POWER OPERATIONS AND DIFFERENTIATIONS ASSOCIATED WITH
SUPERSYMMETRIC POLYNOMIALS ON A BANACH SPACE

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We consider different approaches to constructing power operations on the ring of multisets associated with supersymmetric polynomials of infinitely many variables. Some relations between constructed power operations are established. Also, we study differential operators on algebras of symmetric and supersymmetric analytic functions of bounded type on the Banach space of absolutely summable sequences. We have proved the continuity of such operators and found their evaluations on basis polynomials.

Key words and phrases: supersymmetric polynomial, analytic function on a Banach space, power operation, differential operator, ring of multisets.

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INTRODUCTION AND PRELIMINARIES

Let $X$ be a complex Banach space. A (continuous) map $P: X \to \mathbb{C}$ is said to be a (continuous) $n$-homogeneous polynomial if there exists a (continuous) $n$-linear map $B_P: X^n \to \mathbb{C}$ such that $P(x) = B_P(x, \ldots, x)$. Here 0-homogeneous polynomials are just constant functions. A finite sum of homogeneous polynomials is a polynomial. We denote by $\mathcal{P}(nX)$ the space of all continuous $n$-homogeneous polynomials on $X$ and by $\mathcal{P}(X)$ the algebra of all polynomials on $X$. It is well known that $\mathcal{P}(nX)$ is a Banach space with respect to any of the norms

$$ \|P\|_r = \sup_{\|x\| \leq r} |P(x)|, \quad r > 0. \quad (1) $$

Let us denote by $\tau_b$ the topology on $\mathcal{P}(X)$ generated by the countable family of norms (1) for positive rational numbers $r$. It is clear that $\tau_b$ is metrisable. We denote by $H_b(X)$ the completion of $(\mathcal{P}(X), \tau_b)$. So $H_b(X)$ is a Fréchet algebra which consists of entire analytic functions on $X$ which are bounded on all bounded subsets (so-called entire functions of bounded type). For details on polynomials and analytic functions on Banach spaces we refer the reader to [12]. The spectra (sets of continuous complex homomorphisms = sets of characters) of $H_b(X)$ and its subalgebras were investigated by many authors (see e.g., [1–3, 9, 23, 24]). For every $x \in X$, point evaluation functional $\delta_x: f \mapsto f(x)$, $f \in H_b(X)$, belongs to the spectrum of $H_b(X)$.

A sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ is algebraically independent if for every nonzero polynomial $q$ of $m$ variables with $q(0) = 0$ and any finite subsequence $\{P_{n_1}, \ldots, P_{n_m}\}$, $m \in \mathbb{N}$, the
polynomial \( q(P_n(x), \ldots, P_{m}(x)) \) is nonzero. An algebraically independent sequence of polynomials \( \{P_n\}_{n=1}^{\infty} \) is an algebraic basis of a subalgebra \( A \subseteq P(X) \) if every element in \( A \) can be represented as an algebraic span of a finite number of polynomials from \( \{P_n\}_{n=1}^{\infty} \).

Let \( X = \ell_1 \) and \( G \) be the group of permutations of basis vectors. A function \( f \) on \( \ell_1 \) is said to be symmetric if it is invariant with respect to all permutations in \( G \). We denote by \( P_s(\ell_1) \) the algebra of all continuous symmetric polynomials on \( \ell_1 \) and by \( H_{bs}(\ell_1) \) its closure in \( H_b(\ell_1) \).

There are a lot of algebraic bases in \( P_s(\ell_1) \). The following bases in the algebra of symmetric polynomials are interesting for us:

- the basis of power series:
  \[
  F_k(x) = \sum_{n=1}^{\infty} x_n^k,
  \]

- the basis of elementary symmetric polynomials:
  \[
  G_k(x) = \sum_{n_1 < n_2 < \cdots < n_k} x_{n_1} \cdots x_{n_k};
  \]

- complete symmetric polynomials:
  \[
  H_k(x) = \sum_{n_1 \leq n_2 \leq \cdots \leq n_k} x_{n_1} \cdots x_{n_k}.
  \]

Here \( x = (x_1, x_2, \ldots, x_n, \ldots) \in \ell_1 \). Note that basic properties of classical symmetric polynomials on finite-dimensional spaces are still true for symmetric polynomials on \( \ell_1 \). In particular, the well known Newton formulas hold

\[
\begin{align*}
 mh_m &= H_{m-1}F_1 + H_{m-2}F_2 + \cdots + H_{m-k}F_k + \cdots + F_m, \\
 mg_m &= G_{m-1}F_1 - G_{m-2}F_2 + \cdots + (-1)^{k+1}G_{m-k}F_k + \cdots + (-1)^{m+1}F_m.
\end{align*}
\]

Algebras of symmetric analytic functions with respect to various symmetry groups or subgroups of operators on Banach spaces were studied in [1, 4–11, 13–19, 21].

Denote \( Z_0 = Z \setminus \{0\} \). Let \( \ell_1(Z_0) = \ell_1 \oplus \ell_1 \) be the Banach space of all absolutely summing complex sequences indexed by numbers in \( Z_0 \). Any element \( z \) in \( \ell_1(Z_0) \) has the representation

\[
z = (\ldots, z_{-n}, \ldots, z_{-2}, z_{-1}, z_1, z_2, \ldots, z_n, \ldots) = (y|x) = (\ldots, y_n, \ldots, y_2, y_1|x_1, x_2, \ldots, x_n, \ldots)
\]

with

\[
\|z\| = \sum_{i=-\infty}^{\infty} |z_i|,
\]

where \( x = (x_1, x_2, \ldots, x_n, \ldots) \) and \( y = (y_1, y_2, \ldots, y_n, \ldots) \) are in \( \ell_1 \), \( z_n = x_n, z_{-n} = y_n \) for \( n \in \mathbb{N} \) and

\[
x \mapsto (0|x_1, x_2, \ldots, x_n, \ldots) \quad \text{and} \quad y \mapsto (\ldots, y_{-n}, \ldots, y_{-2}, y_{-1}|0)
\]

are natural isometric embeddings of the copies of \( \ell_1 \) into \( \ell_1(Z_0) \).

Let us define the following polynomials on \( \ell_1(Z_0) \):

\[
T_k(z) = F_k(x) - F_k(y) = \sum_{i=1}^{\infty} x_i^k - \sum_{i=1}^{\infty} y_i^k, \quad k \in \mathbb{N}.
\]
A polynomial $P$ on $\ell_1(\mathbb{Z}_0)$ is said to be *supersymmetric* if it can be represented as an algebraic combination of polynomials $\{T_k\}_{k=1}^\infty$. In other words, $P$ is a finite sum of finite products of polynomials in $\{T_k\}_{k=1}^\infty$ and constants. We denote by $\mathcal{P}_{sup} = \mathcal{P}_{sup}(\ell_1(\mathbb{Z}_0))$ the algebra of all supersymmetric polynomials on $\ell_1(\mathbb{Z}_0)$. Let us denote by $H_b^{sup}$ the closure of $\mathcal{P}_{sup}$ in $H_b(\ell_1(\mathbb{Z}_0))$.

Note that polynomials $T_k$ are algebraically independent because $F_k$ are so. Hence $\{T_k\}_{k=1}^\infty$ forms an algebraic basis in $\mathcal{P}_{sup}$. In [20] it was proved that

$$W_n(y|x) = \sum_{k=0}^n G_k(x) H_{n-k}(-y), \quad n \in \mathbb{N},$$

is another algebraic basis. From [20] it follows that the semigroup of symmetry of $\mathcal{P}_{sup}$ is a minimal semigroup that contains all permutations of coordinates on $(0|x_1, \ldots, x_n,\ldots)$, all permutations of coordinates on $(\ldots, y_n, \ldots, y_1|0)$ and operators of the form:

$$A_\lambda: (\ldots, y_n, \ldots, y_1|x_1, \ldots, x_n,\ldots) \mapsto (\ldots, y_n, \ldots, y_1, \lambda|\lambda, x_1, \ldots, x_n,\ldots), \quad \lambda \in \mathbb{C}.$$  

We say that $z \sim w$ for some $z, w \in \ell_1(\mathbb{Z}_0)$ if $T_k(z) = T_k(w)$ for every $k \in \mathbb{N}$. Let us denote by $\mathcal{M}$ the quotient set $\ell_1(\mathbb{Z}_0)/\sim$ which is a natural domain for supersymmetric polynomials. For a given $z \in \ell_1(\mathbb{Z}_0)$, let $[z] \in \mathcal{M}$ be the class of equivalence which contains $z$. Also, we denote by $\mathcal{M}_+$ the subset of $\mathcal{M}$ consisting of elements $[\{(0|x_1, x_2, \ldots\}]$, $\mathcal{M}_+$ can be considered as a set of multisets. It is clear that every function $f \in H_b^{sup}$ is well-defined on $\mathcal{M}$ and we will write $f(u) = f(y|x)$ for $u = [(y|x)] \in \mathcal{M}$. Note that $z \sim w$ if and only if $\delta_z = \delta_w$. Thus $\mathcal{M}$ can be embedded into the spectrum of $H_b^{sup}$.

In [20] algebraic operations “•” and “◦” on $\ell_1(\mathbb{Z}_0)$ were introduced and extended to $\mathcal{M}$. Let $z = (y|x)$ and $w = (d|b)$ be in $\ell_1(\mathbb{Z}_0)$. Then

$$z \circ w = (y \circ d|x \circ b) = (\ldots, d_n, y_n, \ldots, d_1, y_1|x_1, b_1, \ldots, x_n, b_n, \ldots).$$

Also, we denote $z^\sim = (y|x)^\sim = (x|y)$. Clearly, $(z^\sim)^\sim = z$ and $z \circ z^\sim \sim (0|0)$. These operations can be naturally defined on $\mathcal{M}$ by $[z] \circ [w] = [z \circ w]$ and $[z]^\sim = [z^\sim]$.

Let $x, y \in \ell_1$. By $x \circ y$ (see [11]) we mean the resulting sequence of ordering the set $\{x_iy_j: i, j \in \mathbb{N}\}$ with one single index in some fixed order. If $u = [(0|x)]$ and $v = [(0|y)]$, then $u \circ v = [(0|x \circ y)]$. Let $u = [(y|x)]$ and $v = [(d|b)]$ be in $\mathcal{M}$. We define

$$u \circ v = [((y \circ b) | (x \circ d) | (y \circ d) | (x \circ b))].$$

In [20] it is proved that $(\mathcal{M}, \circ, \triangleright)$ is a commutative ring (so-called the ring of multisets) with zero $0 = [(0|0)]$ and unity $I = [(0|1, 0, \ldots)]$, and $T_k$, $k \in \mathbb{N}$, are ring homomorphisms from $\mathcal{M}$ to $\mathbb{C}$. Note that $(\mathcal{M}, \circ, \triangleright)$ is not a linear space but in [20] it was introduced a “norm” $\| \cdot \|$ on $\mathcal{M}$ satisfying natural conditions by

$$\|u\| = \inf \left\{ \sum_i |x_i| + \sum_j |y_j|: (y|x) \in u \right\}.$$  

Using the norm, in [20] it was introduced a metric $\rho(u, v) = \|u \circ v\|$ and proved that $(\mathcal{M}, \rho)$ is a complete metric space.

In the first section we consider different approaches to construct power operations on $\mathcal{M}$. In Section 2 we study some operators of differentiation on $H_b(\ell_1)$ and $H_b^{sup}$ associated with the symmetric shift $x \mapsto x \circ a$. For combinatorial theory of symmetric polynomials we refer the reader to [22].
1 Power functions

Let $u_1, u_2, \ldots, u_n \in \mathcal{M}$. We will use notation

$$n \bigodot_{k=1}^n u_k \equiv u_1 \bullet u_2 \cdots \bullet u_n.$$ 

Using the operation “$\circ$” we can consider for a given $k \in \mathbb{N}$ a natural power function on $\mathcal{M}$

$$u \mapsto u^k = \underbrace{u \circ \cdots \circ u}_{k}.$$ 

However there are different ways to introduce power function on $\mathcal{M}$ or $\mathcal{M}_+$. Let us define

$$u^{[k]} = [(\ldots, y_{n^k}, \ldots, y_1^k x_1^{k}, \ldots, x_{n^k}^{k}, \ldots)], \quad u = [(y|x)] \in \mathcal{M}, \quad k \in \mathbb{N}.$$ 

If $u = [(0|x)] \in \mathcal{M}_+$, then

$$u^{(k)} = \bigodot_{i_1 < \cdots < i_k} [(0|x_{i_1} \cdots x_{i_k})] = [(0|x_1 x_2 \cdots x_k, x_2 x_3 \cdots x_{k+1}, \ldots)].$$

Finally, for $u = [(0|x)] \in \mathcal{M}_+$, we define $u^{[\overline{k}]} = \bigodot_{i_1 \leq \cdots \leq i_k} [(0|x_{i_1} \cdots x_{i_k})].$

Clearly that

$$T_1(u^{[k]}) = (T_1(u))^k, \quad T_1(u^{[k]} = T_k(u)$$

and

$$T_1([(0|x)]^{(k)}) = G_k(x), \quad T_1([(0|x)]^{[\overline{k}]) = H_k(x).$$

For the general case $u = [(y|x)]$ we set

$$u^{(k)} = \bigodot_{m+n=k} \bigodot_{i_1 \leq \cdots \leq i_m \ \text{if} \ m \text{ is odd}} \bigodot_{j_1 < \cdots < j_n \ \text{if} \ m \text{ is even}} [(y_{i_1} \cdots y_{i_m} x_{j_1} \cdots x_{j_n}, x_{i_1} \cdots x_{i_m}, x_{j_1} \cdots x_{j_n})].$$

**Proposition 1.** Let $u = [(y|x)] \in \mathcal{M}$. Then $T_1(u^{(k)}) = W_k(u)$.

**Proof.** Using (3), we can see that

$$T_1(u^{(k)}) = \sum_{n+m=k, m=2s} H_m(y)G_n(x) - \sum_{n+m=k, m=2s+1} H_m(y)G_n(x)$$

$$= \sum_{n+m=k} H_m(-y)G_n(x) = W_k(u).$$

**Theorem 1.** For every $k \in \mathbb{N}$, power operations $u \mapsto u^{\circ k}$, $u \mapsto u^{[k]}$, $u \mapsto u^{(k)}$ are well-defined on $\mathcal{M}$ and continuous.

**Proof.** Let us suppose that $u = [(y|x)]$, $v = [(d|b)]$ and $(y|x) \sim (d|b)$. Then

$$T_n(u^{\circ k}) = (T_n(u))^k = (T_n(v))^k = T_n(v^{\circ k}), \quad n \in \mathbb{N},$$
that is, \( u^o_k = v^o_k \). The continuity of \( u \mapsto u^o_k \) is proved in [20]. By the similar reason,

\[
T_n(u^{[k]}) = T_{nk}(u) = T_{nk}(v) = T_n(v^{[k]}), \quad n \in \mathbb{N},
\]

and so \( u^{[k]} = v^{[k]} \).

Finally,

\[
T_n(u^{(k)}) = T_1\left((u^{(k)})^{[n]}\right) = T_1\left((u^{[n]})^{(k)}\right) = W_k(u^{[n]}) = W_k(v^{[n]}) = T_n(v^{(k)}).
\]

The continuity of \( u \mapsto u^{[k]} \) and \( u \mapsto u^{(k)} \) follows from the simple fact that if \( \|u_j\| \to 0 \), then \( \|u^{[k]}\| \to 0 \) and \( \|u^{(k)}\| \to 0 \) as \( j \to \infty \).

In [20] it was proved that there exists a continuous homomorphism \( \Phi : H^\sup_{bs} \to H_{bs}(\ell_1) \) with a dense range such that \( \Phi(T_n) = F_n \) and \( \Phi(P_n) = G_n \). Thus the restriction of the inverse map \( \Lambda = \Phi^{-1} \) on the subspace of supersymmetric polynomials \( P_{sup} \) is an algebra isomorphism from \( P_{sup} \) onto \( P_s(\ell_1) \). Applying \( \Lambda \) to the Newton formula, we have

\[
kW_k(u) = \sum_{i=1}^{k} (-1)^{i+1} T_i(u)W_{k-i}(u), \quad u \in M, \quad k \in \mathbb{N}.
\]

**Proposition 2.** For every \( k \in \mathbb{N} \) and \( u \in M \) the following equality holds

\[
\left(\frac{u^{(k)}}{k}\right) = \bigcap_{i=1}^{k} [(1|0)]^{(i+1)} \circ u^{[i]} \circ u^{(k-i)}.
\]

**Proof.** Using (4), we can write

\[
T_n\left(\frac{u^{(k)}}{k}\right) = kT_n\left(u^{(k)}\right) = kW_k(u^{[n]}) = \sum_{i=1}^{k} (-1)^{i+1} T_i(u^{[n]})W_{k-i}(u^{[n]})
\]

\[
= \sum_{i=1}^{k} (-1)^{i+1} T_n(u^{[i]})T_n(u^{(k-i)})
= T_n\left(\bigcap_{i=1}^{k} [(1|0)]^{(i+1)} \circ u^{[i]} \circ u^{(k-i)}\right).
\]

Since it is true for every \( n \in \mathbb{N} \), we have our equality.

Let us denote by \( \nu : \mathbb{Z} \to M \) the following mapping: \( \nu(n) = [(0|1, \ldots, 1, 0, \ldots)] \) if \( n \geq 0 \) and \( \nu(n) = [(\ldots, 0, 1, \ldots, 1|0)] \) if \( n < 0 \). Clearly, \( \nu \) is a ring homomorphism. Let

\[
q(t) = \sum_{k=0}^{m} n_k t^k
\]

be a polynomial with integer coefficients. For a given \( u \in M \) we consider the following transformations from the set \( \mathbb{Z}[t] \) of polynomials with integer coefficients to \( M \):

\[
q_\varphi(u) = \bigcap_{k=0}^{m} \nu(n_k) u^{o_k};
\]
$$q_{[i]}(u) = \bigcirc_{k=0}^{m} v(n_k)u^{[k]};$$

$$q_{\langle \cdot \rangle}(u) = \bigcirc_{k=0}^{m} v(n_k)u^{(k)}.$$ 

From the proved properties of the power operations we have the following proposition.

**Proposition 3.** All mappings $q \mapsto q_{\ast}$, $q \mapsto q_{[i]}$ and $q \mapsto q_{\langle \cdot \rangle}$ are additive maps and $q \mapsto q_{\ast}$ is a multiplicative map between the rings $\mathbb{Z}[t]$ and $\mathcal{M}$.

2. **Derivatives**

Let us consider the following operator on $H_{bs}$ (c.f. [8])

$$\partial f(x) = \lim_{t \to 0} \frac{f(x \bullet (t, 0, 0, \ldots)) - f(x)}{t}.$$ 

Note first that $\partial$ is linear and satisfies the Leibnitz rule for differentiation

$$\partial (fg) = \partial f g + f \partial g$$

for all $f, g$ in the domain of $\partial$. Also, $\partial$ is well defined on polynomials in $H_{bs}$. Indeed

$$F_m(x \bullet (t, 0, 0, \ldots)) - F_m(x) = F_m((t, 0, 0, \ldots)) = t^m.$$ 

Thus $\partial F_1 = 1$ and $\partial F_m(x) = 0$ for $m > 0$. Since polynomials $F_k$, $k \in \mathbb{N}$, form an algebraic basis in the algebra of all symmetric polynomials $P_s(\ell_1)$, operator $\partial$ is defined on $P_s(\ell_1)$ and $\partial P \in P_s(\ell_1)$ for every $P \in P_s(\ell_1)$.

**Proposition 4.** The differential operator $\partial$ is continuous on $H_{bs}(\ell_1)$.

**Proof.** Let $A_0 : \ell_1 \to \ell_1$ be the forward shift operator on $\ell_1$,

$$A_0(x_1, \ldots x_n, \ldots) = (0, x_1, \ldots x_n, \ldots)$$

and $d_1$ be the Gâteaux derivative in direction $(1, 0, 0 \ldots)$ on $H_b(\ell_1)$. Then $\partial$ is the restriction to $H_{bs}(\ell_1)$ of the continuous operator $d_1 \circ C_{A_0}$, where $C_{A_0}(f)(x) = f(A_0(x))$, $f \in H_b(\ell_1)$. Thus $\partial$ is continuous on $H_{bs}(\ell_1)$.

**Proposition 5.** For every $m \in \mathbb{N}$ we have $\partial G_m = G_{m-1}$ and $\partial H_m = H_{m-1}$.

**Proof.** Since

$$G_m(x \bullet z) = \sum_{k+n=m} G_k(x)G_m(z), \quad x, z \in \ell_1,$$

we have

$$G_m(x \bullet (t, 0, 0, \ldots)) = \sum_{k=0}^{m} t^k G_k(x)$$

and so $\partial G_m = G_{m-1}$. The second equality can be obtained by the similar way or using Newton formulas (2) and simple induction.
The differential operator $\partial$ can be extended to the supersymmetric analytic functions. Let $I = [0|1]$ and $I^- = [1|0]$. We set

$$\partial_+ f(u) = \lim_{t \to 0} \frac{f(u \cdot tI) - f(u)}{t} \quad \text{and} \quad \partial_- f(u) = \lim_{t \to 0} \frac{f(u \cdot tI^-) - f(u)}{t},$$

where $u \in M$. Similarly to the symmetric case, both $\partial_+$ and $\partial_-$ are continuous and well-defined on the whole space $H^\text{sup}_b$.

**Theorem 2.** For every $m \in \mathbb{N}$

$$\partial_+ W_m = W_{m-1} \quad \text{and} \quad \partial_- W_m = -W_{m-1},$$

where $W_0 = 1$.

**Proof.** According to (3), we can write

$$\partial_+ W_m(y|x) = \partial_+ \sum_{k=0}^{n} G_k(x) H_{m-k}(-y) = \sum_{k=0}^{n} G_{k+1}(x) H_{m-k}(-y)$$

$$= \sum_{k=0}^{m-1} G_k(x) H_{m-k-1}(-y) = W_{m-1}.$$  

The second equality can be proved by the same way, taking into account that $H_{m-k}(-y) = (-1)^{m-k} H_{m-k}(y)$.

**References**


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