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APPROXIMATION OF POSITIVE OPERATORS BY ANALYTIC VECTORS

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We give the estimates of approximation errors while approximating of a positive operator A in a Banach space by analytic vectors. Our main results are formulated in the form of Bernstein and Jackson type inequalities with explicitly calculated constants. We consider the classes of invariant subspaces $\mathcal{E}^{\nu,\alpha}_{q,p}(A)$ of analytic vectors of A and the special scale of approximation spaces $\mathcal{B}^{s,\alpha}_{q,p,\tau}(A)$ associated with the complex degrees of positive operator. The approximation spaces are determined by E-functional, that plays a similar role as the module of smoothness. We show that the approximation spaces can be considered as interpolation spaces generated by K-method of real interpolation. The constants in the Bernstein and Jackson type inequalities are expressed using the normalization factor.

Key words and phrases: positive operator, approximation space, Bernstein-Jackson-type inequality.

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1 INTRODUCTION

In the paper, we investigate an approximation problem by invariant subspaces $\mathcal{E}^{\nu,\alpha}_{q,p}(A)$ of analytic vectors of positive operator A in a Banach space $\mathfrak{X}$. For this we will use the special scale of approximation spaces $\mathcal{B}^{s,\alpha}_{q,p,\tau}(A)$ associated with complex degrees of A. The defining role in our approach is played by the functional $E(t,x;\mathcal{E}^{\alpha}_{q,p}(A),\mathfrak{X})$, which characterizes the distance from $x\in\mathfrak{X}$ to the subspace $\mathcal{E}^{\nu,\alpha}_{q,p}(A)$.

Analytic vectors of the unbounded linear operator on a Banach space first appear in [11]. It should be noted that the applications of analytic vectors to approximation problems can be found in [5–8] and ect. The results obtained in this direction are formulated in the form of so-called direct and inverse theorems (Jackson and Bernstein inequalities) of the theory of approximation of functions. In this connection, the problem of precise estimates of the constants of such inequalities is very important [1, 6, 9, 14].

The main of our goal is to determine exact estimates of the constants in the Bernstein and Jackson type inequalities, which allow us to estimate of best approximation errors by analytic vectors of positive operator A in a Banach space $\mathfrak{X}$.

We will use the K-method of real interpolation (see, e.g. [2, 13]). Let $(\mathfrak{X}_0, |\cdot|_{\mathfrak{X}_0})$ and $(\mathfrak{X}_1, |\cdot|_{\mathfrak{X}_1})$ are quasi-normed spaces, that form a compatible pair. For every such pair define the K-functional by

$$K(t,x;\mathfrak{X}_0,\mathfrak{X}_1) := \left\{\inf\left(|x_0|^2_{\mathfrak{X}_0} + t^2\,|x_1|^2_{\mathfrak{X}_1}\right)^{1/2} : x_0\in\mathfrak{X}_0, x_1\in\mathfrak{X}_1, x_0+x_1=x\right\}$$

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for $t > 0$ and $x \in \mathfrak{X}_0 + \mathfrak{X}_1$. Elementary properties of this K-functional are noted in [10, Appendix B].

For $0 < \theta < 1$ and $1 \le r \le \infty$, we define an interpolation space

$$(\mathfrak{X}_0, \mathfrak{X}_1)_{\theta,r} = \left\{x \in \mathfrak{X}_0 + \mathfrak{X}_1 \colon |x|_{(\mathfrak{X}_0,\mathfrak{X}_1)_{\theta,r}} < \infty\right\}$$

with the quasi-norm

$$|x|_{(\mathfrak{X}_0,\mathfrak{X}_1)_{\theta,r}} = \begin{cases} \left(\int_0^\infty \left[t^{-\theta}K(t,x;\mathfrak{X}_0,\mathfrak{X}_1)\right]^r dt/t\right)^{1/r}, & 1 \le r < \infty, \\ \sup_{t>0} t^{-\theta}K(t,x;\mathfrak{X}_0,\mathfrak{X}_1), & r = \infty. \end{cases}$$

Following [3], we will use the normalization factor

$$N_{\theta,r} := \begin{cases} \left(\int_0^\infty t^{r(1-\theta)-1}(1+t^2)^{-r/2} dt\right)^{-1/r}, & 1 \le r < \infty, \\ \theta^{-\theta/2}(1-\theta)^{-(1-\theta)/2}, & r = \infty. \end{cases}$$

We note that $N_{\theta,r} = N_{1-\theta,r}$ and $N_{\theta,2} = ((2/\pi)\sin(\pi\theta))^{1/2}$ [10, Exercise B.5].

2 INVARIANT SUBSPACES OF ANALYTIC VECTORS OF POSITIVE OPERATORS

In a Banach space $(\mathfrak{X}, \|\cdot\|)$ we consider a positive operator A with dense domain $\mathfrak{D}^1(A) \subset \mathfrak{X}$. It means that $(-\infty, 0]$ belongs to the resolvent set of A and there exists a number $c > 0$ such that $\|(A - tI)^{-1}\| \le c/(1+|t|)$, $t \in (-\infty, 0]$.

Let $\mathfrak{D}^k(A)$, $k \in \mathbb{Z}_+$, be a domain of A^k with the norm $\|x\|_{\mathfrak{D}^k(A)} = \|A^k x\|$ and $A^0 = I$ is the unit operator on $\mathfrak{X}$.

Let $m, k \in \mathbb{Z}$, $m \ge 0$. As well known [13, Section 1.15.1], for $\alpha \in \mathbb{C}$ such that $-m < \operatorname{Re}\alpha \le \sigma - m$, $0 < \sigma < k$, and all $x \in \left(\mathfrak{X}, \mathfrak{D}^k(A)\right)_{\sigma/k,1}$

$$A^\alpha_\sigma x = \frac{\Gamma(k)}{\Gamma(\alpha+m)\Gamma(k-m-\alpha)} \int_0^\infty t^{\alpha+m-1} A^{k-m}(A+tI)^{-k} x\, dt$$

is an A-convergent integral, independent of the choice of m and k. A^α_σ has the closure A^α that independent of σ. The domain $\mathfrak{D}^\alpha(A)$ of A^α we consider as a Banach space with the norm $\|x\|_{\mathfrak{D}^\alpha(A)} = \|A^\alpha x\|$, $x \in \mathfrak{D}^\alpha(A)$. A^α is a continuous operator for $\operatorname{Re}\alpha < 0$ and A^α is an isomorphic mapping from $\mathfrak{D}^\alpha(A)$ onto $\mathfrak{X}$ for $\operatorname{Re}\alpha > 0$ [13, Theorem 1.15.2]. $\mathfrak{D}^k(A)$ is dense in $\mathfrak{X}$ [13, Lemma 1.14.1]. A density of $\mathfrak{D}^\alpha(A)$ in $\mathfrak{X}$ is a consequence of $\mathfrak{D}^k(A) \subset \left(\mathfrak{X}, \mathfrak{D}^k(A)\right)_{\sigma/k,1} \subset \mathfrak{D}^\alpha(A) \subset \mathfrak{X}$. If $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$ and $\beta > 0$, then $A^\alpha A^\beta = A^\beta A^\alpha = A^{\alpha+\beta}$ and A^β is an isomorphic mapping from $\mathfrak{D}^{\alpha+\beta}(A)$ onto $\mathfrak{D}^\alpha(A)$ [13, Theorem 1.15.2]. So, $\mathfrak{D}^{\alpha+\beta}(A) \subset \mathfrak{D}^\alpha(A)$ and $\mathfrak{D}^\infty(A) \subset \bigcap_{\beta>0} \mathfrak{D}^{\alpha+\beta}(A) \subset \mathfrak{D}^\infty(A)$, where $\mathfrak{D}^\infty(A) := \bigcap_{k\in\mathbb{Z}_+} \mathfrak{D}^k(A)$.

For any $\nu > 0$ and $k \in \mathbb{Z}_+$ we put $x_{k,\nu} := (A/\nu)^k x$, $x \in \mathfrak{D}^\infty(A)$. Let $\{x^*_{k,\nu}\}_{k\in\mathbb{Z}_+}$ denotes the rearrangement of the elements by magnitude of the norms

$$\|x^*_{0,\nu}\|_{\mathfrak{D}^\alpha(A)} \ge \|x^*_{1,\nu}\|_{\mathfrak{D}^\alpha(A)} \ge \ldots \ge \|x^*_{k,\nu}\|_{\mathfrak{D}^\alpha(A)} \ge \ldots.$$

Let $1 < q < \infty$ and $1 \le p \le \infty$. Following [5], we introduce the spaces

$$\mathcal{E}^{\nu,\alpha}_{q,p}(A) := \mathcal{E}^{\nu}_{q,p}(\mathfrak{D}^\alpha(A), \mathfrak{X}) = \left\{x \in \mathfrak{D}^\infty(A) \colon \|x\|_{\mathcal{E}^{\nu,\alpha}_{q,p}(A)} < \infty\right\},$$

where

$$\|x\|_{\mathcal{E}^{\nu,\alpha}_{q,p}(A)} = \begin{cases} \left(\sum_{k\in\mathbb{N}} \|x^*_{k-1,\nu}\|^p_{\mathfrak{D}^\alpha(A)} k^{\frac{p}{q}-1}\right)^{1/p}, & 1 \le p < \infty, \\ \sup_{k\in\mathbb{N}} \|x^*_{k-1,\nu}\|_{\mathfrak{D}^\alpha(A)} k^{\frac{1}{q}}, & p = \infty. \end{cases}$$

If $q = p$, then $\mathcal{E}^{\nu,\alpha}_{q,q}(A) := \mathcal{E}^{\nu,\alpha}_{q}(A)$ and $\|x\|_{\mathcal{E}^{\nu,\alpha}_{q}(A)} = \left(\sum_{k\in\mathbb{Z}_+} \|x_{k,\nu}\|^q_{\mathfrak{D}^\alpha(A)}\right)^{1/q}$. In addition, we consider the spaces

$$\mathcal{E}^{\nu,\alpha}_{1}(A) = \Big\{x \in \mathfrak{D}^\infty(A) \colon \|x\|_{\mathcal{E}^{\nu,\alpha}_{1}(A)} = \sum_{k\in\mathbb{Z}_+} \|x_{k,\nu}\|_{\mathfrak{D}^\alpha(A)} < \infty\Big\},$$

$$\mathcal{E}^{\nu,\alpha}_{\infty}(A) = \Big\{x \in \mathfrak{D}^\infty(A) \colon \|x\|_{\mathcal{E}^{\nu,\alpha}_{\infty}(A)} = \sup_{k\in\mathbb{Z}_+} \|x_{k,\nu}\|_{\mathfrak{D}^\alpha(A)} < \infty\Big\}.$$

For $\alpha = 0$ we obtain the spaces $\mathcal{E}^{\nu}_{q}(A)$ of analytic vectors of A, that have been studied in [4]. Thus, the elements of $\mathcal{E}^{\nu,\alpha}_{q,p}(A)$ extend the class of such spaces. Next we give some of their properties.

Theorem 1. (i) *The following embedding $\mathcal{E}^{\nu,\alpha}_{q,p}(A) \subset \mathcal{E}^{\mu,\alpha}_{q,p}(A)$ with $\mu > \nu$ holds.*

(ii) *The restriction $A|_{\mathcal{E}^{\nu,\alpha}_{q,p}(A)}$ is a bounded operator in $\mathcal{E}^{\nu,\alpha}_{q,p}(A)$ with the norm $\|A|_{\mathcal{E}^{\nu,\alpha}_{q,p}(A)}\| \le \nu$.*

(iii) *Every space $\mathcal{E}^{\nu,\alpha}_{q,p}(A)$ is complete.*

Proof. (i) If $x \in \mathcal{E}^{\nu,\alpha}_{1}(A)$, then $x \in \mathcal{E}^{\nu,\alpha}_{\infty}(A)$, since

$$\|x\|_{\mathcal{E}^{\nu,\alpha}_{\infty}(A)} = \sup_{k\in\mathbb{Z}_+} \|x_{k,\nu}\|_{\mathfrak{D}^\alpha(A)} \le \sum_{k\in\mathbb{Z}_+} \|x_{k,\nu}\|_{\mathfrak{D}^\alpha(A)} = \|x\|_{\mathcal{E}^{\nu,\alpha}_{1}(A)}.$$

On the other hand, if $x \in \mathcal{E}^{\nu,\alpha}_{\infty}(A)$, then $\|A^k x\|_{\mathfrak{D}^\alpha(A)} \le \nu^k \|x\|_{\mathcal{E}^{\nu,\alpha}_{\infty}(A)}$ and $\|A^k x\|^{1/k}_{\mathfrak{D}^\alpha(A)} \le \nu\, \|x\|^{1/k}_{\mathcal{E}^{\nu,\alpha}_{\infty}(A)}$. So, $\limsup\limits_{k\to\infty} \|A^k x\|^{1/k}_{\mathfrak{D}^\alpha(A)} \le \nu$ and the series $\|x\|_{\mathcal{E}^{\mu,\alpha}_{1}(A)} = \sum_k \left\| \left(A/\mu\right)^k x\right\|_{\mathfrak{D}^\alpha(A)}$ is convergent for any $\mu > \nu$. As a result, $x \in \mathcal{E}^{\mu,\alpha}_{1}(A)$. Therefore, for any $\mu > \nu$ we have

$$\mathcal{E}^{\nu,\alpha}_{1}(A) \subset \mathcal{E}^{\nu,\alpha}_{\infty}(A) \subset \mathcal{E}^{\mu,\alpha}_{1}(A). \tag{1}$$

Replacing the 2-norm $\left(|x_0|^2_{\mathfrak{X}_0} + t^2\, |x_1|^2_{\mathfrak{X}_1}\right)^{1/2}$ by the 1-norm $|x_0|_{\mathfrak{X}_0} + t\, |x_1|_{\mathfrak{X}_1}$ in the definition of K-functional, we show that

$$\left(\mathcal{E}^{\nu,\alpha}_{1}(A), \mathcal{E}^{\nu,\alpha}_{\infty}(A)\right)_{\theta,r} = \mathcal{E}^{\nu,\alpha}_{1/(1-\theta),r}(A). \tag{2}$$

The space $\mathcal{E}^{\nu,\alpha}_{q}(A)$ is isometric to $l^{\nu,\alpha}_{q} = \left\{\bar{x} := \{\nu^{-k} A^k x\}_{k\in\mathbb{Z}_+} : \, x \in \mathcal{E}^{\nu,\alpha}_{q}(A)\right\}$ with the norm $\|\bar{x}\|_{l^{\nu,\alpha}_{q}} = \|x\|_{\mathcal{E}^{\nu,\alpha}_{q}(A)}$, as well as, the space $\mathcal{E}^{\nu,\alpha}_{q,p}(A)$ is isometric to $l^{\nu,\alpha}_{q,p} = \left\{\bar{x} := \{\nu^{-k} A^k x\}_{k\in\mathbb{Z}_+} : \right.$ $\left. x \in \mathcal{E}^{\nu,\alpha}_{q,p}(A)\right\}$ with the norm $\|\bar{x}\|_{l^{\nu,\alpha}_{q,p}} = \|x\|_{\mathcal{E}^{\nu,\alpha}_{q,p}(A)}$.

For $0 < \tau \le 1$ we have $K\big(\tau, x; l^{\nu,\alpha}_{1}, l^{\nu,\alpha}_{\infty}\big) = \tau \|x^*_{0,\nu}\|_{\mathfrak{D}^\alpha(A)}$, and for all $s \in \mathbb{N}$ we have $K\big(s, x; l^{\nu,\alpha}_{1}, l^{\nu,\alpha}_{\infty}\big) = \sum_{k=0}^{s-1} \|x^*_{k,\nu}\|_{\mathfrak{D}^\alpha(A)}$. For $1 \le r < \infty$, it follows that

$$\|\bar{x}\|^r_{\left(l^{\nu,\alpha}_{1}, l^{\nu,\alpha}_{\infty}\right)_{\theta,r}} \sim \sum_{s=1}^{\infty} s^{-\theta r - 1} \left(\sum_{k=1}^{s-1} \|x^*_{k,\nu}\|_{\mathfrak{D}^\alpha(A)} \right)^r \ge \sum_{s=1}^{\infty} s^{(1-\theta)r-1} \|x^*_{s-1,\nu}\|^r_{\mathfrak{D}^\alpha(A)},$$

$$\sum_{s=1}^{\infty} s^{-\theta r-1}\left(\sum_{k=1}^{s-1}\|x_{k,\nu}^{*}\|_{\mathfrak{D}^{\alpha}(A)}\right)^{r}\leq c\sum_{k=1}^{\infty}k^{(1-\theta)r-1}\|x_{k-1,\nu}^{*}\|_{\mathfrak{D}^{\alpha}(A)}^{r}.$$

Consequently,

$$\left(l_{1}^{\nu,\alpha},l_{\infty}^{\nu,\alpha}\right)_{\theta,r}=l_{1/(1-\theta),r}^{\nu,\alpha},$$

that is equivalent to (2). In the case $r=\infty$, we have

$$\|\bar{x}\|_{\left(l_{1}^{\nu,\alpha},l_{\infty}^{\nu,\alpha}\right)_{\theta,\infty}}\sim\sup_{s}s^{-\theta}\sum_{k=0}^{s-1}\|x_{k,\nu}^{*}\|_{\mathfrak{D}^{\alpha}(A)}\sim\sup_{s}s^{1-\theta}\|x_{s-1,\nu}^{*}\|_{\mathfrak{D}^{\alpha}(A)}.$$

By (1) and (2), it follows now that

$$\mathcal{E}_{q,p}^{\nu,\alpha}(A)\subset\mathcal{E}_{\infty}^{\nu,\alpha}(A)\subset\mathcal{E}_{1}^{\mu,\alpha}(A)\subset\mathcal{E}_{q,p}^{\mu,\alpha}(A)$$

for any $\mu>\nu$.

(ii) For $x\in\mathcal{E}_{q,p}^{\nu,\alpha}(A)$ and $1\leq p<\infty$ we have

$$\|Ax\|_{\mathcal{E}_{q,p}^{\nu,\alpha}(A)}^{p}=\nu^{p}\sum_{k\in\mathbb{Z}_{+}}(k+1)^{\frac{p}{q}-1}\|\,(A/\nu)^{k}x\|_{\mathfrak{D}^{\alpha}(A)}^{p}\leq\nu^{p}\|x\|_{\mathcal{E}_{q,p}^{\nu,\alpha}(A)}^{p},$$

with the modification when $p=\infty$,

$$\|Ax\|_{\mathcal{E}_{q,\infty}^{\nu,\alpha}(A)}=\nu\sup_{k\in\mathbb{Z}_{+}}(k+1)^{1/q}\|\,(A/\nu)^{k}x\|_{\mathfrak{D}^{\alpha}(A)}\leq\nu\|x\|_{\mathcal{E}_{q,\infty}^{\nu,\alpha}(A)}.$$

It follows the invariance and boundedness of $A|_{\mathcal{E}_{q,p}^{\nu,\alpha}(A)}$ in $\mathcal{E}_{q,p}^{\nu,\alpha}(A)$.

(iii) By [4, Lemma 1.1], we have the completeness of $\mathcal{E}_{q}^{\nu,\alpha}(A)$ for $q=1,\infty$. Now the space $\mathcal{E}_{q,p}^{\nu,\alpha}(A)$ is complete as interpolation space according to (2). □

3 ESTIMATES OF BEST APPROXIMATION ERRORS

In this section we give the estimates of best approximation errors by analytic vectors of positive operator A in a Banach space $\mathfrak{X}$. These estimates are expressed by the Berstein and Jackson type inequalities with exact values of constants.

Following [5], we will define the scale of approximation spaces $\mathcal{B}_{q,p,\tau}^{s,\alpha}(A)$ associated with the complex degrees of A. Let $\mathcal{E}_{q,p}^{\alpha}(A)=\bigcup_{\nu>0}\mathcal{E}_{q,p}^{\nu,\alpha}(A)$ be a subspace with the quasi-norm

$$|x|_{\mathcal{E}_{q,p}^{\alpha}(A)}=\|x\|+\inf\left\{\nu>0\colon x\in\mathcal{E}_{q,p}^{\nu,\alpha}(A)\right\},$$

so that $|x+y|_{\mathcal{E}_{q,p}^{\alpha}(A)}\leq|x|_{\mathcal{E}_{q,p}^{\alpha}(A)}+|y|_{\mathcal{E}_{q,p}^{\alpha}(A)}$ for all $x,y\in\mathcal{E}_{q,p}^{\alpha}(A)$.

For a pair indexes $\{0<s<\infty,0<\tau\leq\infty\}$ or $\{0\leq s<\infty,\tau=\infty\}$ we consider the spaces $\mathcal{B}_{q,p,\tau}^{s,\alpha}(A):=\mathcal{B}_{q,p,\tau}^{s}(\mathfrak{D}^{\alpha}(A),\mathfrak{X})=\{x\in\mathfrak{X}:|x|_{\mathcal{B}_{q,p,\tau}^{s,\alpha}(A)}<\infty\}$, where

$$|x|_{\mathcal{B}_{q,p,\tau}^{s,\alpha}(A)}=\begin{cases}\left(\int_{0}^{\infty}\left[t^{s}E(t,x;\mathcal{E}_{q,p}^{\alpha}(A),\mathfrak{X})\right]^{\tau}dt/t\right)^{1/\tau}, & 0<\tau<\infty,\\ \sup_{t>0}t^{s}E(t,x;\mathcal{E}_{q,p}^{\alpha}(A),\mathfrak{X}), & \tau=\infty,\end{cases}$$

and $E(t,x;\mathcal{E}_{q,p}^{\alpha}(A),\mathfrak{X})=\inf\left\{\|x-x^{0}\|\colon x^{0}\in\mathcal{E}_{q,p}^{\alpha}(A),\ |x^{0}|_{\mathcal{E}_{q,p}^{\alpha}(A)}\leq t\right\}$ for all $x\in\mathfrak{X}$.

Note that for $\alpha=0$ and $q=p$ we obtain the approximation spaces $\mathcal{B}_{q,\tau}^{s}(A)$, which were considered earlier in [4].

Theorem 2. *The following Bernstein-type inequality*

$$|x|_{\mathcal{B}^{s,\alpha}_{q,p,\tau}(A)} \leq c_{s,\tau}\,|x|^s_{\mathcal{E}^\alpha_{q,p}(A)}\|x\|, \quad x \in \mathcal{E}^\alpha_{q,p}(A), \tag{3}$$

holds with $c_{s,\tau} = (\tau^2(1+s))^{1/\tau}N^{-(1+s)}_{1/(1+s),\tau(1+s)}$ *if* $0<\tau<\infty$ *and* $c_{s,\infty} = N^{-(1+s)}_{1/(1+s),\infty}$.

Proof. Let $1 \leq r < \infty$ and $x \in \mathcal{E}^\alpha_{q,p}(A)$. We have

$$|x|^r_{(\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X})_{\theta,r}} \leq |x|^r_{\mathcal{E}^\alpha_{q,p}(A)}\|x\|^r \int_0^\infty \left[\frac{t^{1-\theta}}{(|x|^2_{\mathcal{E}^\alpha_{q,p}(A)} + t^2\|x\|^2)^{1/2}}\right]^r \frac{dt}{t} = N^{-r}_{\theta,r}|x|^{r(1-\theta)}_{\mathcal{E}^\alpha_{q,p}(A)}\|x\|^{r\theta}.$$

Similarly,

$$|x|_{(\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X})_{\theta,\infty}} \leq |x|^{1-\theta}_{\mathcal{E}^\alpha_{q,p}(A)}\|x\|^\theta \sup_{t>0}\frac{t^{1-\theta}}{\sqrt{1+t^2}} = N^{-1}_{\theta,\infty}|x|^{1-\theta}_{\mathcal{E}^\alpha_{q,p}(A)}\|x\|^\theta.$$

So, for $1 \leq r \leq \infty$, we have

$$|x|_{(\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X})_{\theta,r}} \leq N^{-1}_{\theta,r}|x|^{1-\theta}_{\mathcal{E}^\alpha_{q,p}(A)}\|x\|^\theta. \tag{4}$$

Let us define $K_\infty(t,x;\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X}) = \inf\limits_{x=x^0+x^1}\max\left(|x^0|_{\mathcal{E}^\alpha_{q,p}(A)}, t\|x^1\|\right)$. As follows from [12, Rem. 3.1]

$$K_\infty(t,x;\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X}) \leq K(t,x;\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X}) \leq \sqrt{2}K_\infty(t,x;\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X}). \tag{5}$$

Note that $v^{-\theta}K_\infty(v,x;\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X}) \to 0$ as $v \to 0$ or $v \to \infty$ and $t^sE(t,x;\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X}) \to 0$ as $t \to 0$ or $t \to \infty$. Thus

$$\begin{aligned}\int_0^\infty (v^{-\theta}K_\infty(v,x;\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X}))^r dv/v &= -\frac{1}{\theta r}\int_0^\infty K_\infty(v,x;\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X})^r dv^{-\theta r}\\ &= \frac{1}{\theta r}\int_0^\infty v^{-\theta r}dK_\infty(v,x;\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X})^r = \frac{1}{\theta r}\int_0^\infty (t/E(t,x;\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X}))^{-\theta r}dt^r\\ &= \frac{1}{\theta r^2}\int_0^\infty (t^sE(t,x;\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X}))^{\theta r}dt/t \quad \text{with} \quad s = 1/\theta - 1.\end{aligned}$$

Using (5), we have

$$\begin{aligned}\frac{1}{\theta r^2}|x|^{\theta r}_{\mathcal{B}^{s,\alpha}_{q,p,\tau}(A)} &= \frac{1}{\theta r^2}\int_0^\infty (t^sE(t,x;\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X}))^{\theta r}dt/t = \int_0^\infty (v^{-\theta}K_\infty(v,x;\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X}))^r dv/v\\ &\leq \int_0^\infty (v^{-\theta}K(v,x;\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X}))^r dv/v = |x|^r_{\left(\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X}\right)_{\theta,r}}.\end{aligned}$$

From the second inequality (5) it follows that

$$\begin{aligned}|x|^r_{\left(\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X}\right)_{\theta,r}} &\leq 2^{r/2}\int_0^\infty (v^{-\theta}K_\infty(v,x;\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X}))^r dv/v\\ &= 2^{r/2}\frac{1}{\theta r^2}\int_0^\infty (t^sE(t,x;\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X}))^{\theta r}dt/t = 2^{r/2}\frac{1}{\theta r^2}|x|^{\theta r}_{\mathcal{B}^{s,\alpha}_{q,p,\tau}(A)}.\end{aligned}$$

As a result, from the previous inequalities, we get

$$|x|^r_{\left(\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X}\right)_{\theta,r}} \leq 2^{r/2}(\theta r^2)^{-1}|x|^{\theta r}_{\mathcal{B}^{s,\alpha}_{q,p,\tau}(A)} \leq 2^{r/2}|x|^r_{\left(\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X}\right)_{\theta,r}} \quad \text{with} \quad \tau = \theta r. \tag{6}$$

We choose $t > 0$ according to [2, Lemma 7.1.2]. Then we get

$$t^s E(t, x; \mathcal{E}^\alpha_{q,p}(A), \mathfrak{X}))^\theta \le v^{-\theta} K_\infty(v, x; \mathcal{E}^\alpha_{q,p}(A), \mathfrak{X}) \le \Big(t^s E(t-0, x; \mathcal{E}^\alpha_{q,p}(A), \mathfrak{X})\Big)^\theta, \tag{7}$$

which gives $|x|^\theta_{\mathcal{B}^{s,\alpha}_{q,p,\infty}(A)} \le |x|_{(\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X})_{\theta,\infty}}$.

On the other hand, we have $|x|_{(\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X})_{\theta,\infty}} \le \sqrt{2}|x|^\theta_{\mathcal{B}^{s,\alpha}_{q,p,\infty}(A)}$, since

$$\nu^{-\theta} K(\nu, x; \mathcal{E}^\alpha_{q,p}(A), \mathfrak{X}) \le \sqrt{2}(t^s E(t-0, x; \mathcal{E}^\alpha_{q,p}(A), \mathfrak{X}))^\theta \le \sqrt{2}|x|^\theta_{\mathcal{B}^{s,\alpha}_{q,p,\infty}(A)}.$$

Now applying (4), we obtain

$$|x|^\theta_{\mathcal{B}^{s,\alpha}_{q,p,\tau}(A)} \le \begin{cases} (\theta r^2)^{1/r} N^{-1}_{\theta,r} |x|^{1-\theta}_{\mathcal{E}^\alpha_{q,p}(A)} \|x\|^\theta, & 1 \le r < \infty, \\ N^{-1}_{\theta,\infty} |x|^{1-\theta}_{\mathcal{E}^\alpha_{q,p}(A)} \|x\|^\theta, & r = \infty. \end{cases} \tag{8}$$

Setting $s = 1/\theta - 1$ and $\tau = \theta r$ in (8), we get the required inequality (3). ☐

Theorem 3. *The following Jackson-type inequality*

$$E(t, x; \mathcal{E}^\alpha_{q,p}(A), \mathfrak{X}) \le c_{s,\tau} t^{-s} |x|_{\mathcal{B}^{s,\alpha}_{q,p,\tau}(A)}, \quad x \in \mathcal{B}^{s,\alpha}_{q,p,\tau}(A), \tag{9}$$

holds with $c_{s,\tau} = 2^{(1+s)/2}(\tau^2(1+s))^{-1/\tau} N^{(1+s)}_{1/(1+s),\tau(1+s)}$ *if* $0 < \tau < \infty$ *and* $c_{s,\infty} = 1$.

Proof. Let us define the auxiliary function $g(v/t) = (v/t)(1 + (v/t)^2)^{-1/2}$, $t, v > 0$. By integration both sides of $g(v/t)K(t, x; \mathcal{E}^\alpha_{q,p}(A), \mathfrak{X}) \le K(v, x; \mathcal{E}^\alpha_{q,p}(A), \mathfrak{X})$, we get

$$\begin{aligned} \Big(\int_0^\infty \big(v^{-\theta} g(v/t)\big)^r \frac{dv}{v}\Big)^{1/r} K(t, x; \mathcal{E}^\alpha_{q,p}(A), \mathfrak{X}) &\le \Big(\int_0^\infty \big(v^{-\theta} K(v, x; \mathcal{E}^\alpha_{q,p}(A), \mathfrak{X})\big)^r \frac{dv}{v}\Big)^{1/r} \\ &= |x|_{(\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X})_{\theta,r}}, \\ \int_0^\infty \big(v^{-\theta} g(v/t)\big)^r \frac{dv}{v} &= (t^\theta N_{\theta,r})^{-r}. \end{aligned}$$

It follows that

$$K(t, x; \mathcal{E}^\alpha_{q,p}(A), \mathfrak{X}) \le t^\theta N_{\theta,r} |x|_{(\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X})_{\theta,r}}.$$

Taking into account (5), (7), we have

$$v^{1-\theta} E(v, x; \mathcal{E}^\alpha_{q,p}(A), \mathfrak{X})^\theta \le t^{-\theta} K_\infty(t, x; \mathcal{E}^\alpha_{q,p}(A), \mathfrak{X}) \le N_{\theta,r} |x|_{(\mathcal{E}^\alpha_{q,p}(A),\mathfrak{X})_{\theta,r}}.$$

Applying (6), we obtain $v^{1-\theta} E(v, x; \mathcal{E}^\alpha_{q,p}(A), \mathfrak{X})^\theta \le \sqrt{2}(\theta r^2)^{-1/r} N_{\theta,r} |x|^\theta_{\mathcal{B}^{s,\alpha}_{q,p,\tau}(A)}$. If $s = 1/\theta - 1$ and $\tau = \theta r$, we get (9) for $1 \le r < \infty$.

If $r = \infty$, then

$$t^s E(t, x; \mathcal{E}^\alpha_{q,p}(A), \mathfrak{X}) \le \sup_{t>0} t^s E(t, x; \mathcal{E}^\alpha_{q,p}(A), \mathfrak{X}) = |x|_{\mathcal{B}^{s,\alpha}_{q,p,\infty}(A)}$$

for all $x \in \mathcal{B}^{s,\alpha}_{q,p,\infty}(A)$. Thus, we obtain the required inequality (9). ☐

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Встановлено оцінки помилок наближень аналітичними векторами позитивного оператора A в банаховому просторі $\mathfrak{X}$. Основні результати сформульовані у вигляді нерівностей типу Бернштейна і Джексона з обчисленням значень констант. Розглянуто класи інваріантних підпросторів $\mathcal{E}^{\nu,\alpha}_{q,p}(A)$ аналітичних векторів оператора A та спеціальну шкалу апроксимаційних просторів $\mathcal{B}^{s,\alpha}_{q,p,\tau}(A)$, пов'язаних з комплексними степенями позитивного оператора. Апроксимаційні простори визначаються E-функціоналом, який відіграє подібну роль, як модуль гладкості. Показано, що апроксимаційні простори можна розглядати як інтерполяційні простори, породжені K-методом дійсної інтерполяції. Константи в нерівностях типу Бернштейна і Джексона виражаються через коефіцієнт нормалізації.

Ключові слова і фрази: позитивний оператор, апроксимаційний простір, нерівність типу Бернштейна і Джексона.