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Properties of analytic solutions of three similar differential equations of the second order

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An analytic univalent in $\mathbb{D} = \{z : |z| < 1\}$ function $f(z)$ is said to be convex if $f(\mathbb{D})$ is a convex domain. It is well known that the condition $\mathrm{Re}\,\{1 + zf''(z)/f'(z)\} > 0$, $z \in \mathbb{D}$, is necessary and sufficient for the convexity of f. The function f is said to be close-to-convex in $\mathbb{D}$ if there exists a convex in $\mathbb{D}$ function Φ such that $\mathrm{Re}\,(f'(z)/\Phi'(z)) > 0$, $z \in \mathbb{D}$.

S.M. Shah indicated conditions on real parameters β_0, β_1, γ_0, γ_1, γ_2 of the differential equation $z^2w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0$, under which there exists an entire transcendental solution f such that f and all its derivatives are close-to-convex in $\mathbb{D}$.

Let $0 < R \le +\infty$, $\mathbb{D}_R = \{z : |z| < R\}$ and l be a positive continuous function on $[0, R)$, which satisfies $(R - r)l(r) > C$, $C = \mathrm{const} > 1$. An analytic in $\mathbb{D}_R$ function f is said to be of bounded l-index if there exists $N \in \mathbb{Z}_+$ such that for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{D}_R$

$$\frac{|f^{(n)}(z)|}{n!l^n(|z|)} \le \max\left\{\frac{|f^{(k)}(z)|}{k!l^k(|z|)} : 0 \le k \le N\right\}.$$

Here we investigate close-to-convexity and the boundedness of the l-index for analytic in $\mathbb{D}$ solutions of three analogues of Shah differential equation: $z(z-1)w'' + \beta z w' + \gamma w = 0$, $(z-1)^2 w'' + \beta z w' + \gamma w = 0$ and $(1-z)^3 w'' + \beta(1-z)w' + \gamma w = 0$. Despite the similarity of these equations, their solutions have different properties.

Key words and phrases: close-to-convexity, l-index, differential equation.

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Introduction

An analytic univalent in $\mathbb{D} = \{z : |z| < 1\}$ function

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \tag{1}$$

is said to be convex if $f(\mathbb{D})$ is a convex domain. It is well known [4, p. 203] that the condition $\mathrm{Re}\,\{1 + zf''(z)/f'(z)\} > 0$, $z \in \mathbb{D}$, is necessary and sufficient for the convexity of f. By W. Kaplan [6], the function f is said to be close-to-convex in $\mathbb{D}$ (see also [4, p. 583]) if there exists a convex in $\mathbb{D}$ function Φ such that $\mathrm{Re}\,(f'(z)/\Phi'(z)) > 0$, $z \in \mathbb{D}$. Close-to-convex function f has a characteristic property that the complement G of the domain $f(\mathbb{D})$ can be filled with rays which start from ∂G and lie in G. Every close-to-convex in $\mathbb{D}$ function f is univalent in $\mathbb{D}$ and, therefore, $f'(0) \ne 0$. Hence it follows that the function f is close-to-convex

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in $\mathbb{D}$ if and only if the function

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \tag{2}$$

is close-to-convex in $\mathbb{D}$, where $g_n = f_n/f_1$. We remark also, that the function (2) is said to be starlike in $\mathbb{D}$, if $f(\mathbb{D})$ is starlike domain regarding the origin. It is clear, that every starlike function is close-to-convex.

S.M. Shah [8] indicated conditions on real parameters β_0, β_1, γ_0, γ_1, γ_2 of the differential equation

$$z^2 w'' + (\beta_0 z^2 + \beta_1 z) w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2) w = 0, \tag{3}$$

under which there exists an entire transcendental solution (1) such that f and all its derivatives are close-to-convex in $\mathbb{D}$. In particular he obtained the following result.

Theorem 1. *If $-1 \le \beta_0 < 0$, $\beta_1 > 0$ and $\beta_1 + \gamma_2 = \gamma_0 = \gamma_1 = 0$, then the equation* (3) *has an entire solution* (2) *such that all the derivatives $g^{(n)}$, $n \ge 0$, are close-to-convex in $\mathbb{D}$ and $\ln M_g(r) = (1 + o(1))|\beta_0| r$ as $r \to +\infty$, where $M_g(r) = \max\{|g(z)| : |z| = r\}$.*

The investigations are continued in papers [13–18]. In particular in the case of complex parameters β_0, β_1, γ_0, γ_1, γ_2 in [17] it is proved, that if $\gamma_0 = \gamma_1 = \beta_1 + \gamma_2 = 0$, $\beta_0 \ne 0$, $|\beta_1| < 2$ and $2|\beta_1| < (2 - |\beta_1|) \ln 2$, then the equation (3) has an entire solution (2) such that all derivatives $g^{(n)}$, $n \ge 0$, are starlike, thus close-to-convex in $\mathbb{D}$ and $\ln M_g(r) = (1 + o(1))|\beta_0| r$ as $r \to +\infty$. An analogue of this proposition for convex functions is obtained in [18], where it is proved, that if $\gamma_0 = \gamma_1 = \beta_1 + \gamma_2 = 0$, $\beta_0 \ne 0$, $|\beta_1| < 2$ and $4|\beta_1| < (2 - |\beta_1|) \ln 2$, then the equation (3) has an entire solution (2) such that all derivatives $g^{(n)}$, $n \ge 0$, are convex in $\mathbb{D}$. We remark that in this case the differential equation (3) has the form

$$z^2 w'' + (\beta_0 z^2 + \beta_1 z) w' + \gamma_2 w = 0, \quad \gamma_2 = -\beta_1. \tag{4}$$

Let $0 < R \le +\infty$, $\mathbb{D}_R = \{z : |z| < R\}$ and l be a positive continuous function on $[0, R)$, which satisfies $l(r) > \beta/(R - r)$, $\beta = \text{const} > 1$. An analytic in $\mathbb{D}_R$ function f is said to be of bounded l-index [9] if there exists $N \in \mathbb{Z}_+$ such that for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{D}_R$

$$\frac{|f^{(n)}(z)|}{n! l^n(|z|)} \le \max\left\{\frac{|f^{(k)}(z)|}{k! l^k(|z|)} : 0 \le k \le N\right\}. \tag{5}$$

The least such integer is called the l-index of f and is denoted by $N(f; l)$.

If there exists $N \in \mathbb{Z}_+$ such that (5) holds for all $n \in \mathbb{Z}_+$ and for all $z \in G \subset \mathbb{D}_R$, then the function f is said to be of bounded l-index on (or in) G, and the l-index is denoted by $N(f; l, G)$. The l-index boundedness of entire solutions of the equation (3) for certain conditions on parameters β_0, β_1, γ_0, γ_1, γ_2 are studied in [19–24].

Some results from [13–24] are published also in monograph [10].

Here we investigate the properties of analytic in $\mathbb{D}$ solutions of the following analogues of the differential equation (4):

$$\begin{aligned}
&\mathrm{DE}_1 : \ z(z-1) w'' + \beta z w' + \gamma w = 0, \\
&\mathrm{DE}_2 : \ (z-1)^2 w'' + \beta z w' + \gamma w = 0, \\
&\mathrm{DE}_3 : \ (1-z)^3 w'' + \beta(1-z) w' + \gamma w = 0.
\end{aligned}$$

1 Close-to-convexity and growth

Clearly, an analytic in $\mathbb{D}$ function (1) is a solution of DE_1 if and only if

$$z^2 \sum_{n=2}^{\infty} n(n-1) f_n z^{n-2} - z \sum_{n=2}^{\infty} n(n-1) f_n z^{n-2} + \beta z \sum_{n=1}^{\infty} n f_n z^{n-1} + \gamma \sum_{n=0}^{\infty} f_n z^n \equiv 0,$$

that is

$$\sum_{n=2}^{\infty} n(n-1) f_n z^n - \sum_{n=1}^{\infty} n(n+1) f_{n+1} z^n + \sum_{n=1}^{\infty} \beta n f_n z^n + \sum_{n=0}^{\infty} \gamma f_n z^n \equiv 0,$$

whence $\gamma f_0 = 0$ and

$$f_{n+1} = \frac{n(n+\beta-1)+\gamma}{n(n+1)} f_n, \quad n \geq 1. \tag{6}$$

We choose $f_0 = 0$ and $f_1 = 1$. Then the solution of DE_1 has the form

$$f(z) = z + \sum_{n=2}^{\infty} f_n z^n, \tag{7}$$

where the coefficients f_n for $n \geq 2$ are defined by recurrent formula (6).

For the investigation of the close-to-convexity of solution (7) we need the following result of J.F. Alexander [1] (see also [5, p. 9]).

Lemma 1. *If the coefficients of the function* (2) *satisfy the condition*

$$1 \geq 2g_2 \geq \cdots \geq k g_k \geq (k+1) g_{k+1} \geq \cdots > 0,$$

then the function g is close-to-convex in $\mathbb{D}$.

For the use of Lemma 1 it is needed the positiveness of coefficients f_n, and therefore, as in [8], we consider the real parameters β and γ. Since $f_0 = 0$, $f_1 = 1$, from (6) we have $2f_2 = \beta + \gamma$ and, therefore, $1 \geq 2f_2 > 0$ if and only if $0 < \beta + \gamma \leq 1$.

Theorem 2. *If $-1 \leq \beta \leq 1$ and $0 < \beta + \gamma \leq 1$, then DE_1 has an analytic close-to-convex in $\mathbb{D}$ solution* (7)*, for which $M_f(r) \asymp 1$ $(r \uparrow 1)$ if $\beta < 1$ and $M_f(r) \asymp 1/(1-r)$ $(r \uparrow 1)$ if $\beta = 1$.*

Proof. Since $\beta \geq -1$, the sequence $(n(n+\beta-1)+\gamma)$ is nondecreasing, thus we have $n(n+\beta-1)+\gamma \geq \beta+\gamma > 0$ for all $n \geq 1$, and since $\beta \leq 1$, we have $n(n+\beta-1)+\gamma \leq n^2$ for all $n \geq 2$. Therefore, (6) implies $f_n > 0$ for all $n \geq 2$ and

$$(n+1) f_{n+1} = \frac{n(n+\beta-1)+\gamma}{n^2} n f_n \leq n f_n.$$

From (6) it follows also that $f_{n+1} = (1+o(1)) f_n$ as $n \to \infty$, that is the radius of the convergence of (7) is equal to 1. Therefore, in view of Lemma 1 the first part of the theorem is proved.

Further, since $f_1 = 1$, (6) implies

$$f_{n+1} = \prod_{j=1}^{n} \frac{j(j+\beta-1)+\gamma}{j(j+1)}, \quad n \geq 1,$$

whence

$$\begin{aligned}\ln f_{n+1} &= \sum_{j=1}^{n} \ln\left(1+\frac{j(j+\beta-1)+\gamma-j(j+1)}{j(j+1)}\right) = \sum_{j=1}^{n} \ln\left(1+\frac{\beta-2}{j+1}+\frac{\gamma}{j(j+1)}\right)\\ &= \sum_{j=1}^{n}\left(\frac{\beta-2}{j+1}+\frac{\gamma}{j(j+1)}-\frac{(\beta-2)^2}{2(j+1)^2}-\frac{(\beta-2)\gamma}{j(j+1)^2}-\frac{\gamma^2}{2j^2(j+1)^2}+\dots\right)\\ &= \sum_{j=1}^{n}\frac{\beta-2}{j+1}+O(1) = (\beta-2)\ln(n+1)+O(1), \quad n\to\infty,\end{aligned}$$

i.e. $f_n \asymp n^{\beta-2}$ as $n \to \infty$. Therefore, if $\beta < 1$, then $f(r) = O(1)$ as $r \uparrow 1$ and if $\beta = 1$, then $f(r) \asymp \ln(1/(1-r))$ as $r \uparrow 1$. Since $M_f(r) = f(r)$, the proof of Theorem 2 is completed. □

We remark that if $\beta = 1$ and $\gamma = 0$, then DE$_1$ has the form $(z-1)w'' + w' = 0$ and has the solution $f(z) = \ln(1/(1-z))$ such that $f(0) = 0$ and $f'(0) = 1$. This function is convex and, thus, close-to-convex in $\mathbb{D}$.

Now, consider DE$_2$. At first, we remark that if $\gamma = 0$, then $w''/w' = -\beta/(z-1) - -\beta/(z-1)^2$. General solution of this equation has the form

$$w(z) = \int \frac{C_1}{(z-1)^\beta} \exp\left\{\frac{\beta}{z-1}\right\} dz + C_2, \quad C_1 \neq 0.$$

We remark also that every close-to-convex in $\mathbb{D}$ function (1) is univalent in $\mathbb{D}$ and, therefore, by the Bieberbach conjecture proved in [3] $|f_n| \le n|f_1|$ for all $n \ge 1$, i.e. $M_f(r) = = O((1-r)^{-2})$ as $r \uparrow 1$. For every C_1 and C_2 the growth rate of $w(z)$ is essentially faster, i.e. DE$_2$ does not have close-to-convex solution in $\mathbb{D}$.

We will search a solution of DE$_2$ in the form

$$f(z) = F\left(\frac{1}{1-z}\right) = \sum_{n=0}^{\infty} \frac{F_n}{(1-z)^n}. \tag{8}$$

Clearly, (8) satisfies DE$_2$ if and only if

$$\sum_{n=1}^{\infty} \frac{n(n+1)F_n}{(1-z)^n} - \beta \sum_{n=1}^{\infty} \frac{nF_n}{(1-z)^n} + \beta \sum_{n=1}^{\infty} \frac{nF_n}{(1-z)^{n+1}} + \gamma \sum_{n=0}^{\infty} \frac{F_n}{(1-z)^n} \equiv 0,$$

that is

$$\sum_{n=1}^{\infty} \frac{(n(n+1-\beta)+\gamma)F_n}{(1-z)^n} + \gamma F_0 + \sum_{n=2}^{\infty} \frac{\beta(n-1)F_{n-1}}{(1-z)^n} \equiv 0,$$

whence $\gamma F_0 = 0$, $(2-\beta+\gamma)F_1 = 0$ and $(n(n+1-\beta)+\gamma)F_n + \beta(n-1)F_{n-1} = 0$ for $n \ge 2$. As above, we choose $F_0 = 0$ and $F_1 = 1$. Then $2-\beta+\gamma = 0$,

$$F_n = \frac{-\beta(n-1)}{n(n+1-\beta)+\gamma} F_{n-1}, \quad n \ge 2 \tag{9}$$

and

$$F(t) = t + \sum_{n=2}^{\infty} F_n t^n, \quad t = \varrho e^{i\theta}. \tag{10}$$

Theorem 3. *If $-1 \le \beta < 0$ and $2-\beta+\gamma = 0$, then DE$_2$ has an analytic in $\mathbb{D}$ solution* (8) *such that* $\ln M_f(r) = (1+o(1))|\beta|/(1-r)$ *as $r \uparrow 1$ and the function F of the form* (10) *is entire close-to-convex in $\mathbb{D}$.*

Proof. Clearly the function $f(z) = F(1/(1-z))$ satisfies DE$_2$ if and only if

$$\frac{1}{(1-z)^2}F''\Big(\frac{1}{1-z}\Big) + \Big(\frac{\beta}{(1-z)^2} + \frac{2-\beta}{1-z}\Big)F'\Big(\frac{1}{1-z}\Big) + \gamma F\Big(\frac{1}{1-z}\Big) \equiv 0,$$

i.e. F is a solution of the differential equation $t^2w'' + (\beta t^2 + (2-\beta)t)w' + \gamma w = 0$. If we put $\beta_0 = \beta$, $\beta_1 = 2-\beta$ and $\gamma_2 = \gamma$, then we get the differential equation (4) and also $-1 \le \beta_0 < 0$, $\beta_1 > 0$ and $\beta_1 + \gamma_2 = \gamma_0 = \gamma_1 = 0$. Therefore (see the proof of Theorem 1 in [8]), the function (10) is entire and close-to-convex in $\mathbb{D}$ and $\ln M_F(\varrho) = (1+o(1))|\beta|\varrho$ as $\varrho \to +\infty$. Since $M_F(\varrho) = F(\varrho)$ and $M_f(r) = f(r) = F(1/(1-r))$, Theorem 3 is proved. □

Finally, we will search a solution of DE$_3$ also in the form (8). Clearly, (8) satisfies DE$_3$ if and only if

$$\sum_{n=0}^{\infty} \frac{(n+1)(n+2)F_{n+1}}{(1-z)^n} + \beta \sum_{n=1}^{\infty} \frac{nF_n}{(1-z)^n} + \gamma \sum_{n=0}^{\infty} \frac{F_n}{(1-z)^n} \equiv 0,$$

whence $2F_1 + \gamma F_0 = 0$ and $(n+1)(n+2)F_{n+1} + (\beta n + \gamma)F_n = 0$ for $n \ge 1$. If we choose $F_1 = 1$, then $F_0 = -2/\gamma$, that is

$$F(t) = -2/\gamma + t + \sum_{n=2}^{\infty} F_n t^n. \tag{11}$$

This function is close-to-convex if and only if the function (10) is close-to-convex, where

$$F_{n+1} = -\frac{\beta n + \gamma}{(n+1)(n+2)}F_n, \quad n \ge 1. \tag{12}$$

Theorem 4. *If $\beta < 0$, $\gamma < 0$ and $|\beta| + |\gamma| \le 3$, then DE$_3$ has an analytic in $\mathbb{D}$ solution* (8) *such that* $\ln M_f(r) = (1+o(1))|\beta|/(1-r)$ *as $r \uparrow 1$ and the function F of the form* (10) *is entire close-to-convex in $\mathbb{D}$.*

Proof. From the conditions $\beta < 0$, $\gamma < 0$ and $|\beta| + |\gamma| \le 3$ it follows that

$$0 < -\frac{\beta n + \gamma}{n(n+2)} \le 1.$$

Therefore, from (12) we obtain

$$(n+1)F_{n+1} = -\frac{\beta n + \gamma}{n(n+2)} nF_n \le nF_n, \quad n \ge 1,$$

and by Lemma 1 the function F is close-to-convex in $\mathbb{D}$.

Let $\mu_F(\varrho) = \max\{|F_n|\varrho^n :\ n \ge 0\}$ be the maximal term of series (11) and $\nu_F(\varrho) = \max\{n : |F_n|\varrho^n = \mu_F(\varrho)\}$ be its central index. We put $\varrho_n = |F_n|/|F_{n+1}| = (n+1)(n+2)/(|\beta|n + |\gamma|)$. Since $\varrho_n \uparrow +\infty$ as $n \to \infty$, we have [7, p. 13] $\nu_F(\varrho) = n$ for $\varrho_n \le \varrho < \varrho_{n+1}$. Hence it follows that

$$\frac{(\nu_F(\varrho)+1)(\nu_F(\varrho)+2)}{|\beta|\nu_F(\varrho) + |\gamma|} = (1+o(1))\varrho,$$

i.e. $\nu_F(\varrho) = (1+o(1))|\beta|\varrho$ as $\varrho \to +\infty$. Since [7, p. 13]

$$\ln \mu_F(\varrho) = \ln \mu_F(\varrho_0) + \int_{\varrho_0}^{\varrho} \nu_F(x)\, d\ln x,$$

hence it follows that $\ln \mu_F(\varrho) = (1+o(1))|\beta|\varrho$ as $\varrho \to +\infty$ and [7, p. 17] $\ln M_F(\varrho) = (1+o(1))|\beta|\varrho$ as $\varrho \to +\infty$. Thus, $\ln M_f(r) = (1+o(1))|\beta|/(1-r)$ as $r \uparrow 1$. □

2 *l*-index boundedness

We will use the following lemma from [11].

Lemma 2. *Let a function f defined by* (1) *be analytic in the closed disk* $\overline{\mathbb{D}}_R = \{z: \ |z| \leq R\}$, $j = \min\{n \geq 0: \ f_n \neq 0\}$ *and*

$$\sum_{n=1}^{\infty} \frac{(n+j)!}{n!j!} \frac{|f_{n+j}|}{|f_j|} R^n \leq a_j(R) < 1.$$

Then $N(f; l, \mathbb{D}_R) = j$ *with* $l(|z|) = K_j(R)/(R - |z|)$, *where*

$$K_j(R) = \max\left\{1, \frac{1 + a_j(R)}{(1+j)(1 - a_j(R))}\right\}.$$

Hence for the function (2) it follows that if

$$\sum_{n=1}^{\infty} (n+1)|g_{n+1}| R^n \leq a(R) < 1, \tag{13}$$

then

$$\frac{|f^{(n)}(z)|}{n!} \left(\frac{1 - a(R)}{1 + a(R)}(R - |z|)\right)^n \leq \max\left\{\frac{|f'(z)|}{1!} \frac{1 - a(R)}{1 + a(R)}(R - |z|), |f(z)|\right\}$$

for all $z \in \mathbb{D}_R$ and $n \geq 2$.

If $0 < \eta < 1$ and $z \in \mathbb{D}_{\eta R}$, then $R - |z| \geq (1 - \eta)R$ and the last inequality implies $N(f, l; \mathbb{D}_{\eta R}) \leq 1$ with $l(|z|) = (1 + a(R))/((1 - a(R))(1 - \eta)R)$, because if $N(f, l_*, G) \leq N$ and $l^*(r) \leq l_*(r)$, then [9, p. 23] $N(f, l^*, G) \leq N$. Therefore, the following lemma is true.

Lemma 3. *If a function* (2) *is analytic in* $\overline{\mathbb{D}}_R$ *and* (13) *holds, then* $N(f, l; \mathbb{D}_{\eta R}) \leq 1$ *with* $l(|z|) \equiv (1 + a(R))/((1 - \eta)R(1 - a(R)))$.

At first we apply Lemma 3 to the solution of DE_1. By the conditions of Theorem 2 from (6) it follows that for $R \in (0, 1/2)$

$$\begin{aligned}\sum_{n=1}^{\infty} (n+1)|f_{n+1}| R^n &= \sum_{n=1}^{\infty} \frac{n(n+\beta-1)+\gamma}{n} f_n R^n = (\beta+\gamma)R + R\sum_{n=2}^{\infty} \frac{n(n+\beta-1)+\gamma}{n^2} n f_n R^{n-1} \\ &= (\beta+\gamma)R + R\sum_{n=1}^{\infty} \frac{(n+1)n + n\beta + \beta + \gamma}{(n+1)^2}(n+1)|f_{n+1}|R^n \\ &\leq R + R\sum_{n=1}^{\infty} \frac{(n+1)n + n + 1}{(n+1)^2}(n+1)|f_{n+1}|R^n = R + R\sum_{n=1}^{\infty}(n+1)|f_{n+1}|R^n,\end{aligned}$$

whence

$$\sum_{n=1}^{\infty} (n+1)|f_{n+1}| R^n \leq a(R) = \frac{R}{1-R} < 1.$$

Therefore, by Lemma 3 the following proposition is true.

Proposition 1. *By the conditions of Theorem 2 for a solution* (7) *of the equation* DE_1 *we have* $N(f, l; \mathbb{D}_{\eta R}) \leq 1$ *with* $l(|z|) \equiv 1/((1 - \eta)R(1 - 2R))$ *for arbitrary* $R \in (0, 1/2)$ *and* $\eta \in (0, 1)$.

Since f satisfies DE_1, we have

$$z(z-1)f''(z) + \beta z f'(z) + \gamma f(z) \equiv 0. \tag{14}$$

From the conditions $-1 \le \beta \le 1$ and $0 < \beta + \gamma \le 1$ it follows that $-1 < \gamma \le 2$. Therefore, for $1 > |z| \ge \eta R$, $R \in (0,\, 1/2)$, $\eta \in (0,\, 1)$, from (14) we obtain

$$(1-|z|)|f''(z)| \le |\beta||f'(z)| + \frac{|\gamma|}{|z|}|f(z)| \le |f'(z)| + \frac{2}{\eta R}|f(z)|,$$

and if $A \ge (2+\eta R)/(2\eta R) > 1$, then

$$\begin{aligned} \frac{|f''(z)|}{2!}\Big(\frac{1-|z|}{A}\Big)^2 &\le \frac{|f'(z)|}{1!}\Big(\frac{1-|z|}{A}\Big)\frac{1}{2A} + \frac{1}{A^2\eta R}|f(z)| \\ &\le \max\Big\{\frac{|f'(z)|}{1!}\Big(\frac{1-|z|}{A}\Big), |f(z)|\Big\}. \end{aligned} \tag{15}$$

Now we differentiate (14) $n \ge 1$ times. Then

$$z(z-1)f^{(n+2)}(z) + ((2n+\beta)z - n)f^{(n+1)}(z) + (n(n+\beta-1)+\gamma)f^{(n)}(z) \equiv 0,$$

whence for $|z| \ge \eta R$, $|\beta| \le 1$, $-1 < \gamma \le 2$ we obtain

$$\begin{aligned} \frac{|f^{(n+2)}(z)|}{(n+2)!}\Big(\frac{1-|z|}{A}\Big)^{n+2} &\le \frac{(2n+|\beta|)\eta R + n}{A(n+2)\eta R}\frac{|f^{(n+1)}(z)|}{(n+1)!}\Big(\frac{1-|z|}{A}\Big)^{n+1} \\ &\quad + \frac{n(n-1)+n|\beta|+|\gamma|}{A^2(n+2)(n+1)\eta R}\frac{|f^{(n)}(z)|}{n!}\Big(\frac{1-|z|}{A}\Big)^{n} \\ &\le \frac{(2n+1)\eta R + n}{A(n+2)\eta R}\frac{|f^{(n+1)}(z)|}{(n+1)!}\Big(\frac{1-|z|}{A}\Big)^{n+1} \\ &\quad + \frac{n^2+2}{A(n+2)(n+1)\eta R}\frac{|f^{(n)}(z)|}{n!}\Big(\frac{1-|z|}{A}\Big)^{n} \\ &\le \frac{2}{A\eta R}\frac{|f^{(n+1)}(z)|}{(n+1)!}\Big(\frac{1-|z|}{A}\Big)^{n+1} + \frac{1}{A\eta R}\frac{|f^{(n)}(z)|}{n!}\Big(\frac{1-|z|}{A}\Big)^{n} \\ &\le \frac{3}{A\eta R}\max\Big\{\frac{|f^{(n+1)}(z)|}{(n+1)!}\Big(\frac{1-|z|}{A}\Big)^{n+1}, \frac{|f^{(n)}(z)|}{n!}\Big(\frac{1-|z|}{A}\Big)^{n}\Big\} \\ &\le \max\Big\{\frac{|f^{(n+1)}(z)|}{(n+1)!}\Big(\frac{1-|z|}{A}\Big)^{n+1}, \frac{|f^{(n)}(z)|}{n!}\Big(\frac{1-|z|}{A}\Big)^{n}\Big\} \end{aligned} \tag{16}$$

provided $A \ge 3/\eta R$. Since the inequality $A \ge 3/\eta R$ implies $A \ge (2+\eta R)/(2\eta R)$, from (15) and (16) by the condition $A \ge 3/\eta R$ we obtain

$$\frac{|f^{(k)}(z)|}{k!}\Big(\frac{1-|z|}{A}\Big)^{k} \le \max\Big\{\frac{|f'(z)|}{1!}\Big(\frac{1-|z|}{A}\Big), |f(z)|\Big\}$$

for all $k \ge 2$. Therefore, the following proposition is true.

Proposition 2. *By the conditions of Theorem 2 for the solution* (7) *of the equation DE_1 we have $N(f, l; \mathbb{D} \setminus \mathbb{D}_{\eta R}) \le 1$ with $l(|z|) = 3/(\eta R(1-|z|))$ for arbitrary $R \in (0, 1/2)$ and $\eta \in (0, 1)$.*

Uniting Propositions 1 and 2 we get the following theorem.

Theorem 5. *By the conditions of Theorem 2 the solution* (7) *of DE_1 is of bounded l-index $N(f; l, \mathbb{D}) \le 1$ with*

$$l(|z|) = \max\Big\{\frac{3}{\eta R(1-|z|)}, \frac{1}{(1-\eta)R(1-2R)}\Big\}$$

for arbitrary $R \in (0, 1/2)$ and $\eta \in (0, 1)$.

If we choose $\eta = 1/2$ and $R = 1/3$, then $3/(\eta R) = 1/((1-\eta)R(1-2R)) = 18$, and Theorem 5 implies the following corollary.

Corollary 1. *By the conditions of Theorem 2 the solution* (7) *of DE_1 is of bounded l-index $N(f;l,\mathbb{D}) \le 1$ with $l(|z|) = 18/(1-|z|)$.*

Now we consider DE_2. For the solution f of DE_2 we have

$$(1-z)^2 f''(z) + \beta z f'(z) + \gamma f(z) \equiv 0. \tag{17}$$

We put $l(|z|) = A/(1-|z|)^2$. If $-1 \le \beta < 0$ and $2 - \beta + \gamma = 0$, then for all $z \in \mathbb{D}$ and $A \ge 3/2$ from (17) we obtain

$$\begin{aligned} \frac{|f''(z)|}{2!l^2(|z|)} &\le \frac{|\beta||z|}{(1-|z|)^2 2l(|z|)}\frac{|f'(z)|}{1!l(|z|)} + \frac{3}{(1-|z|)^2 2l^2(|z|)}|f(z)| \le \frac{1}{2A}\frac{|f'(z)|}{1!l(|z|)} + \frac{3}{2A^2}|f(z)| \\ &\le \Big(\frac{1}{2A} + \frac{3}{2A^2}\Big)\max\Big\{\frac{|f'(z)|}{1!l(|z|)}, |f(z)|\Big\} \le \max\Big\{\frac{|f'(z)|}{1!l(|z|)}, |f(z)|\Big\}. \end{aligned} \tag{18}$$

For $n \ge 1$ from (17) we have

$$(1-z)^2 f^{(n+2)}(z) - (2n(1-z) - \beta z) f^{(n+1)}(z) + (n(n-1+\beta)+\gamma) f^{(n)}(z) \equiv 0,$$

whence

$$|f^{(n+2)}(z)| \le \Big|\frac{2n(1-z)-\beta z}{(1-z)^2}\Big||f^{(n+1)}(z)| + \Big|\frac{n(n-1+\beta)+\gamma}{(1-z)^2}\Big||f^{(n)}(z)|,$$

then for all $z \in \mathbb{D}$ and $A \ge 1+\sqrt{2}$

$$\begin{aligned} \frac{|f^{(n+2)}(z)|}{(n+2)!l^{n+2}(|z|)} &\le \frac{2n+1}{(1-|z|)^2(n+2)l(|z|)}\frac{|f^{(n+1)}(z)|}{(n+1)!l^{n+1}(|z|)} \\ &\quad + \frac{n^2+3}{(1-|z|)^2(n+2)(n+1)l^2(|z|)}\frac{|f^{(n)}(z)|}{n!l^n(|z|)} \le \frac{2}{A}\frac{|f^{(n+1)}(z)|}{(n+1)!l^{n+1}(|z|)} \\ &\quad + \frac{1}{A^2}\frac{|f^{(n)}(z)|}{n!l^n(|z|)} \le \max\Big\{\frac{|f^{(n+1)}(z)|}{(n+1)!l^{n+1}(|z|)}, \frac{|f^{(n)}(z)|}{n!l^n(|z|)}\Big\}. \end{aligned} \tag{19}$$

In view of (18) and (19) the following theorem is true.

Theorem 6. *By the conditions of Theorem 3 the solution* (8) *of DE_2 is of bounded l-index $N(f,l) \le 1$ with $l(|z|) = (1+\sqrt{2})/(1-|z|)^2$.*

Finally, for the solution of DE_3 we have

$$(1-z)^3 f''(z) + \beta(1-z) f'(z) + \gamma f(z) \equiv 0, \tag{20}$$

whence for all $z \in \mathbb{D}$ and $l(|z|) = A/(1-|z|)^2$, $A \ge 3/2$, by the conditions $\beta < 0$, $\gamma < 0$ and $|\beta| + |\gamma| \le 3$ we obtain

$$\begin{aligned} \frac{|f''(z)|}{2!l^2(|z|)} &\le \frac{|\beta|}{2l(|z|)(1-|z|)^2}\frac{|f'(z)|}{1!l(|z|)} + \frac{|\gamma|}{2(1-|z|)^3 l^2(|z|)}|f(z)| \\ &\le \Big(\frac{|\beta|}{2A} + \frac{|\gamma|}{2A^2}\Big)\max\Big\{\frac{|f'(z)|}{1!l(|z|)}, |f(z)|\Big\} \le \max\Big\{\frac{|f'(z)|}{1!l(|z|)}, |f(z)|\Big\}. \end{aligned} \tag{21}$$

From (20) it follows that $(1-z)^3 f'''(z) - (3(1-z)^2 - \beta(1-z))f''(z) + (\gamma - \beta)f'(z) \equiv 0$, whence for all $z \in \mathbb{D}$ with $l(|z|) = A/(1-|z|)^2$, $A \geq 5/2$, we obtain

$$\begin{aligned} \frac{|f'''(z)|}{3!l^3(|z|)} &\leq \Big(\frac{3}{1-|z|} + \frac{|\beta|}{(1-|z|)^2}\Big)\frac{1}{3l(|z|)}\frac{|f''(z)|}{2!l^2(|z|)} + \frac{|\gamma|+|\beta|}{(1-|z|)^3}\frac{1}{6l^2(|z|)}\frac{|f'(z)|}{1!l(|z|)} \\ &\leq \frac{2}{A}\frac{|f''(z)|}{2!l^2(|z|)} + \frac{1}{2A^2}\frac{|f'(z)|}{1!l(|z|)} \leq \max\Big\{\frac{|f''(z)|}{2!l^2(|z|)}, \frac{|f'(z)|}{1!l(|z|)}\Big\}. \end{aligned} \tag{22}$$

Finally, for $n \geq 2$ from (20) we have

$$\begin{aligned} &(1-z)^3 f^{(n+2)}(z) - (3n(1-z)^2 - \beta(1-z))f^{(n+1)}(z) \\ &\qquad + (3n(n-1)(1-z) - n\beta + \gamma)f^{(n)}(z) + n(n-1)(n-2)f^{(n-1)}(z) \equiv 0, \end{aligned}$$

whence for all $z \in \mathbb{D}$ with $l(|z|) = A/(1-|z|)^2$, $A \geq 4$, we obtain

$$\begin{aligned} \frac{|f^{(n+2)}(z)|}{(n+2)!l^{n+2}(|z|)} &\leq \Big(\frac{3n}{1-|z|} + \frac{|\beta|}{(1-|z|)^2}\Big)\frac{1}{(n+2)l(|z|)}\frac{|f^{(n+1)}(z)|}{(n+1)!l^{n+1}(|z|)} \\ &\quad + \Big(\frac{3n(n-1)}{(1-|z|)^2} + \frac{n|\beta|+|\gamma|}{(1-|z|)^3}\Big)\frac{1}{(n+2)(n+1)l^2(|z|)}\frac{|f^{(n)}(z)|}{n!l^n(|z|)} \\ &\quad + \frac{n(n-1)(n-2)}{(1-|z|)^3}\frac{1}{(n+2)(n+1)nl^3(|z|)}\frac{|f^{(n-1)}(z)|}{(n-1)!l^{n-1}(|z|)} \\ &\leq \frac{3}{A}\frac{|f^{(n+1)}(z)|}{(n+1)!l^{n+1}(|z|)} + \frac{3}{A^2}\frac{|f^{(n)}(z)|}{n!l^n(|z|)} + \frac{1}{A^3}\frac{|f^{(n-1)}(z)|}{(n-1)!l^{n-1}(|z|)} \\ &\leq \Big(\frac{3}{4} + \frac{3}{16} + \frac{1}{64}\Big)\max\Big\{\frac{|f^{(j)}(z)|}{j!l^j(|z|)} :\ n-1 \leq j \leq n+1\Big\} \\ &\leq \max\Big\{\frac{|f^{(j)}(z)|}{j!l^j(|z|)} :\ n-1 \leq j \leq n+1\Big\}. \end{aligned} \tag{23}$$

In view of (21), (22), and (23) the following theorem is true.

Theorem 7. *By the conditions of Theorem 4 the solution* (8) *of* DE_3 *is of bounded* l*-index* $N(f,l) \leq 1$ *with* $l(|z|) = 4/(1-|z|)^2$.

It is known [2, 12] that for an entire function F the function $f(z) = F(q/(1-z)^n)$ is of bounded l-index in $\mathbb{D}$ with $l(|z|) \equiv \beta/(1-|z|)^{n+1}$, $\beta > 1$, if and only if F is of bounded index in $\mathbb{C}$. Since the function $F(z) = e^z$ is of bounded index in $\mathbb{C}$, hence it follows that the function $f_0(z) = \exp\{q/(1-z)\}$, $0 < q \leq 1$, is of bounded l-index in $\mathbb{D}$ with $l(|z|) \equiv \beta/(1-|z|)^2$, $\beta > 1$. We remark that the function f_0 satisfies the differential equation $(1-z)^3 w'' - q(1-z)w' - 2qw = 0$, i.e. f_0 satisfies DE_3 with $\beta = -q < 0$ and $\gamma = -2q < 0$. Since $|\beta| + |\gamma| \leq 3$, by Theorem 7 the function f_0 is of bounded l-index $N(f_0, l) \leq 1$ with $l(|z|) = 4/(1-|z|)^2$.

3 Addition

Here we investigate the l-index boundedness of the entire function (10) and (11) with the coefficients satisfying (9) and (12) respectively.

If $-1 \le \beta < 0$ and $2-\beta+\gamma=0$, then for $R \in (0, 20/13)$ in view of (9) we have

$$\begin{aligned}\sum_{n=1}^{\infty}(n+1)|F_{n+1}|R^n &= R\sum_{n=1}^{\infty}\frac{(n+1)|\beta|n}{(n+1)(n+2+|\beta|)-2-|\beta|}F_nR^{n-1}\\ &= R\frac{2|\beta|}{2(3+|\beta|)-2-|\beta|}+R\sum_{n=1}^{\infty}\frac{|\beta|(n+1)(n+2)}{(n+2)(n+3+|\beta|)-2-|\beta|}F_{n+1}R^n\\ &\le \frac{2R}{5}+R\sum_{n=1}^{\infty}\frac{(n+2)(n+1)F_{n+1}R^n}{(n+2)(n+4)-3}\le\frac{2R}{5}+\sum_{n=1}^{\infty}\frac{R}{4}(n+1)|F_{n+1}|R^n,\end{aligned}$$

whence

$$\sum_{n=1}^{\infty}(n+1)|F_{n+1}|R^n \le a(R)=\frac{8R}{5(4-R)}<1.$$

Therefore, by Lemma 3 for the function (10) for arbitrary $R \in (0, 20/13)$ and $\eta \in (0,1)$ $N(F,l;\mathbb{D}_{\eta R}) \le 1$ with $l(|z|) \equiv (20+3R)/((1-\eta)R(20-13R))$.

Since

$$z^2F''(z)+(\beta z^2+(2-\beta)z)F'(z)+\gamma F(z)\equiv 0, \quad \gamma=\beta-2, \tag{24}$$

for $|z| \ge \eta R$ and $A \ge 1+3/(\eta R) \ge 1/(\eta R)$ we have

$$\begin{aligned}\frac{|F''(z)|}{2!A^2} &\le \frac{\eta R+3}{2A\eta R}\frac{|F'(z)|}{1!A}+\frac{3}{(\eta R)^2 2A^2}|F(z)| \le \frac{\eta R+3}{2A\eta R}\frac{|F'(z)|}{1!A}+\frac{3}{2A\eta R}|F(z)|\\ &\le \frac{\eta R+6}{2A\eta R}\max\left\{\frac{|F'(z)|}{1!A},|F(z)|\right\}\le \max\left\{\frac{|F'(z)|}{1!A},|F(z)|\right\}.\end{aligned} \tag{25}$$

From (24) it follows that $z^2F'''(z)+(\beta z^2+(4-\beta)z)F''(z)+(2\beta z+2-\beta+\gamma)F'(z)\equiv 0$, that is in view of the condition $2-\beta+\gamma=0$ we have $zF'''(z)+(\beta z+4-\beta)F''(z)+2\beta F'(z)\equiv 0$, whence for $|z|\ge \eta R$ and $A \ge 1+3/(\eta R)$

$$\frac{|F'''(z)|}{3!A^3}\le\frac{\eta R+5}{3A\eta R}\frac{|F''(z)|}{2!A^2}+\frac{1}{3A^2\eta R}\frac{|F'(z)|}{1!A}\le\max\left\{\frac{|F''(z)|}{2!A^2},\frac{|F'(z)|}{1!A}\right\}. \tag{26}$$

Now we differentiate (24) $n \ge 2$ times. Then

$$\begin{aligned} z^2F^{(n+2)}(z)&+(\beta z^2+(2n+2-\beta)z)F^{(n+1)}(z)\\ &+(n(2\beta z+n+1-\beta)+\gamma)F^{(n)}(z)+\beta n(n-1)F^{(n-1)}(z)\equiv 0,\end{aligned}$$

whence for $|z|\ge\eta R$ and $A\ge 1+3/\eta R$

$$\begin{aligned}\frac{|F^{(n+2)}(z)|}{(n+2)!A^{n+2}} &\le \frac{\eta R+2n+3}{(n+2)A\eta R}\frac{|F^{(n+1)}(z)|}{(n+1)!A^{n+1}}+\frac{2n\eta R+n(n+2)+3}{(n+2)(n+1)(A\eta R)^2}\frac{|F^{(n)}(z)|}{n!A^n}\\ &\quad+\frac{n(n-1)}{(n+2)(n+1)nA^3(\eta R)^2}\frac{|F^{(n-1)}(z)|}{(n-1)!A^{n-1}}\\ &\le \frac{1}{A\eta R}\left(\frac{\eta R+2n+3}{n+2}+\frac{2n\eta R+n(n+2)+3}{(n+2)(n+1)}+\frac{n-1}{(n+2)(n+1)}\right)\\ &\quad\times\max_{n-1\le j\le n+1}\frac{|F^{(j)}(z)|}{j!A^j}\\ &\le\frac{1}{A\eta R}\frac{7\eta R+33}{12}\max_{n-1\le j\le n+1}\frac{|F^{(j)}(z)|}{j!A^j}\le\max_{n-1\le j\le n+1}\frac{|F^{(j)}(z)|}{j!A^j}.\end{aligned} \tag{27}$$

From (25), (26), (27) for all $k\ge 2$, $|z|\ge\eta R$ and $A=1+3/\eta R$ we obtain

$$\frac{|F^{(k)}(z)|}{k!A^k}\le\max\left\{\frac{|F'(z)|}{1!A},|F(z)|\right\}$$

and, therefore, for the function (10) we have $N(F,l;\mathbb{C}\setminus\mathbb{D}_{\eta R})\le 1$ with $l(|z|)\equiv(\eta R+3)/(\eta R)$ for arbitrary $R\in(0,20/13)$ and $\eta\in(0,1)$.

Thus, we get the following statement.

Proposition 3. *By the conditions of Theorem 3 the function* (10) *with the coefficients satisfying* (9) *for arbitrary* $R\in(0,\,20/13)$ *and* $\eta\in(0,\,1)$ *is of bounded l-index* $N(F,l)\le 1$ *with* $l(|z|)\equiv\max\{(20+3R)/((1-\eta)R(20-13R)),(\eta R+3)/(\eta R)\}$.

If we choose $R=1$ and $\eta=1/2$, then from Proposition 3 we obtain that by the conditions of Theorem 3 the function (10) with the coefficients satisfying (9) is of bounded l-index $N(F,l)\le 1$ with $l(|z|)\equiv 7$.

Finally, we consider the function (11) with the coefficients satisfying (12). Since $F_0=-2/\gamma\neq 0$ Lemma 2 implies the following lemma.

Lemma 4. *If a function* (11) *is analytic in* $\overline{\mathbb{D}}_{\eta R}$ *and*

$$\sum_{n=1}^{\infty}\frac{|F_n|}{|F_0|}R^n\le a_0(R)<1,$$

then $N(F;l,\mathbb{D}_{\eta R})=0$ *with* $l(|z|)\equiv(1+a_0(R))/((1-a_0(R))R(1-\eta))$, $0<\eta<1$.

If $\beta<0$, $\gamma<0$ and $|\beta|+|\gamma|\le 3$, then from (12) we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}|F_n|R^n&=R\sum_{n=1}^{\infty}\frac{|\beta|(n-1)+|\gamma|}{n(n+1)}|F_{n-1}|R^{n-1}\\&=R\frac{|\gamma|}{2}|F_0|+R\sum_{n=1}^{\infty}\frac{|\beta|n+|\gamma|}{(n+1)(n+2)}|F_n|R^n\le\frac{3R|F_0|}{2}+\frac{R}{2}\sum_{n=1}^{\infty}|F_n|R^n,\end{aligned}$$

whence for $R\in(0,1/2)$

$$\sum_{n=1}^{\infty}\frac{|F_n|}{|F_0|}R^n\le a_0(R)=\frac{3R}{2-R}<1$$

and, therefore, by Lemma 4 for function (11) we have $N(F,l;\mathbb{D}_{\eta R})=0$ with

$$l(|z|)\equiv\frac{1+R}{(1-2R)R(1-\eta)}$$

for arbitrary $R\in(0,1/2)$ and $\eta\in(0,1)$.

Since the function $f(z)=F(1/(1-z))$ satisfies DE_3, the function F satisfies the differential equation $tw''+(\beta t+2)w'+\gamma w=0$, i.e.

$$zF''(z)+(\beta z+2)F'(z)+\gamma F(z)\equiv 0. \tag{28}$$

Hence for $|z|\ge\eta R$ and $A\ge 2/\eta R>1$ in view of the conditions $|\beta|+|\gamma|\le 3$ and $\eta R\in(0,1/2)$ we obtain

$$\frac{|F''(z)|}{2!A^2}\le\frac{|\beta|\eta R+2}{2A\eta R}\frac{|F'(z)|}{1!A}+\frac{|\gamma|}{2A^2\eta R}|F(z)|\le\max\left\{\frac{|F'(z)|}{1!A},|F(z)|\right\}, \tag{29}$$

since

$$\frac{|\beta|}{2A}+\frac{1}{A\eta R}+\frac{|\gamma|}{2A^2\eta R}\le\frac{|\beta|\eta R}{4}+\frac{1}{2}+\frac{|\gamma|\eta R}{8}\le\frac{|\beta|}{8}+\frac{|\gamma|}{16}+\frac{1}{2}\le\frac{(|\beta|+|\gamma|)+|\beta|}{16}+\frac{1}{2}\le\frac{7}{8}.$$

If we differentiate (28) $n \geq 1$ times, then we get $tF^{(n+2)}(t) + (\beta t + n + 2)F^{(n+1)}(t) + (n\beta + \gamma)F^{(n)}(t) \equiv 0$, whence for $|z| \geq \eta R$ and $A \geq 2/(\eta R) > 1$

$$\begin{aligned} \frac{|F^{(n+2)}(t)|}{(n+2)!A^{n+2}} &\leq \left(\frac{|\beta|}{(n+2)A} + \frac{1}{A\eta R}\right)\frac{|F^{(n+1)}(t)|}{(n+1)!A^{n+1}} + \frac{n|\beta| + |\gamma|}{(n+2)(n+1)A^2\eta R}\frac{|F^{(n)}(t)|}{n!A^n} \\ &\leq \left(\frac{1}{4} + \frac{1}{2} + \frac{1}{16}\right)\max_{n\leq j\leq n+1}\frac{|F^{(j)}(t)|}{j!A^j} \leq \max_{n\leq j\leq n+1}\frac{|F^{(j)}(t)|}{j!A^j}. \end{aligned} \tag{30}$$

From (29) and (30) it follows that for function (11) we have $N(F, l; \mathbb{C} \setminus \mathbb{D}_{\eta R}) \leq 1$ with $l(|z|) \equiv 2/\eta R$ for arbitrary $R \in (0, 1/2)$ and $\eta \in (0, 1)$.

Thus, we get the following statement.

Proposition 4. *By the conditions of Theorem 3 the function* (11) *with coefficients satisfying* (12) *is of bounded l-index $N(F, l) \leq 1$ with*

$$l(|z|) \equiv \max\left\{\frac{1+R}{(1-2R)R(1-\eta)}, \frac{2}{\eta R}\right\}$$

for arbitrary $R \in (0, 1/2)$ and $\eta \in (0, 1)$.

If we choose $R = 1/5$ and $\eta = 1/2$, then from Proposition 4 we get that by the conditions of Theorem 4 the function (11) with the coefficients satisfying (12) is of bounded l-index $N(F, l) \leq 1$ with $l(|z|) \equiv 20$.

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Однолиста аналітична в $\mathbb{D} = \{z : \ |z| < 1\}$ функція $f(z)$ називається опуклою, якщо $f(\mathbb{D})$ — опукла область. Добре відомо, що умова $\mathrm{Re}\,\{1 + zf''(z)/f'(z)\} > 0$, $z \in \mathbb{D}$, є необхідною і достатньою для опуклості f. Функція f називається близькою до опуклої в $\mathbb{D}$, якщо існує опукла в $\mathbb{D}$ функція Φ така, що $\mathrm{Re}\,(f'(z)/\Phi'(z)) > 0$, $z \in \mathbb{D}$.

С.М. Шах вказав умови на дійсні параметри β_0, β_1, γ_0, γ_1, γ_2 диференціального рівняння $z^2w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0$, за яких існує цілий трансцендентний розв'язок f такий, що f і всі його похідні є близькими до опуклих в $\mathbb{D}$.

Нехай $0 < R \le +\infty$, $\mathbb{D}_R = \{z : \ |z| < R\}$ і l — додатна неперервна функція на $[0, R)$ така, що $(R - r)l(r) > C$, $C = \mathrm{const} > 1$. Аналітична в $\mathbb{D}_R$ функція f називається обмеженого l-індексу, якщо існує $N \in \mathbb{Z}_+$ таке, що

$$\frac{|f^{(n)}(z)|}{n!l^n(|z|)} \le \max\left\{\frac{|f^{(k)}(z)|}{k!l^k(|z|)} : \ 0 \le k \le N\right\}$$

для всіх $n \in \mathbb{Z}_+$ і $z \in \mathbb{D}_R$.

Досліджено близькість до опуклості та обмеженість l-індексу для аналітичних в $\mathbb{D}$ розв'язків трьох аналогічних Шаху диференціальних рівнянь: $z(z-1)w'' + \beta z w' + \gamma w = 0$, $(z-1)^2 w'' + \beta z w' + \gamma w = 0$ і $(1-z)^3 w'' + \beta(1-z)w' + \gamma w = 0$. Незважаючи на подібність цих рівнянь, їх розв'язки мають різні властивості.

Ключові слова і фрази: близькість до опуклості, l-індекс, диференціальне рівняння.