



A note on triadjoints of biorthomorphisms

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Let X be a vector lattice and a bilinear mapping $T : X \times X \rightarrow X$ be a biorthomorphism. The triadjoint of T is a biorthomorphism and the space of all biorthomorphisms on bidual of X is an f -algebra. If $X = A$ is an f -algebra, then the set of all orthomorphisms on the order bidual of A is an order ideal of the set of all biorthomorphisms on order bidual of A . In particular case, it is a band of it.

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1 Introduction

Let X be an Archimedean vector lattice (Riesz space). A subset E of X is called solid if $|x| \leq |y|$ and $y \in E$ imply $x \in E$. A solid linear subspace of X is said to be an order ideal. A space X is called Dedekind complete if every subset of X , which is bounded from above, has a supremum and X is called σ -Dedekind complete if every sequence of X , which is bounded from above, has a supremum. A subset E of X is called order bounded if E is contained in an order interval. An order closed ideal is called a band. We say that a linear operator $T : X \rightarrow Y$ between vector lattices is order bounded if it maps order bounded subsets into order bounded subsets. A linear operator T from a vector lattice X into another vector lattice Y is called a lattice homomorphism if $T(x \vee y) = Tx \vee Ty$ for all $x, y \in X$. A linear operator $T : X \rightarrow X$ on a vector lattice is called an orthomorphism if T is both order bounded and band preserving. We denote the set of all orthomorphisms by $Orth(X)$. We say that an order-bounded bilinear mapping $T : X \times X \rightarrow X$ is called a biorthomorphism if T is both band preserving and order bounded in each variable. We denote the set of all biorthomorphisms by $Orth(X, X)$. The set of all biorthomorphisms $Orth(X, X)$ is vector lattice by lattice operations defined by $(T \vee S)(x, y) = T(x, y) \vee S(x, y)$ and $(T \wedge S)(x, y) = T(x, y) \wedge S(x, y)$ for all $(x, y) \in X^+ \times X^+$, where X^+ stands for a positive cone (see [4]). By X' we denote the order dual of a vector lattice, that is, the set of all order bounded linear functionals on X . By X'' we denote the second order dual or order bidual of X .

In this paper, we show that the set of all biorthomorphisms $Orth(X'', X'')$ on the bidual X'' of an Archimedean vector lattice X is an f -algebra. Also, we show that the set of all orthomorphisms $Orth(A'')$ on the order bidual of an f -algebra A is an order ideal in the set of all biorthomorphisms $Orth(A'', A'')$ on the order bidual of A . For the unexplained notion and terminology we refer to the classical book [1].

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2 f -algebra of biorthomorphisms

Definition 1. A lattice ordered algebra A is called an f -algebra if $a \wedge b = 0$, $c \in A^+$ imply $ac \wedge b = ca \wedge b = 0$. An f -algebra A is said to be semi-prime if the only nilpotent element of A is zero. If A has unit element, then it is semiprime.

We establish the following bilinear mappings by using Arens product (see [2]):

$$\begin{aligned} A' \times A \ni (f, a) &\mapsto f.a(b) := f(ab) \in A', & A'' \times A' \ni (F, f) &\mapsto F.f(a) := F(f.a) \in A', \\ A'' \times A'' \ni (F, G) &\mapsto F.G(f) = F(G.f) \in A''. \end{aligned} \quad (1)$$

Definition 2. Let X be an Archimedean vector lattice. A mapping $T''' : X'' \times X'' \rightarrow X''$ given by the Arens product in (1) is called the triadjoint biorthomorphism of T if $T : X \times X \rightarrow X$ is a biorthomorphism.

Theorem 1. Let X be an Archimedean vector lattice and $T \in \text{Orth}(X, X)$. Then the triadjoint T''' of T belongs to $\text{Orth}(X'', X'')$.

Proof. Proof of this result is given by Arens multiplication (see [7]). $\square$

Theorem 2 ([3]). Let X be an Archimedean vector lattice. If $p \in X^+$, then $\text{Orth}(X, X)$ is an Archimedean f -algebra with respect to the multiplication defined by $(T \star_p S)(x, y) = T(x, S(p, y))$ for all $T, S \in \text{Orth}(X, X)$, $x, y \in X$.

Theorem 3. Let X be an Archimedean vector lattice with separating order dual X' . If $p \in X^+$, then $\text{Orth}(X'', X'')$ is a Dedekind complete f -algebra with respect to the multiplication defined by $(T \star_p S)(E, F) = T(E, S(p, F))$ for all $T, S \in \text{Orth}(X'', X'')$, $E, F \in X''$.

Proof. Let $p \in X^+$. By using canonical embedding $i : X \rightarrow X''$, we get that p belongs to X''^+ , since embedding is one to one positive lattice homomorphism. Let $T, S \in \text{Orth}(X'', X'')$. The map defined by $(T \star_p S)(E, F) = T(E, S(p, F))$ for all $T, S \in \text{Orth}(X'', X'')$, $E, F \in X''$ is bilinear map. It is separately orthomorphism on X'' . So, $T \star_p S \in \text{Orth}(X'', X'')$. Hence, $\text{Orth}(X'', X'')$ is a lattice ordered algebra with respect to the multiplication $\star_p$. By using the symmetry of biorthomorphisms and commutativity of $\star_p$, we get that $\text{Orth}(X'', X'')$ is an f -algebra. Since X'' is Dedekind complete, $\text{Orth}(X'', X'')$ is also a Dedekind complete f -algebra. $\square$

Let A be an Archimedean semi-prime f -algebra. Then the second order dual A'' of A is a Dedekind complete f -algebra by using Arens multiplication (see [5]). There are three cases to be the semiprime of A'' : a) A has unit; b) A has weak approximate unit and satisfies Stone condition; c) A is σ -bounded approximate unit.

Now, we give the definitions of b) and c).

Definition 3. Let A be a lattice ordered algebra. An element x in the positive cone A^+ is called totally bounded if $x^2 \leq x$. The lattice ordered algebra A is said to have a σ -bounded approximate unit if for all $f \in A'$ the set $\{f(x) : x \text{ is totally bounded}\}$ is bounded in $\mathbb{R}$.

Definition 4. We say that the semi-prime f -algebra A satisfies the Stone condition whenever $x \wedge I \in A$ for all $x \in A^+$.

Definition 5. The upward directed net a_α in A^+ is said to be a weak approximate unit if $f(b) = \sup_\alpha f(a_\alpha b)$ for all $b \in A^+$ and all $f \in (A')^+$.

Let $m : \text{Orth}(A'') \rightarrow \text{Orth}(A'', A'')$ be a mapping defined by $m(\pi)(F, G) = \pi(F).G$ for all $F, G \in A'', \pi \in \text{Orth}(A'')$. Note, that m is one to one lattice homomorphism. By $(A')'_n$ we denote the band of all continuous order bidual of A . If A has unit, then we have the equality $A'' = (A')'_n$. The following theorem is related to the mapping m .

Theorem 4. *Let A be an Archimedean f -algebra satisfying one of the conditions $a), b), c)$. Then $\text{Orth}(A'')$ is an order ideal in $\text{Orth}(A'', A'')$.*

Proof. For the conditions $a), b)$, it is known that $(A')'_n$ is a semi-prime f -algebra (see [5]). This implies that A'' is semi-prime. Since A'' is a Dedekind complete semi-prime f -algebra, by using [3, 4, 8], it is proved. For the condition $c)$, we note that A'' is semi-prime by [6]. Again by using the results of the papers [3, 4, 8], it is an order ideal in $\text{Orth}(A'', A'')$. $\square$

If we suppose that the mapping m is lattice and algebra isometry, then $\text{Orth}(A'')$ is a band in $\text{Orth}(A'', A'')$.

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Нехай X є векторною ґраткою, а білінійне відображення $T : X \times X \rightarrow X$ є біорторморфізмом. Потрібне спряження відображення T також є біорторморфізмом, а простір усіх біорторморфізмів на другому спряженому просторі X є f -алгеброю. Якщо $X = A$ є f -алгеброю, то множина всіх ортоморфізмів на порядковому другому спряженому просторі A є порядковим ідеалом множини всіх біорторморфізмів на порядковому другому спряженому просторі A . Зокрема, вона є його смугою.

Ключові слова і фрази: ортоморфізм, біорторморфізм, потрібне спряжене відображення, f -алгебра.