EXTREME AND EXPOSED SYMMETRIC BILINEAR FORMS ON THE SPACE $L_2(\ell_\infty^n)$

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We classify extreme points and exposed points of the unit ball of the space of bilinear symmetric forms on the real Banach space of bilinear symmetric forms on $\ell_\infty^n$. It is shown that for this case, the set of extreme points is equal to the set of exposed points.

**Key words and phrases:** extreme point, exposed point.

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**INTRODUCTION**

Throughout the paper, we let $n \in \mathbb{N}, n \geq 2$. We write $B_E$ for the closed unit ball of a real Banach space $E$ and the dual space of $E$ is denoted by $E^*$. An element $x \in B_E$ is called an extreme point of $B_E$ if $y, z \in B_E$ with $x = \frac{1}{2}(y + z)$ implies $x = y = z$. We denote by $\text{ext} B_E$ the set of all extreme points of $B_E$. An element $x \in B_E$ is called an exposed point of $B_E$ if there is a functional $f \in E^*$ such that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of $B_E$ is an extreme point. We denote by $\text{exp} B_E$ the set of exposed points of $B_E$.

A mapping $P : E \to \mathbb{R}$ is a continuous $n$-homogeneous polynomial if there exists a continuous $n$-linear form $T$ on the product $E \times \cdots \times E$ such that $P(x) = T(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(nE)$ the Banach space of all continuous $n$-homogeneous polynomials from $E$ into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. We denote by $\mathcal{L}(nE)$ the Banach space of all continuous $n$-linear forms on $E$ endowed with the norm $\|T\| = \sup_{\|x_i\|=1} |T(x_1, \ldots, x_n)|$. $\mathcal{L}_n(E)$ denotes the closed subspace of all continuous symmetric $n$-linear forms on $E$. For more details about the theory of polynomials and multilinear mappings on Banach spaces, we refer to [8].

Let us introduce the history of classification problems of extreme and exposed points of the unit ball of continuous $n$-homogeneous polynomials on a Banach space. We let $l_p^n = \mathbb{R}^n$ for every $1 \leq p \leq \infty$ equipped with the $l_p$-norm. Choi et al. ([3, 4]) initiated and classified $\text{ext} B_{l_p^n}$ for $p = 1, 2$. Choi and Kim [7] classified $\text{ext} B_{l_2^n}$ for $p = 1, 2, \infty$. Later, B. Grecu [12] classified the sets $\text{ext} B_{l_2^n}$ for $1 < p < 2$ or $2 < p < \infty$. Kim et al. [37] showed that if $E$ is a separable real Hilbert space with $\dim(E) \geq 2$, then, $\text{ext} B_{(2E)}$ is equal to $\text{exp} B_{(2E)}$.

Kim [16] classified $\text{exp} B_{l_2^n}$ for every $1 \leq p \leq \infty$. Kim [18] characterized $\text{ext} B_{l_2^n}$, where $d_+(1, w)^2$ denotes $\mathbb{R}^2$ equipped with the octagonal norm

$$\|(x, y)\|_w = \max \left\{ |x|, |y|, \frac{|x| + |y|}{1 + \bar{w}} \right\}$$

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for $0 < w < 1$. Kim [25] classified $\exp B_{\mathcal{P}(2d_1,w^2)}$ and showed that $\exp B_{\mathcal{P}(2d_1,w^2)}$ is a proper subset of $\text{ext } B_{\mathcal{P}(2d_1,w^2)}$. Recently, Kim ([30, 33]) classified $\text{ext } B_{\mathcal{P}(2\mathbb{R}^2_{\beta(\frac{1}{2})})}$ and $\text{ext } B_{\mathcal{P}(2\mathbb{R}^2_{\beta(\frac{1}{2})})}$, where $\mathbb{R}^2_{\beta(\frac{1}{2})}$ denotes $\mathbb{R}^2$ endowed with a hexagonal norm

$$
\|(x,y)\|_{\beta(\frac{1}{2})} = \max \left\{ |y|, |x| + \frac{1}{2}|y| \right\}.
$$

Parallel to the classification problems of $\text{ext } B_{\mathcal{P}(\Phi_E)}$ and $\exp B_{\mathcal{P}(\Phi_E)}$, it seems to be very natural to study the classification problems of extreme and exposed points of the unit ball of continuous (symmetric) multilinear forms on a Banach space. Kim [17] initiated and classified $\text{ext } B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$ and $\exp B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$. Kim ([19, 21, 22, 24]) classified $\text{ext } B_{\mathcal{L}_s(2d_1,w^2)}$, $\text{ext } B_{\mathcal{L}_s(2d_2,w^2)}$, and $\text{ext } B_{\mathcal{L}_s(2d_3,w^2)}$. Kim ([28, 29]) also classified $\text{ext } B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$ and $\text{ext } B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$. It was shown that $\text{ext } B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$ and $\text{ext } B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$ are equal to $\exp B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$ and $\exp B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$ respectively. Kim [32] classified $\text{ext } B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$ and $\text{ext } B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$. Kim [34] characterized $\text{ext } B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$, $\text{ext } B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$, $\exp B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$ and showed that $\exp B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$ and $\exp B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$ are equal to $\exp B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$ and $\exp B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$ respectively. Recently, Kim [35] characterized for $m \geq 2$, $\text{ext } B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$, $\text{ext } B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$, $\exp B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$ and showed that $\exp B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$ and $\exp B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$ are equal to $\text{ext } B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$ and $\text{ext } B_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$ respectively.

We refer to [1, 2, 5, 6, 9–11, 13–15, 20, 23, 26, 27, 31, 36, 38–47] for some recent work about extremal properties of homogeneous polynomials and multilinear forms on Banach spaces.

In this paper, we classify $\text{ext } B_{\mathcal{L}_s(2\mathbb{L}_n^{\Phi}))}$ and $\exp B_{\mathcal{L}_s(2\mathbb{L}_n^{\Phi}))}$. It is shown that

$$
\text{ext } B_{\mathcal{L}_s(2\mathbb{L}_n^{\Phi}))} = \exp B_{\mathcal{L}_s(2\mathbb{L}_n^{\Phi}))}.
$$

1 RESULTS

Throughout the paper, $\mathbb{R}^6_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$ denotes $\mathbb{R}^6$ with the $\mathcal{L}_s(2\mathbb{R}^2_{\beta})$-norm

$$
\|(a,b,c,d,e,f)\|_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})} = \max \left\{ |a|, |b|, |d|, \frac{1}{2}(|a|+|d|), \frac{1}{2}(|b|+|e|), \frac{1}{4}(|a+b-2d|+|c|), \frac{1}{4}(|a+b-2d|-|c|)+\frac{1}{2}|e-f| \right\}.
$$

Notice that if $(a,b,c,d,e,f) \in \mathbb{R}^6_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$ with $\|(a,b,c,d,e,f)\|_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})} = 1$, then $|a| \leq 1$, $|b| \leq 1$, $|d| \leq 1$, $|c| \leq 4$, $|e| \leq 2$, $|f| \leq 2$. Notice that

$$
\|(a,b,c,d,e,f)\|_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})} = \|(b,a,c,d,e,f)\|_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})} = \|(a,b,-c,d,e,f)\|_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})} = \|(a,b,c,d,-e,-f)\|_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})} = \|(-a,-b,c,-d,e,f)\|_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}.
$$

Therefore, without loss of generality we may assume that $a \geq |b|$, $c \geq 0$ and $e \geq 0$.

In [36] it was shown that the space $\mathbb{R}^6_{\mathcal{L}_s(2\mathbb{R}^2_{\beta})}$ is isometrically isomorphic to the space $\mathcal{L}_s(2\mathbb{L}_n^{\Phi}))$. 
Theorem 1. Let \((a, b, c, d, e, f) \in \mathbb{R}^6\). Then, the following statements are equivalent:

1. \((a, b, c, d, e, f) \in \text{ext}\ B_{\mathbb{R}^6}^{L_s(2\delta_{\mathbb{R}})}\);
2. \((b, a, c, d, f, e) \in \text{ext}\ B_{\mathbb{R}^6}^{L_s(2\delta_{\mathbb{R}})}\);
3. \((a, b, -c, d, e, f) \in \text{ext}\ B_{\mathbb{R}^6}^{L_s(2\delta_{\mathbb{R}})}\);
4. \((a, b, c, d, -e, -f) \in \text{ext}\ B_{\mathbb{R}^6}^{L_s(2\delta_{\mathbb{R}})}\);
5. \((-a, -b, c, -d, e, f) \in \text{ext}\ B_{\mathbb{R}^6}^{L_s(2\delta_{\mathbb{R}})}\).

Proof. It is obvious. □

Lemma 1. Let \(a, b \in \mathbb{R}\) be such that \(|a| + |b| = 1\). Then the following are equivalent:

1. \(|a| = 1, b = 0\) or \((a = 0, |b| = 1)\);
2. if \(\varepsilon, \delta \in \mathbb{R}\) satisfies \(|a + \varepsilon| + |b + \delta| \leq 1\) and \(|a - \varepsilon| + |b - \delta| \leq 1\), then \(\varepsilon = \delta = 0\).

Proof. By symmetry, we may assume that \(|a| \geq |b|\).

\[\begin{align*}
(1) &\implies (2). \text{ Suppose that } |a| = 1, b = 0 \text{ and let } \varepsilon, \delta \in \mathbb{R} \text{ be such that } |a + \varepsilon| + |b + \delta| \leq 1 \text{ and } |a - \varepsilon| + |b - \delta| \leq 1. \text{ Then } |a + \varepsilon| + |\delta| \leq 1 \text{ and } |a - \varepsilon| + |\delta| \leq 1, \text{ which shows that } 1 \geq |a| + |\varepsilon| + |\delta| = 1 + |\varepsilon| + |\delta|. \text{ Therefore, } \varepsilon = \delta = 0. \\
(2) &\implies (1). \text{ Assume otherwise. Then } 0 < |b| \leq |a| < 1. \text{ Let } t > 0 \text{ be such that } t|a| < |b|. \text{ Let } \varepsilon := t|a|\text{sign}(a) \text{ and } \delta := -t|a|\text{sign}(b). \text{ Notice that } \varepsilon \neq 0 \text{ and } \delta \neq 0. \text{ It follows that} \\
|a + \varepsilon| + |b + \delta| &= (|a| + t|a|) + (|b| - t|a|) = |a| + |b| = 1 \text{ and} \\
|a - \varepsilon| + |b - \delta| &= (|a| - t|a|) + (|b| + t|a|) = |a| + |b| = 1.
\end{align*}\]

This is a contradiction. Therefore, \((2) \implies (1)\) is true. □

We are in position to classify the extreme points of \(B_{\mathbb{R}^6}^{L_s(2\delta_{\mathbb{R}})}\).

Theorem 2.

\[
\text{ext}\ B_{\mathbb{R}^6}^{L_s(2\delta_{\mathbb{R}})} = \left\{ \pm (1, 1, \pm 4, 1, 2, 2), \pm (1, 1, \pm 4, 1, -2, 2), \pm (1, -1, \pm 4, 0, 1, 1), \right.
\]

\[
\pm (1, -1, \pm 4, 0, -1, 1), \pm (1, -1, \pm 2, 2, 0), \pm (1, -1, \pm 2, 1, -2, 0), \pm (1, -1, \pm 2, 1, 2, 0), \right.
\]

\[
\pm (1, 1, \pm 2, 0, -1, 1), \pm (1, 1, 0, 1, \pm 2, 0), \pm (1, 1, 1, 0, 1, \pm 2), \right.
\]

\[
\pm (1, 1, 0, -1, 0, 0) \right\}.
\]

Proof. Let \(T = (a, b, c, d, e, f) \in \text{ext}\ B_{\mathbb{R}^6}^{L_s(2\delta_{\mathbb{R}})}\). Without loss of generality we may assume that \(a \geq |b|, c \geq 0\) and \(e \geq 0\).

Claim: \(a = 1\).

Assume otherwise. Then, \(a < 1\). We claim that \(|d| < 1\). Assume that \(|d| = 1\). Since \(T = (a, b, c, d, e, f) \in \text{ext}\ B_{\mathbb{R}^6}^{L_s(2\delta_{\mathbb{R}})}\), by Lemma 1,

\[
\frac{1}{2}(a - d + e) = \frac{1}{2}(|b - d| + |f|) = \frac{1}{4}(|a + b - 2d| + c) = 1, |a + b - 2d| = c, |e - f| = 2.
\]
Hence, $c = 2$. Since $2 = |2d| = 2 + a + b \geq 2$, $a + b = 0$, so $a = b = 0$. Hence,

$$1 = \frac{1}{2}(|a - d| + e) = \frac{1}{2}(1 + e), \quad 1 = \frac{1}{2}(|b - d| + |f|) = \frac{1}{2}(1 + |f|),$$

which shows that $e = |f| = 1$. Since $|e - f| = 2$, $e = -f = 1$. Hence, $T = (0, 0, 2, \pm 1, 1, -1)$. We will show that $T$ is not extreme. Notice that for $n \in \mathbb{N}$,

$$(0, 0, 2, 1, 1, -1) = \frac{1}{2}\left(\left(\frac{1}{n'} - \frac{1}{n}, 2, 1, 1 + \frac{1}{n'}, -1 + \frac{1}{n}\right) + ( - \frac{1}{n} + \frac{1}{n}, 2, 1 - \frac{1}{n}, -1 - \frac{1}{n}\right)$$

and $\|\pm \frac{1}{n'} \mp \frac{1}{n}, 2, 1, 1 \pm \frac{1}{n'}, -1 \mp \frac{1}{n}\|_{L_2(\mathbb{R}^n)} = 1$. Notice that for $n \in \mathbb{N}$,

$$(0, 0, 2, -1, 1, -1) = \frac{1}{2}\left(\left(\frac{1}{n'} - \frac{1}{n}, 2, -1, 1 - \frac{1}{n}, -1 - \frac{1}{n}\right) + ( - \frac{1}{n} + \frac{1}{n}, 2, -1, 1 + \frac{1}{n}, -1 + \frac{1}{n}\right)$$

and $\|\pm \frac{1}{n'} \mp \frac{1}{n}, 2, -1, 1 \pm \frac{1}{n'}, -1 \mp \frac{1}{n}\|_{L_2(\mathbb{R}^n)} = 1$. This is a contradiction. Therefore, $|d| < 1$. Since $|b| \leq a < 1$, $|d| < 1$, choose $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \min\{1 - a, 1 - |d|\}.$$

Then,

$$\left\|\left(a \pm \frac{1}{N}, b \pm \frac{1}{N}, c, d \pm \frac{1}{N}, e, f\right)\right\|_{L_2(\mathbb{R}^n)} = 1$$

and

$$T = \frac{1}{2}\left(\left(\left(a + \frac{1}{N}, b + \frac{1}{N}, c, d + \frac{1}{N}, e, f\right) + (a - \frac{1}{N}, b - \frac{1}{N}, c, d - \frac{1}{N}, e, f)\right),

$$

which shows that $T$ is not extreme. This is a contradiction. Therefore, the claim holds.

Claim: $c = 0$ or 2 or 4.

Assume otherwise. Then, $0 < c < 2$ or $2 < c < 4$. We will reach to a contradiction.

Suppose that $0 < c < 2$. Let $|d| < 1$. Notice that if $b = 1$, then, by Lemma 1,

$$1 = \frac{1}{2}(1 - d + e) = \frac{1}{2}(1 - d + |f|) = \frac{1}{4}(2 - 2d + c) = \frac{1}{4}|2 - 2d + c| + \frac{1}{2}|e - f|,$$

so, $d = 0$ and $c = 2 + d = 2$, which is a contradiction. Notice that if $b = -1$, then, by Lemma 1,

$$1 = \frac{1}{2}(1 - d + e) = \frac{1}{2}(1 + d + |f|) = \frac{1}{4}(2|d| + c) = \frac{1}{4}|2|d| - c| + \frac{1}{2}|e - f|,$$

so, $c = 4 - 2|d| > 2$, which is a contradiction. Let $|b| < 1$. Notice that if $\frac{1}{4}(1 - d + e) = 1$, then, by Lemma 1,

$$b - d = 0, \quad |f| = 2, \quad |1 + b - 2d| = c, \quad |e - f| = 2,$$

which shows that $e = 0$ and $d = -1$, which is a contradiction. Let us note that if $\frac{1}{4}(|1 + b - 2d| + c) = 1$, then, by Lemma 1,

$$b - d = 0, \quad |f| = 2, \quad |1 + b - 2d| = c, \quad |e - f| = 2,$$

which shows that $c = 2$, which is a contradiction. Let us note that if $\frac{1}{4}(1 - d + e) = \frac{1}{4}(|1 + b - 2d| + c) = 1$, then, by Lemma 1,

$$b - d = 0, \quad |f| = 2, \quad \frac{1}{4}|1 + b - 2d| - c| + \frac{1}{2}|e - f| = 1,$$
which shows that $c = 3 + d > 2$, which is a contradiction. Suppose that
\[
\frac{1}{2}(1 - d + e) = \frac{1}{4}((1 + b - 2d) + c) = 1.\]
If $b - d = 0$, $|f| = 2$, $\frac{1}{4} |1 + b - 2d| - c + \frac{1}{2} |e - f| = 1$, then $c = 3 + d > 2$, which is a contradiction. If $|1 + b - 2d| = c$, $|e - f| = 2$, $\frac{1}{4}(1 - d + |f|) = 1$, then $c = 2, which is a contradiction.

Let $d = 1$. Suppose $e < 2$. If $|b| < 1$, then, by Lemma 1,
\[
1 = \frac{1}{2}(1 - b + |f|) = \frac{1}{4}(1 - b + c),
\]
which shows that $c = 3 + b > 2$, which is a contradiction. If $b = 1$, then, by Lemma 1,
\[
|f| = 2, \frac{1}{4} c + \frac{1}{2} |e - f| = 1,
\]
so $T = (1, 1, c, 1, \frac{1}{2} c, 2)$ or $(1, 1, c, 1, -\frac{1}{2} c, -2)$ for $0 < c < 2$. Hence, $T$ is not extreme. This is a contradiction. If $b = -1$, then, by Lemma 1,
\[
f = 0, \frac{1}{4}(2 - c) + \frac{1}{2} |e - f| = 1,
\]
which shows that $e = 2 + \frac{1}{2} c$. Hence, $c = 0$, which is a contradiction. Suppose $e = 2$. If $|b| < 1$, then, by Lemma 1,
\[
1 = \frac{1}{2}(1 - d + |f|) = \frac{1}{4}(1 - b + c)
\]
or
\[
1 = \frac{1}{2}(1 - d + |f|) = \frac{1}{4}|1 - b - c| + \frac{1}{2}(2 - f)
\]
or
\[
1 = \frac{1}{4}(1 - b + c) = \frac{1}{4}|1 - b - c| + \frac{1}{2}(2 - f).
\]
If
\[
1 = \frac{1}{2}(1 - d + |f|) = \frac{1}{4}(1 - b + c) \quad \text{or} \quad 1 = \frac{1}{4}(1 - b + c) = \frac{1}{4}|1 - b - c| + \frac{1}{2}(2 - f),
\]
then $c = 3 + b > 2$, which is a contradiction. If
\[
1 = \frac{1}{2}(1 - d + |f|) = \frac{1}{4}|1 - b - c| + \frac{1}{2}(2 - f),
\]
then $T = (1, b, -(1 + 3b), 1, 2, 1 + b)$ for $-1 < b < -\frac{1}{3}$. Hence, $T$ is not extreme. This is a contradiction. If $b = 1$, then $f = 2$ or $\frac{1}{4} b + \frac{1}{2}(2 - f) = 1$. If $f = 2$, then $T = (1, 1, c, 1, 2, 2)$ for $0 < c < 2$. Hence, $T$ is not extreme. This is a contradiction. If $\frac{1}{4} c + \frac{1}{2}(2 - f) = 1$, then $T = (1, 1, c, 1, 2, 1 + c)$ for $0 < c < 2$. Hence, $T$ is not extreme. This is a contradiction. If $b = -1$, then $f = 0$ and $c \geq 2$, which is a contradiction. Let $d = -1$. If $|b| < 1$, then, by Lemma 1,
\[
1 = \frac{1}{2}(1 + b + |f|) = \frac{1}{4}(3 + b + c)
\]
or
\[
1 = \frac{1}{2}(1 + d + |f|) = \frac{1}{4}(3 + b - c) + \frac{1}{2}|f|
\]
or
\[
1 = \frac{1}{4}(3 + b + c) = \frac{1}{4}(3 + b - c) + \frac{1}{2}|f|.\]
Hence, $T = (1, b, 1 - b, -1, 0, \pm(1 - b))$ for $-1 < b < 1$. Hence, $T$ is not extreme. This is a contradiction. If $b = 1$, then $f = 0$ and $1 \geq \frac{1}{4}(|a + b - 2d| + c) = 1 + \frac{c}{4}$. Hence, $c = 0$, which is a contradiction. If $b = -1$, then $f = 0$ and $\frac{1}{4}(2 - c) + \frac{1}{2}|f| = 1$. Hence, $T = (1, -1, c, -1, 0, \pm(1 + \frac{c}{2}))$ for $0 < c < 2$. Hence, $T$ is not extreme. This is a contradiction.

We have shown that if $0 < c < 2$, then $T$ is not extreme.

Suppose that $2 < c < 4$. Let $|d| < 1$. If $|b| < 1$, then, by Lemma 1,

$$b - d, \ |f| = 2, \ |1 + b - 2d| = c, \ |e - f| = 2.$$ 

If $\frac{1}{2}(1 - d + 2) = 1$, then $e = 0$ and $d = -1$, which is a contradiction. If $\frac{1}{4}(|1 + b - 2d| + c) = 1$, then $c = 1 - d < 2$, which is a contradiction. If $b = -1$, then, by Lemma 1,

$$1 = \frac{1}{2}(1 - d + e) = \frac{1}{2}(1 + d + |f|) = \frac{1}{4}(2d + c) = \frac{1}{4}|2d - c| + \frac{1}{2}|e - f|.$$ 

Hence, $T = (1, -1, c, 2 - \frac{1}{2}c, 3 - \frac{1}{2}c, -1 + \frac{1}{2}c)$ for $2 < c < 4$. Hence, $T$ is not extreme. This is a contradiction. If $b = 1$, then

$$1 = \frac{1}{2}(1 - d + e) = \frac{1}{2}(1 - d + |f|) = \frac{1}{4}(2 - 2d + c) = \frac{1}{4}|2 - 2d - c| + \frac{1}{2}|e - f|.$$ 

Hence, $d = \frac{-c^2}{2}$, $e = \frac{1}{2}c = |f|$. If $f = \frac{1}{2}c$, then $1 = \frac{1}{4}|2 - 2d - c| = \frac{c}{2} - 1$, so $c = 4$. This is a contradiction. If $f = -\frac{1}{2}c$, then $1 = c - 1$, so $c = 2$. This is a contradiction. Let $|d| = 1$. Suppose that $e < 2$. If $|b| < 1$, then, by Lemma 1,

$$1 = \frac{1}{2}(1 - b + |f|) = \frac{1}{4}(1 - b + c), \ 1 - b - c = 0, \ |e - f| = 2.$$ 

Hence $b = -1$. This is a contradiction. If $b = 1$, then $T = (1, 1, c, \pm 1, \pm \frac{1}{2}c, \pm 2)$ for $2 < c < 4$. Hence, $T$ is not extreme. This is a contradiction. If $b = -1$, then $f = 0$ and $c \leq 2$. This is a contradiction. Suppose that $e = 2$. If $|b| < 1$, then, by Lemma 1,

$$1 = \frac{1}{2}(1 - b + |f|) = \frac{1}{4}(1 - b + c) or 1 - b - c = f = 0.$$ 

If $1 = \frac{1}{2}(1 - b + |f|) = \frac{1}{4}(1 - b + c)$, then $T = (1, b, 3 + b, 1, 2, \pm(1 + b))$ for $-1 < b < 1$. Hence, $T$ is not extreme. This is a contradiction. If $1 - b - c = f = 0$, then $c = 1 - b < 2$. This is a contradiction. Let $d = 1$. If $b = 1$, then, by Lemma 1,

$$f = 0 \ or \ \frac{1}{4}c + \frac{1}{2}(2 - f) = 1.$$ 

If $f = 0$, then $T = (1, 1, c, 1, 2, 0)$ for $2 < c < 4$. Hence, $T$ is not extreme. This is a contradiction. If $\frac{1}{2}c + \frac{1}{2}(2 - f) = 1$, then $T = (1, 1, c, 1, 2, \frac{1}{2}c)$ for $2 < c < 4$. Hence, $T$ is not extreme. This is a contradiction.

Let $d = -1$. If $|b| < 1$, then we reach to a contradiction as in the proof of the case $d = 1$. If $b = 1$, then, by Lemma 1, $f = 0$ and $1 \geq \frac{1}{4}(|a + b - 2d| + |c|) = \frac{1}{4}(4 + c)$, so $c = 0$. This is a contradiction. If $b = -1$, then, by Lemma 1,

$$\frac{1}{2} + \frac{1}{4}c = \frac{1}{4}(c - 2) + \frac{1}{2}|f| = 1,$$
so $c = 2$. This is a contradiction. We have shown that if $2 < c < 4$, then $T$ is not extreme.

**Case 1: $c = 0$.**

**Claim:** $|b| = |d| = 1$.

Assume otherwise. Then, $|b| < 1, |d| < 1$ or $|b| = 1, |d| < 1$ or $|b| < 1, |d| = 1$.

Assume that $|b| < 1$ and $|d| < 1$. By Lemma 1,

$$
\frac{1}{2}(1 - d + e) = \frac{1}{2}(|b - d| + |f|) = 1, 1 + b - 2d = 0, |e - f| = 2.
$$

Hence, $b = -1$, which is a contradiction. Assume that $|b| = 1$ and $|d| < 1$. If $b = 1$, then, by Lemma 1,

$$
\frac{1}{2}(1 - d + e) = \frac{1}{2}(1 - d + |f|) = \frac{1}{2}(1 - d + |e - f|) = 1.
$$

Hence, $d = -1$, which is a contradiction. If $b = -1$, then, by Lemma 1,

$$
\frac{1}{2}(1 - d + e) = \frac{1}{2}(1 + d + |f|) = 1, d = 0, |e - f| = 2.
$$

Hence, $T = (1, -1, 0, 0, 1, -1)$. Notice that $T$ is not extreme since

$$
T = \frac{1}{2}\left( (1, -1, \frac{2}{n'}, \frac{1}{n'}, \frac{3}{n'}, \frac{2}{n'}, -1, -1, 1, 1, 1) \right)
$$

and $\| (1 - 1, \pm \frac{2}{n'}, \pm \frac{1}{n'}, 1 \pm \frac{1}{n'}) \|_{L_2(\mathbb{R})} = 1$ for every $n \in \mathbb{N}$. Assume that $|b| < 1$ and $|d| = 1$. If $d = 1$, then, by Lemma 1,

$$
e = 2, \frac{1}{2}(1 - b + |f|) = \frac{1}{4}(1 - b) + \frac{1}{2}|e - f| = 1.
$$

Hence, $T = (1, -\frac{1}{3}, 0, 1, 2, \frac{2}{3})$. Notice that $T$ is not extreme since

$$
T = \frac{1}{2}\left( (1, -\frac{1}{3} + \frac{1}{n'}, \frac{3}{n'}, \frac{2}{3} + \frac{1}{n'}) \right)
$$

and $\| (1, -\frac{1}{3} \pm \frac{1}{n'}, 1 \pm \frac{1}{n'}, 1, 2, \frac{2}{3} \pm \frac{1}{n'}) \|_{L_2(\mathbb{R})} = 1$

for every $n > 3$. If $d = -1$, then, by Lemma 1,

$$
e = 0, \frac{1}{2}(1 + b + |f|) = \frac{1}{4}(3 + b) + \frac{1}{2}|f| = 1.
$$

Hence, $b = 1$, which is a contradiction. We have shown that the claim holds.

Suppose that $b = d = 1$. By Lemma 1,

$$(e = |f| = 2) \text{ or } (e = |e - f| = 2) \text{ or } (|f| = |e - f| = 2).$$

If $e = |f| = 2$, then $T = (1, 1, 0, 1, 2, 2)$. Notice that $T$ is not extreme since

$$
T = \frac{1}{2}\left( (1, 1, \frac{1}{n'}, 1, 2, 2) + (1, 1, -\frac{1}{n'}, 1, 2, 2) \right)
$$

and $\| (1, 1, \pm \frac{1}{n'}, 1, 2, 2) \|_{L_2(\mathbb{R})} = 1$ for every $n \in \mathbb{N}$. This is a contradiction. If $e = |e - f| = 2$, then $T = (1, 1, 0, 1, 2, 0) \in \text{ext } B_{\mathbb{R}}^{1, 1, 2, 2}$. Indeed, let $T_1 := T + (\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3)$ and
$T_2 := T - (\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3)$ for some $\varepsilon_j$, $\beta_j \in \mathbb{R}$ ($j = 1, 2, 3$). Obviously, $\varepsilon_1 = \varepsilon_2 = \delta_1 = 0$. Since $|2 \pm \delta_2| \leq 2$, we have $\delta_2 = 0$. Since  
$$\frac{1}{4}|\varepsilon_3| + \frac{1}{2}|2 - \delta_3| \leq 2, \frac{1}{4} - \varepsilon_3| + \frac{1}{2}|2 + \delta_3| \leq 2,$$
we have $\delta_3 = \varepsilon_3 = 0$. Therefore, $T_1 = T_2 = T$. Hence, $T$ is extreme. If $|f| = |e - f| = 2$, then $T = (1, 1, 0, 1, 0, \pm 2)$. By Theorem 1, $T$ is extreme. If $b = -d = 1$, then, by Lemma 1, $T = (1, 1, 0, -1, 0, 0)$. We claim that $T$ is extreme. Let $T_1 := T + (\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3)$ and $T_2 := T - (\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3)$ for some $\varepsilon_j$, $\beta_j \in \mathbb{R}$ ($j = 1, 2, 3$). Obviously, $\varepsilon_1 = \varepsilon_2 = \delta_1 = 0$. Since $|2 \pm \delta_2| \leq 2, |2 \pm \delta_3| \leq 2$, we have $\delta_2 = \delta_3 = 0$. Since $\frac{1}{4}(4 + |\varepsilon_3|) \leq 1$, we have $\varepsilon_3 = 0$. Therefore, $T_1 = T_2 = T$. Hence, $T$ is extreme. Notice that $(1, 1, 0, -1, 0, 0)$. If $-b = -d = 1$, then $|f| = 1$ and $T = (1, -1, 0, -1, 0, \pm 1)$. Notice that $T$ is not extreme since  
$$T = \frac{1}{2}\left((1, -1, \frac{2}{n}, -1, 0, 1 + \frac{1}{n}) + (1, -1, -\frac{2}{n}, -1, 0, 1 - \frac{1}{n})\right)$$
and $\|(1, -1, \pm \frac{2}{n}, -1, 0, 1 \pm \frac{1}{n})\|_{L_2(\mathbb{R}_+^n)} = 1$ for every $n \in \mathbb{N}$. This is a contradiction. If $-b = d = 1$, then $c = 2$. This is a contradiction.

Case 2: $c = 2$.

Claim: $|d| = 0$ or 1.

Assume that $0 < |d| < 1$. If $b = d$, by Lemma 1,  
$$|f| = 2, \frac{1}{2}(1 - d + e) = \frac{1}{4}(1 - d) + \frac{1}{2} = 1.$$ 
Hence, $d = -1$, which is a contradiction. Assume that $b \neq d$. If $|b| < 1$, by Lemma 1,  
$$\frac{1}{2}(1 - d + e) = \frac{1}{2}|b - d + |f|| = \frac{1}{4}(1 + b - 2d) + \frac{1}{2} = 1, \quad |1 + b - 2d| - 2| = 4, \quad e - f = 0$$
or  
$$\frac{1}{2}(1 - d + e) = \frac{1}{2}|b - d + |f|| = \frac{1}{4}(1 + b - 2d) + \frac{1}{2} = 1, \quad |1 + b - 2d| = 2, \quad |e - f| = 2.$$ 
If $\frac{1}{2}(1 - d + e) = \frac{1}{2}|b - d + |f|| = \frac{1}{4}(1 + b - 2d) + \frac{1}{2} = 1, \quad |1 + b - 2d| = 2, \quad |e - f| = 2$, then $b = -1$, which is a contradiction. If $\frac{1}{2}(1 - d + e) = \frac{1}{2}|b - d + |f|| = \frac{1}{4}(1 + b - 2d) + \frac{1}{2} = 1, \quad |1 + b - 2d| - 2| = 4, \quad e - f = 0$, then $|d| = 2$, which is a contradiction. If $|b| = 1$, then, by Lemma 1,  
$$\frac{1}{2}(1 - d + e) = \frac{1}{4}(1 + b - 2d) + \frac{1}{2} = 1, \quad |1 + b - 2d| = |e - f| = 2.$$ 
If $b = 1$, then $d = 0$, which is a contradiction. If $b = -1$, then $d = 1$, which is a contradiction. Therefore, we have shown that $|d| = 0$ or 1.

Suppose that $d = 0$. If $|b| < 1$, then, by Lemma 1,  
$$e = 1, \quad \frac{1}{2}|b + |f|| = \frac{1}{4}(1 + b) + \frac{1}{2} = 1.$$ 
Hence, $b = 1$, which is a contradiction. Let $|b| = 1$. Suppose that $\frac{1}{2} + \frac{1}{2}e = 1$. Then, $e = 1$ and $T = (1, 1, 2, 0, 1, -1)$ or $(1, 1, 2, 0, -1, 1)$. We claim that $(1, 1, 2, 0, 1, -1)$ is extreme. Indeed, let
\[ T_1 := T + (\epsilon_1, \epsilon_2, \epsilon_3, \delta_1, \delta_2, \delta_3) \] and \[ T_2 := T - (\epsilon_1, \epsilon_2, \epsilon_3, \delta_1, \delta_2, \delta_3) \] for some \( \epsilon_j, \beta_j \in \mathbb{R} \) \( j = 1, 2, 3 \). Observe, \( \epsilon_1 = \epsilon_2 = 0 \). Since

\[
|1 \pm \delta_1| + |1 \pm \delta_2| \leq 2, \quad |1 \pm \delta_1| + |1 \pm \delta_3| \leq 2, \quad |2 \pm 2\delta_1| + |2 \pm \epsilon_3| \leq 4,
\]

we have \( \delta_1 = \delta_2 = -\delta_3 = \frac{1}{2} \epsilon_3 \). Since

\[
\frac{3}{4} |\pm \delta_1| + |1 \pm \delta_1| \leq 1,
\]

we have \( \delta_1 = \delta_2 = -\delta_3 = \frac{1}{2} \epsilon_3 = 0 \). Therefore, \( T_1 = T_2 = T \). Hence, \( T \) is extreme. By Theorem 1, \( (1, 1, 2, 0, -1, 1) \) is extreme.

Suppose that \( \frac{1}{2} + \frac{1}{4} \epsilon < 1 \). By Lemma 1, \( |f| = 1 \). If \( f = 1 \), then \( T = (1, 1, 2, 0, e, 1) \) for \( 0 \leq e < 1 \). Notice that such \( (1, 1, 2, 0, e, 1) \) is not extreme. If \( f = -1 \), then \( T = (1, 1, 2, 0, 0, -1) \). Notice that \( (1, 1, 2, 0, 0, -1) \) is not extreme since

\[
T = \frac{1}{2} \left( (1, 1, 2 + \frac{2}{n}, \frac{1}{n}, \frac{1}{n}, -1 + \frac{1}{n}) + (1, 1, 2 - \frac{2}{n}, -1 - \frac{1}{n}, -1 - \frac{2}{n}) \right)
\]

and \( \|(1, 1, 2 \pm \frac{2}{n}, \pm \frac{1}{n}, \pm \frac{1}{n}, -1 \pm \frac{1}{n})\|_{L_1(\mathbb{R}^3)} = 1 \) for every \( n > 2 \). This is a contradiction.

Suppose that \( |d| = 1 \). We claim that \( |b| = 1 \). Assume that \( |b| < 1 \). If \( d = 1 \), then, by Lemma 1,

\[
\frac{1}{4} (1 - b + |f|) = \frac{1}{4} (1 - b) + \frac{1}{2} = 1
\]

or

\[
\frac{1}{4} (1 - b) + \frac{1}{2} = \frac{1}{4} (1 - b) + \frac{1}{2} = 1
\]

or

\[
\frac{1}{2} (1 - b + |f|) = \frac{1}{4} (1 + b) + \frac{1}{2} |e - f| = 1.
\]

Hence, \( b = -1 \), which is a contradiction. If \( d = -1 \), then, by Lemma 1,

\[
e = 0, \quad \frac{1}{2} (1 + b + |f|) = \frac{1}{4} (3 + b) + \frac{1}{2} = 1.
\]

Hence, \( b = -1 \), which is a contradiction. Therefore, \( |b| = 1 \). Suppose that \( b = d = 1 \). If \( \frac{1}{2} + \frac{1}{4} |e - f| < 1 \), then \( T = (1, 1, 2, 1, 2, \pm 2) \). Notice that \( (1, 1, 2, 1, 2, \pm 2) \) is not extreme since

\[
T = \frac{1}{2} \left( (1, 1, 2 \pm \frac{1}{n}, 1, 2, 2) + (1, 1, 2 - \frac{1}{n}, 1, 2, 2) \right)
\]

and \( \|(1, 1, 2 \pm \frac{1}{n}, 1, 2, 2)\|_{L_1(\mathbb{R}^3)} = 1 \) for every \( n > 2 \). This is a contradiction. Suppose that \( \frac{1}{2} + \frac{1}{4} |e - f| = 1 \). If \( e = 2 \), then \( T = (1, 1, 2, 1, 2, 0) \). Notice that \( (1, 1, 2, 1, 2, 0) \) is not extreme since

\[
T = \frac{1}{2} \left( (1, 1, 2 \pm \frac{1}{n}, 1, 2, \frac{1}{2n}) + (1, 1, 2 - \frac{1}{n}, 1, 2, -\frac{1}{2n}) \right)
\]

and \( \|(1, 1, 2 \pm \frac{1}{n}, 1, 2, \pm \frac{1}{2n})\|_{L_1(\mathbb{R}^3)} = 1 \) for every \( n \in \mathbb{N} \). This is a contradiction.

If \( |f| = 2 \), then \( T = (1, 1, 2, 1, 0, 2) \). By Theorem 1, \( (1, 1, 2, 1, 0, 2) \) is not extreme. Suppose that \( -b = d = 1 \). Then \( T = (1, -1, 2, 1, e, 0) \) for \( 0 \leq e \leq 2 \). Since \( T \) is extreme, \( e = 0 \) or \( 2 \). Notice that \( (1, -1, 2, 1, 0, 0) \) is not extreme. We claim that \( T = (1, -1, 2, 1, 0, 2) \) is extreme. Let

\[
T_1 := T + (\epsilon_1, \epsilon_2, \epsilon_3, \delta_1, \delta_2, \delta_3) \quad \text{and} \quad T_2 := T - (\epsilon_1, \epsilon_2, \epsilon_3, \delta_1, \delta_2, \delta_3)
\] for some \( \epsilon_j, \beta_j \in \mathbb{R} \) \( j = 1, 2, 3 \).
3). Obviously, \( \epsilon_1 = \epsilon_2 = \delta_1 = \delta_2 = \delta_3 = 0 \). Since \( 2 + |2 \pm \epsilon_3| \leq 4 \), we have \( \epsilon_3 = 0 \). Therefore, \( T_1 = T_2 = T \). Hence, \( T \) is extreme.

Suppose that \( b = d = -1 \). Then \( T = (1, -1, 2, -1, 0, f) \) for \( -2 \leq f \leq 2 \). Since \( T \) is extreme, \( f = \pm 2 \). By Theorem 1, \( T = (1, -1, 2, -1, \alpha, \pm 2) \) is extreme. Suppose that \( b = -d = 1 \). Then,

\[
1 \geq \frac{1}{4}(|1 + b - 2d| + c) = \frac{3}{2},
\]

which is a contradiction.

**Case 3:** \( c = 4 \).

Claim: \( |b| = 1 \).

Assume that \( |b| < 1 \). By Lemma 1, we have \( 0 < d < 1 \), \( \frac{1}{2}(1 - d + e) = 1 \). Hence, \( T = (1, 2d - 1, 4, d, 1 + d, 1 + d) \) for \( 0 < d < 1 \). Hence, \( T \) is not extreme. This is a contradiction. Therefore, \( |b| = 1 \). If \( b = 1 \), then \( T = (1, 1, 4, 1, e, e) \) for \( 0 \leq e \leq 2 \). Since \( T \) is extreme, \( e = 0 \) or \( 2 \). We claim that \( (1, 1, 4, 1, 2, 2) \) is extreme. Let \( T_1 := T + (e_1, e_2, \epsilon_3, \delta_1, \delta_2, \delta_3) \) and \( T_2 := T - (e_1, e_2, \epsilon_3, \delta_1, \delta_2, \delta_3) \) for some \( e_j, \beta_j \in \mathbb{R} \) \( (j = 1, 2, 3) \). Obviously, \( e_1 = e_2 = \epsilon_3 = \delta_1 = 0 \), \( \delta_3 = \beta_3 \). Since \( |2 \pm \beta_3| \leq 2 \), we have \( \delta_2 = 0 \). Therefore, \( T_1 = T_2 = T \). Hence, \( T \) is extreme.

Notice that \( (1, 1, 4, 1, 0, 0) \) is not extreme since

\[
T = \frac{1}{2} \left( (1, 1, 4, 1, \frac{1}{n}, \frac{1}{n}) + (1, 1, 4, 1, -\frac{1}{n}, -\frac{1}{n}) \right)
\]

and \( \| (1, 1, 4, 1, \pm \frac{1}{n}, \pm \frac{1}{n}) \|_{L^4(\mathbb{R}^n)} = 1 \) for every \( n \in \mathbb{N} \). This is a contradiction.

If \( b = -1 \), then \( d = 0 \), \( e = f \), \( 0 \leq e \leq 1 \). Hence, \( T = (1, -1, 4, 0, e, e) \) for \( 0 \leq e \leq 1 \). Since \( T \) is extreme, \( e = 0 \) or \( 1 \). Notice that \( (1, -1, 4, 0, 0, 0) \) is not extreme since

\[
(1, -1, 4, 0, 0, 0, 0) = \frac{1}{2} \left( (1, -1, 4, 0, \frac{1}{n}, \frac{1}{n}) + (1, -1, 4, 0, -\frac{1}{n}, -\frac{1}{n}) \right)
\]

and \( \| (1, -1, 4, 0, \pm \frac{1}{n}, \pm \frac{1}{n}) \|_{L^4(\mathbb{R}^n)} = 1 \) for every \( n \in \mathbb{N} \). This is a contradiction. We claim that \( T = (1, -1, 4, 0, 1, 1) \) is extreme. Let

\[
T_1 := T + (\epsilon_1, \epsilon_2, \epsilon_3, \delta_1, \delta_2, \delta_3) \quad \text{and} \quad T_2 := T - (\epsilon_1, \epsilon_2, \epsilon_3, \delta_1, \delta_2, \delta_3)
\]

for some \( \epsilon_j, \beta_j \in \mathbb{R} \) \( (j = 1, 2, 3) \). Obviously, \( \epsilon_j = 0 \) for \( j = 1, 2, 3 \). Since

\[
2|\delta_1| + 4 \leq 4, \quad 1 + |1 \pm \delta_2| \leq 2, \quad 1 + |1 \pm \delta_3| \leq 2,
\]

we have \( \delta_j = 0 \) for \( j = 1, 2, 3 \). Therefore, \( T_1 = T_2 = T \). Hence, \( T \) is extreme.

Therefore, we complete the proof.

**Theorem 3 ([22]).** Let \( E \) be a real Banach space such that \( \text{ext} B_E \) is finite. Suppose that \( x \in \text{ext} B_E \) satisfies that there exists \( f \in E^* \) with \( f(x) = 1 = \| f \| \) and \( |f(y)| < 1 \) for every \( y \in \text{ext} B_E \{ \pm x \} \). Then, \( x \in \text{exp} B_E \).

The following theorem gives the explicit formula for the norm of every linear functional on \( \mathbb{R}^6_{L^4(\mathbb{R}^n)} \):

**Theorem 4.** Let \( f \in (\mathbb{R}^6_{L^4(\mathbb{R}^n)})^* \). Let \( a_1 := f(e_1), a_2 := f(e_2), a_3 := f(e_4), \beta := f(e_3), \gamma_1 := f(e_5), \gamma_2 := f(e_6) \). Then,

\[
\| f \| = \left\{ |a_1 + a_2 + a_3| + 4|\beta| + 2|\gamma_1 + \gamma_2|, \ |a_1 - a_2 + 4|\beta| + |\gamma_1 + \gamma_2|, \ |a_1 - a_2 + a_3| + 2|\beta| + 2|\gamma_1|, \ |a_1 - a_2 - a_3| + 2|\beta| + 2|\gamma_2|, \ |a_1 + a_2| + 2|\beta| + |\gamma_1 - \gamma_2|, \ |a_1 + a_2 + a_3| + 2|\gamma_1|, \ |a_1 + a_2 - a_3| \right\}.
\]
Proof. It follows from the Krein-Milman Theorem and the fact that

$$\|f\| = \sup_{T \in \text{ext} B_{R^6_{L_4(\mathbb{R}^2)}}} |f(T)|.$$ 

Notice that if \( f \in (\mathbb{R}^6_{L_4(\mathbb{R}^2)})^* \) and \( \|f\| = 1 \), then \( |\alpha_j| \leq 1, |\beta| \leq \frac{1}{2}, |\gamma_k| \leq \frac{1}{2} \) for \( j = 1, 2, 3 \) and \( k = 1, 2, 3 \).

**Theorem 5.** \( \text{ext} B_{R^6_{L_4(\mathbb{R}^2)}} = \exp B_{R^6_{L_4(\mathbb{R}^2)}} \).

Proof. It is enough to show that if \( T = (a, b, c, d, e, f) \in \text{ext} B_{R^6_{L_4(\mathbb{R}^2)}} \), then \( T \) is exposed.

**Claim:** \( T = (1, 1, 4, 1, 2, 2) \) is exposed.

Let \( f \in (\mathbb{R}^6_{L_4(\mathbb{R}^2)})^* \) be such that \( \alpha_1 = \alpha_2 = \alpha_3 = 0, \beta = \gamma_1 = \gamma_2 = \frac{1}{8} \). By Theorem 4, \( f(T) = \|f\| = 1 \) and \( |f(R)| < 1 \) for every \( R \in \text{ext} B_{R^6_{L_4(\mathbb{R}^2)}} \setminus \{ \pm T \} \). By Theorem 3, \( T \) is exposed.

By Theorem 1, \( \pm (1, 1, -4, 1, 2, 2), \pm (1, 1, \pm 4, 1, -2, -2) \) are exposed.

**Claim:** \( T = (1, -1, 4, 0, 1, 1) \) is exposed.

Let \( f \in (\mathbb{R}^6_{L_4(\mathbb{R}^2)})^* \) be such that \( \alpha_1 = -\alpha_2 = \frac{1}{8}, \alpha_3 = 0, \beta = \frac{3}{16}, \gamma_1 = \gamma_2 = 0 \). By Theorem 4, \( f(T) = \|f\| = 1 \) and \( |f(R)| < 1 \) for every \( R \in \text{ext} B_{R^6_{L_4(\mathbb{R}^2)}} \setminus \{ \pm T \} \). By Theorem 3, \( T \) is exposed.

By Theorem 1, \( \pm (1, 1, -4, 0, 1, 1), \pm (1, 1, \pm 4, 0, -1, -1) \) are exposed.

**Claim:** \( T = (1, -1, 2, 1, 2, 0) \) is exposed.

Let \( f \in (\mathbb{R}^6_{L_4(\mathbb{R}^2)})^* \) be such that \( \alpha_1 = -\alpha_2 = -\alpha_3 = \frac{1}{8}, \beta = \gamma_1 = \gamma_2 = 0 \). By Theorem 4, \( f(T) = \|f\| = 1 \) and \( |f(R)| < 1 \) for every \( R \in \text{ext} B_{R^6_{L_4(\mathbb{R}^2)}} \setminus \{ \pm T \} \). By Theorem 3, \( T \) is exposed.

By Theorem 1, \( \pm (1, -1, -2, 1, 2, 0), \pm (1, 1, -2, 1, -2, 0), \pm (1, 1, -2, -1, 0, -2) \) are exposed.

**Claim:** \( T = (1, 1, 2, 0, 1, -1) \) is exposed.

Let \( f \in (\mathbb{R}^6_{L_4(\mathbb{R}^2)})^* \) be such that \( \alpha_1 = \alpha_2 = -\alpha_3 = \frac{1}{8}, \beta = 0, \gamma_1 = -\gamma_2 = \frac{1}{4} \). By Theorem 4, \( f(T) = \|f\| = 1 \) and \( |f(R)| < 1 \) for every \( R \in \text{ext} B_{R^6_{L_4(\mathbb{R}^2)}} \setminus \{ \pm T \} \). By Theorem 3, \( T \) is exposed.

By Theorem 1, \( \pm (1, 1, -2, 0, 1, -1), \pm (1, 1, \pm 2, 0, -1, 1) \) are exposed.

**Claim:** \( T = (1, 1, 0, 1, 2, 0) \) is exposed.

Let \( f \in (\mathbb{R}^6_{L_4(\mathbb{R}^2)})^* \) be such that \( \alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{8}, \beta = 0, \gamma_1 = -\gamma_2 = \frac{1}{4} \). By Theorem 4, \( f(T) = \|f\| = 1 \) and \( |f(R)| < 1 \) for every \( R \in \text{ext} B_{R^6_{L_4(\mathbb{R}^2)}} \setminus \{ \pm T \} \). By Theorem 3, \( T \) is exposed.

By Theorem 1, \( \pm (1, 1, 0, 1, \pm 2, 0), \pm (1, 1, \pm 1, 0, 1, 2) \) are exposed.

**Claim:** \( T = (1, 1, 0, -1, 0, 0) \) is exposed.

Let \( f \in (\mathbb{R}^6_{L_4(\mathbb{R}^2)})^* \) be such that \( \alpha_1 = \alpha_2 = -\alpha_3 = \frac{1}{8}, \beta = \gamma_1 = \gamma_2 = 0 \). By Theorem 4, \( f(T) = \|f\| = 1 \) and \( |f(R)| < 1 \) for every \( R \in \text{ext} B_{R^6_{L_4(\mathbb{R}^2)}} \setminus \{ \pm T \} \). By Theorem 3, \( T \) is exposed.

By Theorem 1, \( \pm (1, -1, 0, -1, 0, 0) \) is exposed.

Therefore, we complete the proof. \( \square \)

**References**


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Кім С.Г. Екстремальнi та виставленi симетричнi бiлiнiйнi форми на просторi \( L_s^{(2) l_\infty} \) // Карпатськi матем. публ. — 2020. — Т.12, №2. — С. 340–352.

Класифіковано екстремальнi точки та виставленi точки одниничної кули простору бiлiнiйних симетричних форм на дiйсному банаховому просторi бiлiнiйних симетричних форм на \( l_\infty \).

Показано, що в цьому випадку множина екстремальних точок дорiвнює множинi виставлених точок.

Ключовi слова i фрази: екстремальна точка, виставлена точка.