ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2021, **13** (1), 58–67
doi:10.15330/cmp.13.1.58-67

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2021, Т.13, №1, С.58–67

Some fixed point theorems on α-β-G-complete G-metric spaces

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In this manuscript, we initiate the concept of rectangular α-G-admissible mappings with respect to β and we consider related type contractions in the setting of G-metric spaces. We provide some fixed point results. Also, some examples are given to support the obtained results.

Key words and phrases: G-metric space, fixed point, almost perfect function, rectangular α-admissible mapping.

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1 Introduction

Banach contraction principle (BCP) [6] is a well-known result in fixed point theory. It is one of the main tools for both the theoretical and computational aspects in mathematical sciences. This theorem has witnessed numerous generalizations and prolongations in the literature due to its simplicity and constructive approach. Recently, the idea of an almost perfect function, which was considered as a generalization of an altering distance function, has been commenced by Shatanawi W. et al. [23]. The concept of α-admissibility was initiated by Samet B. et al. [19]. Later, it is generalized to triangular α-admissibility by Karapinar E. et al. [8]. Recently, Shatanawi W. and Abodayeh K. [22] introduced the notion of triangular α-admissibility with respect to another function β. In this paper, we present new fixed point results via the idea of rectangular α-admissible mappings.

2 Preliminaries

Definition 1 ([11]). *Let Ω be a nonempty set and $G : \Omega^3 := \Omega \times \Omega \times \Omega \to [0, +\infty)$ be a function satisfying the following:*

1) $G(s, t, u) = 0$ *if* $s = t = u$;
2) $G(s, s, t) > 0$ *for all* $s, t \in \Omega$ *with* $s \neq t$;
3) $G(s, s, t) \leq G(s, t, u)$ *for all* $s, t, u \in \Omega$ *with* $t \neq u$;
4) $G(s, t, u) = G(t, u, s) = G(u, s, t) = \cdots$ *(symmetry in all three variables);*
5) $G(s, t, u) \leq G(s, v, v) + G(v, t, u)$ *for all* $s, t, u, v \in \Omega$ *(rectangular inequality).*

УДК 515.126.4, 517.988.523
2020 *Mathematics Subject Classification:* 54H25, 47H10.

Then the function G is called a generalized metric or, more specifically, a G-metric on Ω and the pair (Ω, G) is a G-metric space.

Example 1 ([11]). *If Ω is a nonempty subset of $\mathbb{R}$, then the function $G : \Omega^3 \to [0, +\infty)$ given by $G(s, t, u) = |s - t| + |t - u| + |u - s|$ for all $s, t, u \in \Omega$ is a G-metric on Ω.*

Definition 2 ([16]). *Let (Ω, G) be a G-metric space and $\{y_n\}$ be a sequence in Ω. An element $y \in \Omega$ is said to be the limit of $\{y_n\}$, if $\lim\limits_{n,m\to+\infty} G(y, y_n, y_m) = 0$ and we say that the sequence $\{y_n\}$ is G-convergent to y. In this case, for any $\varepsilon > 0$ there exists a positive integer N such that $G(y, y_n, y_m) < \varepsilon$ for all $n, m \geq N$.*

Definition 3 ([11]). *Let (Ω, G) be a G-metric space. The sequence $\{y_n\}$ is said to be G-Cauchy, if for every $\varepsilon > 0$ there exists a positive integer N such that $G(y_n, y_m, y_l) < \varepsilon$ for all $n, m, l \geq N$.*

Definition 4 ([12]). *A G-metric space (Ω, G) is said to be G-complete (or complete G-metric space), if every G-Cauchy sequence in (Ω, G) is G-convergent in (Ω, G).*

Lemma 1 ([11]). *Let (Ω, G) be a G-metric space. Then $G(\varsigma, \iota, \iota) \leq 2G(\varsigma, \iota, \nu)$, for all $\varsigma, \iota, \nu \in \Omega$.*

For other related properties and fixed point results on G-metric spaces, see [1,2,4,5,7,8,10, 13–15,17,18,20,21].

Definition 5 ([9]). *A function $\zeta : [0, +\infty) \to [0, +\infty)$ is called an altering distance if:*

1) $\zeta(s) = 0$ if and only if $s = 0$;

2) ζ is nondecreasing and continuous.

Definition 6 ([22]). *A function $\zeta : [0, +\infty) \to [0, +\infty)$ is called an almost perfect function, if it is nondecreasing and satisfies the following:*

1) $\zeta(s) = 0$ if and only if $s = 0$;

2) if $\{s_n\}$ is a sequence in $[0, +\infty)$ such that $\zeta(s_n) \to 0$, then $\{s_n\}$ converges to 0.

3 Main Results

Here, we introduce the notions of α-β-G-complete G-metric space, α-β-G-continuous functions and rectangular α-G-admissible mappings with respect to a given function β. We will establish related fixed point results in α-β-G-complete G-metric spaces, which are not necessarily G-complete.

Definition 7. *Let (Ω, G) be a G-metric space. The sequence $\{y_n\}$ in the set Ω is said to be α-β-G-Cauchy, if for every $\varepsilon > 0$ there exists a positive integer N such that $G(y_n, y_m, y_l) < \varepsilon$ for all $n, m, l \geq N$ with $\alpha(y_n, y_{n+1}, y_{n+1}) \geq \beta(y_n, y_{n+1}, y_{n+1})$.*

Definition 8. *Let (Ω, G) be a G-metric space and $\alpha, \beta : \Omega^3 \to [0, +\infty)$ be two functions. Then Ω is said to be an α-β-G-complete G-metric space if and only if every Cauchy sequence $\{y_n\}$ in Ω with $\alpha(y_n, y_{n+1}, y_{n+1}) \geq \beta(y_n, y_{n+1}, y_{n+1})$ for all $n \in \mathbb{N}$ converges in Ω.*

In the above definition, if we take $\beta(y_n, y_{n+1}, y_{n+1}) = 1$, then it is called α-G-complete G-metric space.

Example 2. *Let $\Omega = (0, +\infty)$ and $G(s,t,u) = |s-t| + |t-u| + |u-s|$ be a G-metric on Ω. Let A be a closed subset of Ω. Define $\alpha, \beta : \Omega^3 \to [0, +\infty)$ by*

$$\alpha(\varsigma, \iota, \nu) = \begin{cases} (\varsigma + \iota + \nu)^2, & \varsigma, \iota, \nu \in A, \\ 0, & \text{otherwise}, \end{cases}$$

and $\beta(\varsigma, \iota, \nu) = 2\varsigma\iota\nu$. Clearly, (Ω, G) is not a G-complete G-metric space, but (Ω, G) is an α-β-G-complete G-metric space. Indeed, if $\{y_n\}$ is a G-Cauchy sequence in Ω such that $\alpha(y_n, y_{n+1}, y_{n+1}) \geq \beta(y_n, y_{n+1}, y_{n+1})$ for all $n \in \mathbb{N}$, then $y_n \in A$ for all $n \in \mathbb{N}$. Now, since A is closed, then there exists $y^* \in A$ such that $y_n \to y^*$ as $n \to +\infty$.

Definition 9. *Let (Ω, G) be a G-metric space and $\alpha, \beta : \Omega^3 \to [0, +\infty)$ be two functions. A mapping $R : \Omega \to \Omega$ is said to be α-β-G-continuous mapping, if each sequence $\{y_n\}$ on Ω with $y_n \to y$ as $n \to +\infty$ and $\alpha(y_n, y_{n+1}, y_{n+1}) \geq \beta(y_n, y_{n+1}, y_{n+1})$ for all $n \in \mathbb{N}$, implies $Ry_n \to Ry$ as $n \to +\infty$.*

In the above definition, if we take $\beta(y_n, y_{n+1}, y_{n+1}) = 1$, then it is called α-G-continuous mapping.

The following example is in support of Definition 9.

Example 3. *Let $\Omega = (0, +\infty)$ and $G(s,t,u) = |s-t| + |t-u| + |u-s|$ be a G-metric on Ω. Assume that $R : \Omega \to \Omega$ and $\alpha, \beta : \Omega^3 \to [0, +\infty)$ are defined by*

$$R(\varsigma) = \begin{cases} (\varsigma)^5, & \varsigma \in [0,1], \\ \sin \pi\varsigma + 2, & \varsigma \in (1, +\infty), \end{cases}$$

$$\alpha(\varsigma, \iota, \nu) = \begin{cases} (\varsigma^2 + \iota^2 + \nu^2 + 1), & \varsigma, \iota, \nu \in [0,1], \\ 0, & \text{otherwise}, \end{cases}$$

and $\beta(\varsigma, \iota, \nu) = \varsigma^2$. Clearly, R is not G-continuous, but R is α-β-G-continuous on (Ω, G). Indeed, if $y_n \to y$ as $n \to +\infty$ and $\alpha(y_n, y_{n+1}, y_{n+1}) \geq \beta(y_n, y_{n+1}, y_{n+1})$, then $y_n \in [0,1]$ for all $n \in \mathbb{N}$, and so $\lim_{n \to +\infty} Ry_n = \lim_{n \to +\infty} y_n^5 = y^5 = Ry$.

In 2013, Alghamdi M.A. and Karapinar E. [3] introduced the concept of α-G-admissible mappings.

Definition 10 ([3]). *Let (Ω, G) be a G-metric space, R be a self-mapping on Ω and $\alpha : \Omega^3 \to [0, +\infty)$ be a given function. We say that R is an α-G-admissible mapping, if for all $\varsigma, \iota, \nu \in \Omega$ with $\alpha(\varsigma, \iota, \nu) \geq 1$ the inequality $\alpha(R\varsigma, R\iota, R\nu) \geq 1$ holds.*

As a generalization of Definition 10, we state the following.

Definition 11. *Let R be a self-mapping on a G-metric space (Ω, G). Let $\alpha, \beta : \Omega^3 \to [0, +\infty)$ be two functions. We say that R is α-G-admissible with respect to β, if for all $\varsigma, \iota, \nu \in \Omega$ with $\alpha(\varsigma, \iota, \nu) \geq \beta(\varsigma, \iota, \nu)$ the inequality $\alpha(R\varsigma, R\iota, R\nu) \geq \beta(R\varsigma, R\iota, R\nu)$ holds.*

Still in same direction, the notion of rectangular α-G-admissibility is as follows.

Definition 12. *Let $R : \Omega \to \Omega$ and α be a nonnegative function on Ω^3. Such R is called a rectangular α-G-admissible mapping, if it satisfies the following conditions:*

1) R is α-G-admissible;

2) $\alpha(\varsigma, s, s) \geq 1$ and $\alpha(s, \iota, \nu) \geq 1$ imply that $\alpha(\varsigma, \iota, \nu) \geq 1$.

Definition 13. *Let α, β be two nonnegative functions on Ω^3. A self-mapping R on Ω is said to be rectangular α-G-admissible with respect to β, if the following conditions are satisfied:*

1) if $\varsigma, \iota, \nu \in \Omega$ with $\alpha(\varsigma, \iota, \nu) \geq \beta(\varsigma, \iota, \nu)$, then $\alpha(R\varsigma, R\iota, R\nu) \geq \beta(R\varsigma, R\iota, R\nu)$;

2) if $\varsigma, \iota, \nu \in \Omega$ with $\alpha(\varsigma, t, t) \geq \beta(\varsigma, t, t)$ and $\alpha(t, \iota, \nu) \geq \beta(t, \iota, \nu)$, then $\alpha(\varsigma, \iota, \nu) \geq \beta(\varsigma, \iota, \nu)$.

The following example supports Definition 13.

Example 4. *Let $\Omega = [0, +\infty)$. Consider the mapping $R : \Omega \to \Omega$ defined by $R\varsigma = \varsigma^2$. Also, $\alpha, \beta : \Omega^3 \to [0, +\infty)$ are defined by*

$$\alpha(\varsigma, \iota, \nu) = \begin{cases} \varsigma + \iota + \nu, & \iota, \nu \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\beta(\varsigma, \iota, \nu) = \begin{cases} \varsigma^2 + \iota + \nu, & \iota, \nu \geq 1, \\ 1, & \text{otherwise.} \end{cases}$$

Then R is rectangular α-G-admissible with respect to β.

Proof. To prove the condition 1) of Definition 13, take $\varsigma, \iota, \nu \in \Omega$ such that $\alpha(\varsigma, \iota, \nu) \geq \beta(\varsigma, \iota, \nu)$. Then $\iota, \nu \geq 1$ and hence $\varsigma + \iota + \nu \geq \varsigma^2 + \iota + \nu$. So, we conclude that $\varsigma \in [0, 1]$. Therefore, $\varsigma^2 + \iota^2 + \nu^2 \geq \varsigma^4 + \iota^2 + \nu^2$ and so $\alpha(R\varsigma, R\iota, R\nu) \geq \beta(R\varsigma, R\iota, R\nu)$.

To verify condition 2) of Definition 13, take $\varsigma, \iota, \nu \in \Omega$ such that $\alpha(\varsigma, t, t) \geq \beta(\varsigma, t, t)$ and $\alpha(t, \iota, \nu) \geq \beta(t, \iota, \nu)$. Then, we conclude that $\varsigma \in [0, 1]$, $t = 1$, $\iota, \nu \geq 1$. Therefore, $\varsigma + \iota + \nu \geq \varsigma^2 + \iota + \nu$ and hence $\alpha(\varsigma, \iota, \nu) \geq \beta(\varsigma, \iota, \nu)$. □

Definition 14. *Let (Ω, G) be a G-metric space, and α, β be two nonnegative functions on Ω^3. We say that the self-mapping R on Ω is an α-β-G-contraction, if there exist $k \in [0, 1)$ and an almost perfect function ζ such that, for all $\varsigma, \iota, \nu \in \Omega$ with $\alpha(\varsigma, \iota, \nu) \geq \beta(\varsigma, \iota, \nu)$ we have*

$$\begin{aligned} \zeta(G(R\varsigma, R\iota, R\nu)) \leq \max \Big\{ & k\zeta(G(\varsigma, \iota, \nu)), k\zeta(G(\varsigma, R\varsigma, R\varsigma)), k\zeta(G(\iota, R\iota, R\iota)), \\ & k\zeta(G(\nu, R\nu, R\nu)), k\zeta\Big(\frac{1}{4}\big[G(R\varsigma, \iota, \nu) + G(\varsigma, R\iota, R\nu)\big]\Big) \Big\}. \end{aligned}$$

Our essential new result is as follows.

Theorem 1. *Let (Ω, G) be a G-metric space, α and β be two nonnegative functions on Ω^3, and $R : \Omega \to \Omega$ be a mapping. Assume that:*

(i) (Ω, G) is α-β-G-complete;

(ii) there exists $\varsigma_0 \in \Omega$ such that $\alpha(\varsigma_0, R\varsigma_0, R\varsigma_0) \geq \beta(\varsigma_0, R\varsigma_0, R\varsigma_0)$;

(iii) R is an α-β-G-contraction;

(iv) R is rectangular α-G-admissible with respect to β;

(v) R is α-β-G-continuous.

Then R has a fixed point.

Proof. From *(ii)*, we take $\varsigma_0 \in \Omega$ with $\alpha(\varsigma_0, R\varsigma_0, R\varsigma_0) \geq \beta(\varsigma_0, R\varsigma_0, R\varsigma_0)$. Consider $\varsigma_{n+1} = R\varsigma_n$ for all $n \in \mathbb{N}$. According to the α-G-admissibility property of the mapping R with respect to β, we have $\alpha(\varsigma_1, \varsigma_2, \varsigma_2) = \alpha(R\varsigma_0, R\varsigma_1, R\varsigma_1) \geq \beta(R\varsigma_0, R\varsigma_1, R\varsigma_1) = \beta(\varsigma_1, \varsigma_2, \varsigma_2)$. Similarly, $\alpha(\varsigma_2, \varsigma_3, \varsigma_3) = \alpha(R\varsigma_1, R\varsigma_2, R\varsigma_2) \geq \beta(R\varsigma_1, R\varsigma_2, R\varsigma_2) = \beta(\varsigma_2, \varsigma_3, \varsigma_3)$. By induction, one writes $\alpha(\varsigma_n, \varsigma_{n+1}, \varsigma_{n+1}) \geq \beta(\varsigma_n, \varsigma_{n+1}, \varsigma_{n+1})$ for all $n \in \mathbb{N}$. Also, since R is rectangular α-G-admissible with respect to β, we have $\alpha(\varsigma_n, \varsigma_m, \varsigma_m) \geq \beta(\varsigma_n, \varsigma_m, \varsigma_m)$ for all $n, m \in \mathbb{N}$ with $n < m$.

If there exists $m \in \mathbb{N}$ such that $\varsigma_m = \varsigma_{m+1}$, then $\varsigma_m = R\varsigma_m$ and so ς_m is a fixed point of R. Thus, let $\varsigma_n \neq \varsigma_{n+1}$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
\zeta(G(\varsigma_{n+1}, \varsigma_{n+2}, \varsigma_{n+2})) &= \zeta(G(R\varsigma_n, R\varsigma_{n+1}, R\varsigma_{n+1})) \\
&\leq \max\Big\{k\zeta(G(\varsigma_n, \varsigma_{n+1}, \varsigma_{n+1})), k\zeta(G(\varsigma_n, R\varsigma_n, R\varsigma_n)), \\
&\qquad k\zeta(G(\varsigma_{n+1}, R\varsigma_{n+1}, R\varsigma_{n+1})), k\zeta(G(\varsigma_{n+1}, R\varsigma_{n+1}, R\varsigma_{n+1})), \\
&\qquad k\zeta(\frac{1}{4}[G(R\varsigma_n, \varsigma_{n+1}, \varsigma_{n+1}) + G(\varsigma_n, R\varsigma_{n+1}, R\varsigma_{n+1})])\Big\} \\
&= \max\Big\{k\zeta(G(\varsigma_n, \varsigma_{n+1}, \varsigma_{n+1})), k\zeta(G(\varsigma_n, \varsigma_{n+1}, \varsigma_{n+1})), \\
&\qquad k\zeta(G(\varsigma_{n+1}, \varsigma_{n+2}, \varsigma_{n+2})), k\zeta(G(\varsigma_{n+1}, \varsigma_{n+2}, \varsigma_{n+2})), \\
&\qquad k\zeta(\frac{1}{4}[G(\varsigma_{n+1}, \varsigma_{n+1}, \varsigma_{n+1}) + G(\varsigma_n, \varsigma_{n+2}, \varsigma_{n+2})])\Big\} \\
&= \max\Big\{k\zeta(G(\varsigma_n, \varsigma_{n+1}, \varsigma_{n+1})), k\zeta(G(\varsigma_{n+1}, \varsigma_{n+2}, \varsigma_{n+2})), \\
&\qquad k\zeta(\frac{G(\varsigma_n, \varsigma_{n+2}, \varsigma_{n+2})}{4})\Big\}.
\end{aligned} \tag{1}$$

Note that

$$\frac{G(\varsigma_n, \varsigma_{n+2}, \varsigma_{n+2})}{4} \leq \frac{G(\varsigma_n, \varsigma_{n+2}, \varsigma_{n+2})}{2} \leq \max\{G(\varsigma_n, \varsigma_{n+1}, \varsigma_{n+1}), G(\varsigma_{n+1}, \varsigma_{n+2}, \varsigma_{n+2})\}.$$

From (1), we get

$$\zeta(G(\varsigma_{n+1}, \varsigma_{n+2}, \varsigma_{n+2})) \leq \max\{k\zeta(G(\varsigma_n, \varsigma_{n+1}, \varsigma_{n+1})), k\zeta(G(\varsigma_{n+1}, \varsigma_{n+2}, \varsigma_{n+2}))\}.$$

If $\max\{k\zeta(G(\varsigma_n, \varsigma_{n+1}, \varsigma_{n+1})), k\zeta(G(\varsigma_{n+1}, \varsigma_{n+2}, \varsigma_{n+2}))\} = k\zeta(G(\varsigma_{n+1}, \varsigma_{n+2}, \varsigma_{n+2}))$ for some $n \in \mathbb{N}$, then we reach a contradiction with respect to $\varsigma_n = \varsigma_{n+1}$. So,

$$\max\{k\zeta(G(\varsigma_n, \varsigma_{n+1}, \varsigma_{n+1})), k\zeta(G(\varsigma_{n+1}, \varsigma_{n+2}, \varsigma_{n+2}))\} = k\zeta(G(\varsigma_n, \varsigma_{n+1}, \varsigma_{n+1})).$$

Hence,

$$\zeta(G(\varsigma_{n+1},\varsigma_{n+2},\varsigma_{n+2})) \le k\zeta(G(\varsigma_n,\varsigma_{n+1},\varsigma_{n+1})).$$

Repeating this manner n-times, we have

$$\begin{aligned}\zeta(G(\varsigma_n,\varsigma_{n+1},\varsigma_{n+1})) &\le k\zeta(G(\varsigma_{n-1},\varsigma_n,\varsigma_n))\\ &\le k^2\zeta(G(\varsigma_{n-2},\varsigma_{n-1},\varsigma_{n-1})) \le \cdots \le k^n\zeta(G(\varsigma_0,\varsigma_1,\varsigma_1)).\end{aligned} \tag{2}$$

Letting $n \to +\infty$ in (2), we get

$$\lim_{n\to+\infty}\zeta(G(\varsigma_n,\varsigma_{n+1},\varsigma_{n+1})) = 0.$$

Since ζ is an almost perfect function, we get

$$\lim_{n\to+\infty} G(\varsigma_n,\varsigma_{n+1},\varsigma_{n+1}) = 0.$$

Now, we shall show that $\{\varsigma_n\}$ is a G-Cauchy sequence in X. Take $n, m \in \mathbb{N}$ with $m > n$. Since $\alpha(\varsigma_n,\varsigma_m,\varsigma_m) \ge \beta(\varsigma_n,\varsigma_m,\varsigma_m)$, we have

$$\begin{aligned}\zeta(G(\varsigma_n,\varsigma_m,\varsigma_m)) =& \zeta(G(R\varsigma_{n-1},R\varsigma_{m-1},R\varsigma_{m-1}))\\ \le& \max\Big\{k\zeta(G(\varsigma_{n-1},\varsigma_{m-1},\varsigma_{m-1})), k\zeta(G(\varsigma_{n-1},R\varsigma_{n-1},R\varsigma_{n-1})),\\ & k\zeta(G(\varsigma_{m-1},R\varsigma_{m-1},R\varsigma_{m-1})), k\zeta(G(\varsigma_{m-1},R\varsigma_{m-1},R\varsigma_{m-1})),\\ & k\zeta(\frac{1}{4}\big[G(R\varsigma_{n-1},\varsigma_{m-1},\varsigma_{m-1}) + G(\varsigma_{n-1},R\varsigma_{m-1},R\varsigma_{m-1})\big])\Big\}\\ =&\max\Big\{k\zeta(G(\varsigma_{n-1},\varsigma_{m-1},\varsigma_{m-1})), k\zeta(G(\varsigma_{n-1},\varsigma_n,\varsigma_n)), k\zeta(G(\varsigma_{m-1},\varsigma_m,\varsigma_m)),\\ & k\zeta(\frac{1}{4}\big[G(\varsigma_n,\varsigma_{m-1},\varsigma_{m-1}) + G(\varsigma_{n-1},\varsigma_m,\varsigma_m)\big])\Big\}.\end{aligned} \tag{3}$$

Using the rectangular inequality, we obtain

$$G(\varsigma_{n-1},\varsigma_m,\varsigma_m) \le G(\varsigma_{n-1},\varsigma_n,\varsigma_n) + G(\varsigma_n,\varsigma_m,\varsigma_m),$$

and

$$G(\varsigma_n,\varsigma_{m-1},\varsigma_{m-1}) \le G(\varsigma_n,\varsigma_m,\varsigma_m) + G(\varsigma_m,\varsigma_{m-1},\varsigma_{m-1}).$$

Since $G(\varsigma_m,\varsigma_{m-1},\varsigma_{m-1}) \le 2G(\varsigma_{m-1},\varsigma_m,\varsigma_m)$, we get

$$\begin{aligned}&\frac{1}{4}\big[G(\varsigma_n,\varsigma_{m-1},\varsigma_{m-1}) + G(\varsigma_{n-1},\varsigma_m,\varsigma_m)\big]\\ &\qquad\le \max\big\{G(\varsigma_{n-1},\varsigma_n,\varsigma_n), G(\varsigma_n,\varsigma_m,\varsigma_m), G(\varsigma_{m-1},\varsigma_m,\varsigma_m)\big\}.\end{aligned}$$

Thus, (3) becomes

$$\begin{aligned}\zeta(G(\varsigma_n,\varsigma_m,\varsigma_m)) \le \max\big\{&k\zeta(G(\varsigma_{n-1},\varsigma_{m-1},\varsigma_{m-1})), k\zeta(G(\varsigma_{n-1},\varsigma_n,\varsigma_n)),\\ &k\zeta(G(\varsigma_{m-1},\varsigma_m,\varsigma_m)), k\zeta(G(\varsigma_n,\varsigma_m,\varsigma_m))\big\}.\end{aligned} \tag{4}$$

If

$$\begin{aligned}\max\big\{&k\zeta(G(\varsigma_{n-1},\varsigma_{m-1},\varsigma_{m-1})), k\zeta(G(\varsigma_{n-1},\varsigma_n,\varsigma_n)),\\ &k\zeta(G(\varsigma_{m-1},\varsigma_m,\varsigma_m)), k\zeta(G(\varsigma_n,\varsigma_m,\varsigma_m))\big\} = k\zeta(G(\varsigma_n,\varsigma_m,\varsigma_m)),\end{aligned}$$

then by (4), we have $\zeta(G(\varsigma_n,\varsigma_m,\varsigma_m)) \le k\zeta(G(\varsigma_n,\varsigma_m,\varsigma_m))$, which is a contradiction. So,

$$\max\left\{k\zeta(G(\varsigma_{n-1},\varsigma_{m-1},\varsigma_{m-1})),k\zeta(G(\varsigma_{n-1},\varsigma_n,\varsigma_n)),\right.$$
$$\left.k\zeta(G(\varsigma_{m-1},\varsigma_m,p\varsigma_m)),k\zeta(G(\varsigma_n,\varsigma_m,\varsigma_m))\right\} \neq k\zeta(G(\varsigma_n,\varsigma_m,\varsigma_m)).$$

Therefore, (4) becomes

$$\begin{aligned}&\zeta(G(\varsigma_n,\varsigma_m,\varsigma_m))\\ &\qquad\le \max\left\{k\zeta(G(\varsigma_{n-1},\varsigma_{m-1},\varsigma_{m-1})),k\zeta(G(\varsigma_{n-1},\varsigma_n,\varsigma_n)),k\zeta(G(\varsigma_{m-1},\varsigma_m,\varsigma_m))\right\}.\end{aligned} \tag{5}$$

Using (2) and (5), we have

$$\begin{aligned}&\zeta(G(\varsigma_n,\varsigma_m,\varsigma_m))\\ &\qquad\le \max\{k\zeta(G(\varsigma_{n-1},\varsigma_{n+s},\varsigma_{n+s})),k^n\zeta(G(\varsigma_0,\varsigma_1,\varsigma_1)),k^m\zeta(G(\varsigma_0,\varsigma_1,\varsigma_1))\}\\ &\qquad\le \max\{k^2\zeta(G(\varsigma_{n-2},\varsigma_{n-1+s},\varsigma_{n-1+s})),k^n\zeta(G(\varsigma_0,\varsigma_1,\varsigma_1)),k^m\zeta(G(\varsigma_0,\varsigma_1,\varsigma_1))\}\\ &\qquad\vdots\\ &\qquad\le \max\{k^n\zeta(G(\varsigma_0,\varsigma_{s+1},\varsigma_{s+1})),k^n\zeta(G(\varsigma_0,\varsigma_1,\varsigma_1)),k^m\zeta(G(\varsigma_0,\varsigma_1,\varsigma_1))\}.\end{aligned} \tag{6}$$

On letting $n,m \to +\infty$ in (6), we get

$$\lim_{n,m\to+\infty}\zeta(G(\varsigma_n,\varsigma_m,\varsigma_m)) = 0.$$

Since ζ is an almost perfect function, we get

$$\lim_{n,m\to+\infty} G(\varsigma_n,\varsigma_m,\varsigma_m) = 0.$$

This implies that the sequence $\{\varsigma_n\}$ is G-Cauchy in Ω. The α-β-G-completeness of the G-metric space (Ω, G) implies that there exists $\varsigma \in \Omega$ such that $\varsigma_n \to \varsigma$. The α-β-G-continuity of the mapping R implies that $R\varsigma_n \to R\varsigma$. Since the limit is unique, we have $R\varsigma = \varsigma$ and hence ς is a fixed point of R. □

Example 5. *Let $\Omega = [0,+\infty)$. Define $G : \Omega^3 \to [0,+\infty)$ by*

$$G(\varsigma,\iota,\nu) = \mid \varsigma-\iota \mid + \mid \iota-\nu \mid + \mid \nu-\varsigma \mid .$$

Let $R : \Omega \to \Omega$ be defined by $R(\varsigma) = \frac{1}{4}\sin^2\varsigma$. Consider the function ζ defined on $[0,+\infty)$ as

$$\zeta(v) = \begin{cases}\frac{v}{1+v}, & 0 \le v \le \frac{1}{2},\\ 1, & \frac{1}{2} < v.\end{cases}$$

Consider two nonnegative functions α, β on Ω^3 given as

$$\alpha(\varsigma,\iota,\nu) = \begin{cases} e^{-(\varsigma^2+\iota^2+\nu^2)}, & \varsigma,\iota,\nu \in [0,1],\\ 0, & otherwise,\end{cases}$$

and

$$\beta(\varsigma,\iota,\nu) = \begin{cases} e^{-(\varsigma+\iota+\nu)}, & \varsigma,\iota,\nu \in [0,1],\\ 1, & otherwise.\end{cases}$$

Then

(i) ζ is an almost perfect function;

(ii) R is α-β-G-continuous;

(iii) there exists $\varsigma_0 \in X$ such that $\alpha(\varsigma_0, R\varsigma_0, R\varsigma_0) \geq \beta(\varsigma_0, R\varsigma_0, R\varsigma_0)$;

(iv) R is a rectangular α-G-admissible mapping with respect to β;

(v) (Ω, G) is an α-β-G-complete G-metric space;

(vi) for all $\varsigma, \iota, \nu \in \Omega$ with $\alpha(\varsigma, \iota, \nu) \geq \beta(\varsigma, \iota, \nu)$, we have

$$\zeta(G(R\varsigma, R\iota, R\nu)) \leq \max\big\{k\zeta(G(\varsigma,\iota,\nu)), k\zeta(G(\varsigma, R\varsigma, R\varsigma)), k\zeta(G(\iota, R\iota, R\iota)), k\zeta(G(\nu, R\nu, R\nu)), k\zeta\big(\frac{1}{4}\big[G(R\varsigma,\iota,\nu) + G(\varsigma, R\iota, R\nu)\big]\big)\big\}.$$

Proof. We can easily prove *(i), (ii), (iii)*. To prove *(iv)*, let $\varsigma, \iota, \nu \in \Omega$ with $\alpha(\varsigma, \iota, \nu) \geq \beta(\varsigma, \iota, \nu)$. Then $\varsigma, \iota, \nu \in [0,1]$. Since $R(\varsigma) = \frac{1}{4}\sin^2\varsigma \in [0,1]$, $R(\iota) = \frac{1}{4}\sin^2\iota \in [0,1]$ and $R(\nu) = \frac{1}{4}\sin^2\nu \in [0,1]$, one writes

$$\begin{aligned}\alpha(R\varsigma, R\iota, R\nu) &= \alpha(\frac{1}{4}\sin^2\varsigma, \frac{1}{4}\sin^2\iota, \frac{1}{4}\sin^2\nu) = e^{-(\frac{1}{16}\sin^4\varsigma + \frac{1}{16}\sin^4\iota + \frac{1}{16}\sin^4\nu)}\\ &\geq e^{-(\frac{1}{4}\sin^2\varsigma + \frac{1}{4}\sin^2\iota + \frac{1}{4}\sin^2\nu)} = \beta(\frac{1}{4}\sin^2\varsigma, \frac{1}{4}\sin^2\iota, \frac{1}{4}\sin^2\nu) = \beta(R\varsigma, R\iota, R\nu).\end{aligned}$$

Also, given $\varsigma, \iota, \nu \in \Omega$ such that $\alpha(\varsigma, t, t) \geq \beta(\varsigma, t, t)$ and $\alpha(t, \iota, \nu) \geq \beta(t, \iota, \nu)$, then $\varsigma, \iota, \nu, t \in [0,1]$. So, $\varsigma + \iota + \nu \geq \varsigma^2 + \iota^2 + \nu^2$. Hence, $e^{-(\varsigma^2+\iota^2+\nu^2)} \geq e^{-(\varsigma+\iota+\nu)}$. Therefore, $\alpha(\varsigma, \iota, \nu) \geq \beta(\varsigma, \iota, \nu)$. Thus, R is rectangular α-admissible with respect to β.

To prove *(v)*, let $\{\varsigma_n\}$ be a G-Cauchy sequence in Ω such that $\alpha(\varsigma_n, \varsigma_{n+1}, \varsigma_{n+1}) \geq \beta(\varsigma_n, \varsigma_{n+1}, \varsigma_{n+1})$. Then $\varsigma_n \in [0,1]$ for all $n \in \mathbb{N}$. Since $[0,1]$ is closed, we conclude that $\{\varsigma_n\}$ is G-convergent in $[0,1]$ and hence (Ω, G) is an α-β-G-complete G-metric space.

To prove *(vi)*, given $\varsigma, \iota, \nu \in \Omega$ such that $\alpha(\varsigma, \iota, \nu) \geq \beta(\varsigma, \iota, \nu)$. Then $\varsigma, \iota, \nu \in [0,1]$.

So, $\frac{|\varsigma-\iota|+|\iota-\nu|+|\nu-\varsigma|}{1+|\varsigma-\iota|+|\iota-\nu|+|\nu-\varsigma|} \leq 1$. Since $\frac{1}{4} \mid \sin^2\varsigma - \sin^2\iota \mid \leq \frac{1}{2}$, we have

$$\begin{aligned}\zeta(G(R\varsigma, R\iota, R\nu)) &= \zeta\Big(G(\frac{1}{4}\sin^2\varsigma, \frac{1}{4}\sin^2\iota, \frac{1}{4}\sin^2\nu)\Big)\\ &= \zeta\Big(\mid \frac{1}{4}\sin^2\varsigma - \frac{1}{4}\sin^2\iota \mid + \mid \frac{1}{4}\sin^2\iota - \frac{1}{4}\sin^2\nu \mid + \mid \frac{1}{4}\sin^2\nu - \frac{1}{4}\sin^2\varsigma \mid \Big)\\ &= \frac{\mid \frac{1}{4}\sin^2\varsigma - \frac{1}{4}\sin^2\iota \mid + \mid \frac{1}{4}\sin^2\iota - \frac{1}{4}\sin^2\nu \mid + \mid \frac{1}{4}\sin^2\nu - \frac{1}{4}\sin^2\varsigma \mid}{1+ \mid \frac{1}{4}\sin^2\varsigma - \frac{1}{4}\sin^2\iota \mid + \mid \frac{1}{4}\sin^2\iota - \frac{1}{4}\sin^2\nu \mid + \mid \frac{1}{4}\sin^2\nu - \frac{1}{4}\sin^2\varsigma \mid}\\ &\leq \frac{\mid \frac{1}{4}\varsigma - \frac{1}{4}\iota \mid + \mid \frac{1}{4}\iota - \frac{1}{4}\nu \mid + \mid \frac{1}{4}\nu - \frac{1}{4}\varsigma \mid}{1+ \mid \frac{1}{4}\varsigma - \frac{1}{4}\iota \mid + \mid \frac{1}{4}\iota - \frac{1}{4}\nu \mid + \mid \frac{1}{4}\nu - \frac{1}{4}\varsigma \mid}\\ &= \frac{\mid \varsigma - \iota \mid + \mid \iota - \nu \mid + \mid \nu - \varsigma \mid}{4+ \mid \varsigma - \iota \mid + \mid \iota - \nu \mid + \mid \nu - \varsigma \mid}\\ &= \frac{\mid \varsigma - \iota \mid}{4+ \mid \varsigma - \iota \mid + \mid \iota - \nu \mid + \mid \nu - \varsigma \mid} + \frac{\mid \iota - \nu \mid}{4+ \mid \varsigma - \iota \mid + \mid \iota - \nu \mid + \mid \nu - \varsigma \mid}\\ &\quad + \frac{\mid \nu - \varsigma \mid}{4+ \mid \varsigma - \iota \mid + \mid \iota - \nu \mid + \mid \nu - \varsigma \mid},\end{aligned}$$

and so

$$\begin{aligned}
&\zeta(G(R\varsigma, R\iota, R\nu))\\
&\quad\le \frac{|\varsigma-\iota|}{\frac{4}{3}+\frac{4}{3}\Big[|\varsigma-\iota|+|\iota-\nu|+|\nu-\varsigma|\Big]}+\frac{|\iota-\nu|}{\frac{4}{3}+\frac{4}{3}\Big[|\varsigma-\iota|+|\iota-\nu|+|\nu-\varsigma|\Big]}\\
&\qquad+\frac{|\nu-\varsigma|}{\frac{4}{3}+\frac{4}{3}\Big[|\varsigma-\iota|+|\iota-\nu|+|\nu-\varsigma|\Big]}\\
&\quad=\frac{3}{4}\Bigg(\frac{|\varsigma-\iota|}{1+|\varsigma-\iota|+|\iota-\nu|+|\nu-\varsigma|}+\frac{|\iota-\nu|}{1+|\varsigma-\iota|+|\iota-\nu|+|\nu-\varsigma|}\\
&\qquad+\frac{|\nu-\varsigma|}{1+|\varsigma-\iota|+|\iota-\nu|+|\nu-\varsigma|}\Bigg)\\
&\quad=\frac{3}{4}\left(\frac{|\varsigma-\iota|+|\iota-\nu|+|\nu-\varsigma|}{1+|\varsigma-\iota|+|\iota-\nu|+|\nu-\varsigma|}\right)\\
&\quad=\frac{3}{4}\zeta(G(\varsigma,\iota,\nu))\\
&\quad\le \max\Big\{\frac{3}{4}\zeta(G(\varsigma,\iota,\nu)),\frac{3}{4}\zeta(G(\varsigma,R\varsigma,R\varsigma)),\frac{3}{4}\zeta(G(\iota,R\iota,R\iota)),\\
&\qquad\qquad\frac{3}{4}\zeta(G(\nu,R\nu,R\nu)),\frac{3}{4}\zeta\big(\frac{1}{4}\big[G(R\varsigma,\iota,\nu)+G(\varsigma,R\iota,R\nu)\big]\big)\Big\}.
\end{aligned}$$

Therefore, all conditions of Theorem 1 are satisfied. Hence, R has a fixed point which is 0. □

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Received 19.02.2020

Чері В.С., Редді Ґ.С.М., Ішік Г., Айді Г., Чері Д.С. *Деякі теореми про нерухому точку на α-β-G-повних G-метричних просторах* // Карпатські матем. публ. — 2021. — Т.13, №1. — С. 58–67.

У цій статті ми вводимо поняття прямокутних α-G-допустимих відображень по відношенню до функції β і розглядаємо відповідного типу стиски у контексті G-метричних просторів. Ми доводимо деякі результати про нерухому точку. Також наведено деякі приклади для ілюстрації отриманих результатів.

Ключові слова і фрази: G-метричний простір, нерухома точка, майже ідеальна функція, прямокутне α-допустиме відображення.