ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2020, **12** (2), 499–503
doi:10.15330/cmp.12.2.499-503

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2020, Т.12, №2, С.499–503

A NOTE ON A GENERALIZATION OF INJECTIVE MODULES

TÜRKMEN B.N., TÜRKMEN E.

As a proper generalization of injective modules in term of supplements, we say that a module M has the property (ME) if, whenever $M \subseteq N$, M has a supplement K in N, where K has a mutual supplement in N. In this study, we obtain that (1) a semisimple R-module M has the property (E) if and only if M has the property (ME); (2) a semisimple left R-module M over a commutative Noetherian ring R has the property (ME) if and only if M is algebraically compact if and only if almost all isotopic components of M are zero; (3) a module M over a von Neumann regular ring has the property (ME) if and only if it is injective; (4) a principal ideal domain R is left perfect if every free left R-module has the property (ME)

Key words and phrases: supplement, mutual supplement, module with the property (ME), left perfect ring.

Faculty of Science and Literature, Amasya University, 16160, Amasya, Turkey
E-mail: burcunisancie@hotmail.com (Türkmen B.N.), ergulturkmen@hotmail.com (Türkmen E.)

Introduction

In this paper, all rings are associative with identity and all modules are unital left modules. Let R be such a ring and let M be an R-module. The notation $K \subseteq M$ ($K \subset M$) means that K is a (proper) submodule of M. A non-zero submodule $K \subseteq M$ is called *essential* in M, written as $K \trianglelefteq M$, if $K \cap L \neq 0$ for every non-zero submodule of M. Dually, a proper submodule $S \subset M$ is called *small* (in M), denoted by $S \ll M$, if $M \neq S + K$ for every proper submodule K of M. A module M is called *hollow* if every submodule of M is small in M. By $Rad(M)$, namely *radical*, we will denote the sum of all small submodules of M. Equivalently, $Rad(M)$ is the intersection of all maximal submodules of M [9]. Following [9], a module M is called *supplemented* if every submodule of M has a supplement in M. A submodule $K \subseteq M$ is a supplement of a submodule L in M if and only if $M = L + K$ and $L \cap K \ll K$.

In [1], a supplement submodule X of M is then defined when X is a supplement of some submodule of M. Every direct summand of a module M is a supplement submodule of M, and supplemented modules are a generalization of semisimple modules. In addition, every factor module of a supplemented module is again supplemented. A module M is called $\oplus$-supplemented if every submodule N of M has a supplement that is a direct summand of M [5]. Clearly every $\oplus$-supplemented module is supplemented, but a supplemented module need not be $\oplus$-supplemented in general (see [5, Lemma A.4 (2)]). It is shown in [5, Proposition A.7 and Proposition A.8] that if R is a Dedekind domain, every supplemented R-module is $\oplus$-supplemented. Hollow modules are $\oplus$-supplemented.

УДК 512.552
2010 *Mathematics Subject Classification:* 16D10,16D50,16N80.

Let M be a module. A module N is said to be *extension* of M provided $M \subseteq N$. As a generalization of injective modules, since every direct summand is a supplement, Zöschinger defined in [10] a module M with *the property* (E) if it has a supplement in every extension. He studied the various properties of a module M with the property (E) in the same paper. For a module N, two submodules K and $K^{'}$ of N are called *mutual supplements* if $N = K + K^{'}$, $K \cap K^{'} \ll K$ and $K \cap K^{'} \ll K^{'}$. We consider the following condition for a module M:

(ME) in any extension N of M, M has a supplement K in N and there exists a submodule $K^{'}$ of N such that K and $K^{'}$ are mutual supplements in N.

Now we have these implications on modules:

$$\text{injective} \Rightarrow \text{module with the property } (ME) \Rightarrow \text{module with the property } (E).$$

Some examples are given to show that these inclusions are proper. In the section 2, we obtain some elementary facts about the property (ME). We prove that a semisimple R-module M has the property (E) if and only if M has the property (ME). We also prove that M has the property (ME) if and only if M is algebraically compact if and only if almost all isotopic components of M are zero for a semisimple left R-module M over a commutative Noetherian ring R. We obtain that a module M over a von Neumann regular ring has the property (ME) if and only if it is injective. We show that any factor module of a module with the property (ME) doesn't have the property (ME). Finally, we also show that R is left perfect if every free left R-module has the property (ME) over a principal ideal domain R.

1 Modules with the property (ME)

Proposition 1. *Let M be a semisimple R-module. Then, the following statements are equivalent.*

(1) M has the property (E).

(2) M has the property (ME).

Proof. $(1) \Longrightarrow (2)$ Let N be any extension of M. By (1), we have $N = M + K$ and $M \cap K \ll K$ for some submodule $K \subseteq M$. Since M is a semisimple module, there exists a submodule X of M such that $M = (M \cap K) \oplus X$. So $(M \cap K) \cap X = K \cap X = 0$. Therefore $N = M + K = [(M \cap K) \oplus X] + K = K \oplus X$. This means that K and X are mutual supplements in N. Thus M has the property (ME).

$(2) \Longrightarrow (1)$ is trivial. □

Let R be a ring and M be a left R-module. Take two sets I and J, and for every $i \in I$ and $j \in J$, an element r_{i_j} of R such that, for every $i \in I$, only finitely many r_{i_j} are non-zero. Furthermore, take an element m_i of M for every $i \in I$. These data describe a system of linear equations in M:

$$\sum_{j \in J} r_{i_j} x_j = m_i \text{ for every } i \in I.$$

The goal is to decide whether this system has a solution, i.e. whether there exist elements x_j of M for every $j \in J$ such that all the equations of the system are simultaneously satisfied (note that we do not require that only finitely many of the x_j are non-zero here). Now consider such a system of linear equations, and assume that any subsystem consisting of only finitely many equations is solvable (the solutions to the various subsystems may be different). If every such "finitely-solvable" system is itself solvable, then the module M is called *algebraically compact*. For example, every injective module is algebraically compact.

Corollary 1. *Let R be a commutative Noetherian ring. Then, the following three statements are equivalent for a semisimple left R-module M.*

(1) M has the property (ME).

(2) M is algebraically compact.

(3) Almost all isotopic components of M are zero.

Proof. It follows from Proposition 1 and [10, Proposition 1.6]. □

It is clear that every injective module has the property (ME), but the following example shows that a module with the property (ME) need not be injective. Firstly, we need the following crucial lemma.

Lemma 1. *Every simple module has the property (ME).*

Proof. Let M be a simple module and N be any extension of M. Since M is simple, then $M \ll N$ or $M \oplus K = N$ for a submodule K of N. In the first case, N is a supplement of M in N such that N and 0 are mutual supplements in N. In the second case, K is a supplement of M in N such that K and M are mutual supplements in N. So, in each case M has the property (ME). □

Recall from [2] that a ring R is *von Neumann regular* if every element $a \in R$ can be written in the form axa, for some $x \in R$. More formally, a ring R is regular in the sense of von Neumann if and only if the following equivalent conditions hold:

(1) $\frac{R}{I}$ is a projective R-module for every finitely generated ideal I,

(2) every finitely generated left ideal is generated by an idempotent,

(3) every finitely generated left ideal is a direct summand of R.

Example 1 ([3, 6.1]). (1) *Let V be a countably infinite-dimensional left vector space over a division ring S. Let $R = End({}_S V)$ be the ring of left linear operators on V. Then R is a von Neumann regular ring. Claim that the simple left R-module V is not injective. Assume the contrary that ${}_R V$ is injective. Consider a basis $\{v_i | i \in \mathbb{N}\}$ of V. For each $i \in \mathbb{N}$, let us define $f_i \in R$ by $f_i(v_i) = v_i$ and $f_i(v_i) = 0$ for $i \neq j$. Set $A = \sum_i Rf_i$. Then A is a left ideal of R. Consider a left R-homomorphism $\varphi : A \longrightarrow_R V$ defined by $\varphi(\sum_i r_i f_i) = \sum_i r_i v_i$, where $r_i \in R$ is zero for all but finitely many i. Since ${}_R V$ is injective, there exists $v \in V$ such that $\varphi(f_i) = f_i v$ for every $i \in \mathbb{N}$. This gives $v_i = f_i v$ for every $i \in \mathbb{N}$. Now if $v = d_1 v_1 + d_2 v_2 + \cdots + d_n v_n$, then any $i \in \mathbb{N} \backslash \{1, 2, \ldots, n\}$, we have $f_i v = 0$, a contradiction. This shows ${}_R V$ is not injective. Thus R is not a left V-ring as the simple left R-module V is not injective. By Lemma 1, the left R-module V has the property (ME).*

(2) *Consider the simple $\mathbb{Z}$-module $\frac{\mathbb{Z}}{p\mathbb{Z}}$, where p is prime. By Lemma 1, M has the property (ME). On the other hand, it is not injective.*

Recall from [9, 41.13] that an R-module M is *π-projective* (or *co-continious*) if for every two submodules U, V of M with $U + V = M$ there exists $f \in End_R(M)$ with $Im(f) \subset U$ and $Im(1 - f) \subset V$.

Lemma 2. *Let M be a module with the property (ME) and N be an extension of M such that N is π-projective or $Rad(N) = 0$. Then, M is a direct summand of N.*

Proof. Let N be any extension of M. Since M has the property (ME), there exist submodules K and $K^{'}$ of N such that $N = M + K$, $M \cap K \ll K$ and K, $K^{'}$ are mutual supplements in N. It follows from [9, 41.14(2)] that $N = M \oplus K$.

If $Rad(N) = 0$, then $M \cap K \subseteq Rad(N) = 0$. We have $N = M \oplus K$. □

A ring R is said to be a *left V-ring* if every simple left R-module is injective. It is well known that R is left V-ring if and only if $Rad(M) = 0$ for every left R-module M.

Proposition 2. *For a module M over a left V-ring R, the module M is injective if and only if M has the property (ME).*

Proof. ($\Longrightarrow$) It is clear.

($\Longleftarrow$) It follows from Lemma 2. □

Corollary 2. *Let R be a commutative von Neumann regular ring. Then, an R-module M has the property (ME) if and only if it is injective.*

Proof. Since R is a commutative von Neumann regular ring, it is a left V-ring. Hence, the proof follows from Proposition 2. □

Recall that a ring R is *left hereditary* if every factor module of an injective left R-module is injective [8].

Example 2 ([10])**.** *Let $R = \prod_{i \in I} F_i$ be a ring, where each F_i is field for an infinite index set I. Then R is a commutative von Neumann regular ring. Since R is not Noetherian, it is not semisimple and so, by the Theorem of Osofsky [6], there is a cyclic R-module (which is clearly a factor module of R) which is not injective and hence doesn't have the property (ME) by Corollary 2.*

Theorem 1. *If every free left R-module has the property (ME) over a principal ideal domain R, then R is left perfect.*

Proof. Let M be any free R-module. By the hypothesis and [7, Theorem 9.8], every submodule of M has the property (ME). There exist submodules K and K' of M such that $M = U + K$, $U \cap K \ll K$, and K, K' mutual supplements in M for any submodule U of M, $M = K \oplus K^{'}$. So M is $\oplus$-supplemented. It follows from [4, Corollary 2.11] that R is left perfect. □

REFERENCES

[1] Clark J., Lomp C., Vanaja N., Wisbauer R. Lifting Modules. Supplements and Projectivity in Module Theory. Frontiers in Mathematics, Birkhäuser-Basel, 2006. doi:10.1007/3-7643-7573-6

[2] Jacobson N. Basic Algebra II. W. H. Freeman and Company. Yale University, New York, 1989.

[3] Jain S.K., Srivastava A.K., Tuganbaev A.A. Cyclic Modules and The Structure of Rings. Oxford Mathematical Monograps, Oxford Science Publications, 2012.

[4] Keskin D., Smith P.F., Xue W. *Rings whose modules are* $\oplus$*-supplemented*. J. Algebra 1999, **218**, 470–487.

[5] Mohamed S.H., Müller B.J. Continuous and Discrete Modules. London Math. Soc. Lecture Note Ser. 147, Cambridge University Press, Cambridge, 1990. doi:10.1017/CBO9780511600692

[6] Osofsky B.L. *Rings all of whose finitely generated modules are injective*. Pacific J. Math. 1964, **14**, 645–650.

[7] Rotman J.J. Advanced Modern Algebra. Prentice Hall, 2003.

[8] Sharpe D.W., Vamos P. Injective Modules. Cambridge Uiversity Press, Cambridge, 1972.

[9] Wisbauer R. Foundations of Modules and Rings. Gordon and Breach, 1991.

[10] Zöschinger H. *Moduln, die in jeder erweiterung ein komplement haben*. Math. Scand. 1974, **35**, 267–287.

Received 06.02.2020

Revised 23.04.2020

Туркмен Б.Н., Туркмен Е. *Замітка про узагальнення ін'єктивних модулів* // Карпатські матем. публ. — 2020. — Т.12, №2. — С. 499–503.

Як належне узагальнення ін'єктивних модулів у термінах доповнень скажемо, що модуль M має властивість (ME), якщо як тільки $M \subseteq N$, то M має доповнення K в N, де K має взаємне доповнення в N. У цьому дослідженні ми отримуємо, що (1) напівпростий R-модуль M має властивість (E) тоді і тільки тоді, коли M має властивість (ME); (2) напівпростий лівий R-модуль M над комутативним нетеровим кільцем R має властивість (ME) тоді і тільки тоді, коли M алгебраїчно компактний та тоді і тільки тоді, коли майже всі ізотопні компоненти M є нульовими; (3) модуль M над регулярним кільцем фон Неймана має властивість (ME) тоді і тільки тоді, коли він ін'єктивний; (4) основна область ідеалу R є досконалою зліва, якщо кожен вільний лівий R-модуль має властивість (ME)

Ключові слова і фрази: доповнення, взаємне доповнення, модуль з властивістю (ME), ліве досконале кільце.