will prove the boundedness on other functional spaces.

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Перетворення Гільберта вздовж кривих має велике значення в гармонічному аналізі. Відомо, що його обмеженість на $L^p(\mathbb{R}^n)$ широко досліджувалось різними авторами в різних контекстах і автори отримували позитивні результати для деяких або всіх $p, 1 < p < \infty$. Теорія Літлвуда-Пелі надає альтернативні методи вивчення сингулярних інтегралів. Перетворення Гільберта взовж кривих, як класичний приклад сингулярного інтеграла, призвело до появи сучасної теорії операторів Кальдерона-Зіґмунда, які здебільшого вивчені на лебегових просторах L^p. У цій статті ми використовуємо теорію Літлвуда-Пелі щоб доведести, що обмеженість перетворення Гільберта вздовж кривої Γ на просторах Бєсова $B^s_{p,q}(\mathbb{R}^n)$ може бути отримана з його L^p-обмеженості, де $s \in \mathbb{R}, p, q \in]1, +\infty[$, і $\Gamma(t)$ — відповідна крива в $\mathbb{R}^n$. Відомо, що простори Бєсова $B^s_{p,q}(\mathbb{R}^n)$ вкладені в простори $L^p(\mathbb{R}^n)$ для $s > 0$ (тобто $B^s_{p,q}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n), s > 0$). Отже, наш результат можна розглядати як продовження відомих результатів на простори Бєсова $B^s_{p,q}(\mathbb{R}^n)$ для довільних значень s в $\mathbb{R}$.

Ключові слова і фрази: перетворення Гільберта, розклад Літлвуда-Пелі, простори Бєсова.