Smooth symmetric bilinear forms on $L_s(2^2|L_\infty^2)$

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In [Carpathian Math. Publ. 2020, 12 (2), 340–352], the author classified the extreme points and exposed points of the unit ball of the space of symmetric bilinear forms on the space $L_s(2^2|L_\infty^2)$, where $L_s(2^2|L_\infty^2)$ is the space of symmetric bilinear forms on the plane with the supremum norm. Motivated by this paper, we classify the smooth points of the unit ball of the space of symmetric bilinear forms on $L_s(2^2|L_\infty^2)$.

Key words and phrases: smooth point, space of symmetric bilinear forms.

Introduction

Throughout the paper, we let $n, m \in \mathbb{N}, n, m \geq 2$. We write $B_E$ for the closed unit ball of a real Banach space $E$ and the dual space of $E$ is denoted by $E^\ast$. An element $x \in B_E$ is called an extreme point of $B_E$ if $y, z \in B_E$ with $x = \frac{1}{2}(y + z)$ implies $x = y = z$. An element $x \in B_E$ is called an exposed point of $B_E$ if there is an $f \in E^\ast$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of $B_E$ is an extreme point. A point $x \in B_E$ is called a smooth point of $B_E$ if there is a unique $f \in E^\ast$ so that $f(x) = 1 = \|f\|$. We denote by $\text{ext } B_E$, $\text{exp } B_E$ and $\text{sm } B_E$ the set of extreme points, the set of exposed points and the set of smooth points of $B_E$, respectively. We denote by $L^n(E)$ the Banach space of all continuous $n$-linear forms on $E$ endowed with the norm $\|T\| = \sup_{\|x_i\|=1} |T(x_1, \ldots, x_n)|$. An $n$-linear form $T$ is symmetric if $T(x_1, \ldots, x_n) = T(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for every permutation $\sigma$ on $\{1, \ldots, n\}$. $L_s^n(E)$ denotes the closed subspace of all continuous symmetric $n$-linear forms on $E$. Notice that $L^n(E)$ is identified with the dual of $n$-fold projective tensor product $\hat{\otimes}_{\pi,n} E$. With this identification, the action of a continuous $n$-linear form $T$ as a bounded linear functional on $\hat{\otimes}_{\pi,n} E$ is given by

$$\langle \sum_{i=1}^k x^{(1),i} \otimes \cdots \otimes x^{(n),i}, T \rangle = \sum_{i=1}^k T(x^{(1),i}, \ldots, x^{(n),i}).$$

Notice also that $L_s^n(E)$ is identified with the dual of $n$-fold symmetric projective tensor product $\hat{\otimes}_{s,\pi,n} E$. With this identification, the action of a continuous symmetric $n$-linear form $T$ as a bounded linear functional on $\hat{\otimes}_{s,\pi,n} E$ is given by

$$\langle \sum_{i=1}^k \frac{1}{n!} \left( \sum_{\sigma} x^{(1),i} \otimes \cdots \otimes x^{(n),i} \right), T \rangle = \sum_{i=1}^k T(x^{(1),i}, \ldots, x^{(n),i}),$$

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Smooth symmetric bilinear forms on $L_s^{(2l_2^\infty)}$

where $\sigma$ goes over all permutations on $\{1, \ldots, n\}$. A mapping $P : E \rightarrow \mathbb{R}$ is a continuous $n$-homogeneous polynomial if there exists a unique $T \in L_s(nE)$ such that $P(x) = T(x_1, \ldots, x_n)$ for every $x \in E$. We denote by $P(nE)$ the Banach space of all continuous $n$-homogeneous polynomials from $E$ into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\| = 1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [3].

The main result about smooth points is known as the Mazur density theorem. Recall that the Mazur density theorem (see [5, p. 171]) says that the set of all the smooth points of a solid closed convex subset of a separable Banach space is a residual subset of its boundary. Y.S. Choi and S.G. Kim [1, 2] initiated and characterized the smooth points of the unit balls of $L_1$-homogeneous polynomial if there exists a unique $(s'y_1)_{y_1} \in L_1$ that the Mazur density theorem (see [5, p. 171]) says that the set of all the smooth points of $L_1$-homogeneous polynomial if there exists a unique $(s'y_1)_{y_1} \in L_1$

Let $l_1^m = \mathbb{R}^m$ with the supremum norm. S.G. Kim [12] characterized the smooth points of the unit balls of $L_s(nL_1)$ and $L_s(nl_2^\infty)$ for $n, m \geq 2$. In [13], S.G. Kim classified the smooth points of the unit ball of $P(2\mathbb{R}^2_{h(\frac{1}{2})})$, where $\mathbb{R}^2_{h(\frac{1}{2})} = \mathbb{R}^2$ with the hexagonal norm $\|(x,y)\|_{h(\frac{1}{2})} = \max \left\{ |y|, \|x\| + \frac{1}{2} |y| \right\}$, and in [10], he classified the extreme and exposed points of the unit ball of $L_s(2L_s(2l_2^\infty))$. Recently, S.G. Kim [11] classified the smooth points of the unit ball of $L_s(nL_1)$ and $L_s(nl_1)$ for $n \geq 2$.

In this paper we classify the smooth points of the unit ball of $L_s(2L_s(2l_2^\infty))$.

1 Results

Let $n \geq 2$. For $j = 0, \ldots, n$, we let

$$F_j \left( (x_1, y_1), \ldots, (x_n, y_n) \right) = \sum_{\{l_1, \ldots, l_j, k_1, \ldots, k_{n-j} \} = \{1, \ldots, n\}} x_{l_1} \cdots x_{l_j} y_{k_1} \cdots y_{k_{n-j}}.$$ 

Then,

$$\{ F_0 \left( (x_1, y_1), \ldots, (x_n, y_n) \right), \ldots, F_n \left( (x_1, y_1), \ldots, (x_n, y_n) \right) \}$$

is a basis for $L_s(nL_1^2)$. Hence, $\dim(L_s(nL_1^2)) = n + 1$. If $S \in L_s(nL_1^2)$, then

$$S \left( (x_1, y_1), \ldots, (x_n, y_n) \right) = \sum_{j=0}^{n} a_j F_j \left( (x_1, y_1), \ldots, (x_n, y_n) \right)$$

for some $a_0, \ldots, a_n \in \mathbb{R}$. By simplicity we denote $S = (a_0, \ldots, a_n)^t$.

Let $\mathbb{R}^{n+1}_{L_s(nL_1^2)} := \mathbb{R}^{n+1}$ with the $L_s(nL_1^2)$-norm

$$\left\| (a_0, \ldots, a_n) \right\|_{L_s(nL_1^2)} := \sup_{\|(x_k, y_k)\|_{l_\infty} = 1, k=1, \ldots, n} \left| \sum_{j=0}^{n} a_j F_j \left( (x_1, y_1), \ldots, (x_n, y_n) \right) \right| = \left\| (a_0, \ldots, a_n)^t \right\|.$$

We have the following identification.

**Theorem 1.** For $m, n \geq 2$, the equality $L_s(mL_s(nL_1^2)) = L_s(m\mathbb{R}^{n+1}_{L_s(nL_1^2)})$ holds.
Notice that \( \text{ext } B_{\mathcal{L}_s(\mathcal{L}_s^2(\ell_2^n)))} = \text{ext } B_{\mathcal{L}_s(\mathcal{L}_s^2(\ell_2^n)))} \), \( \exp B_{\mathcal{L}_s(\mathcal{L}_s^2(\ell_2^n)))} = \exp B_{\mathcal{L}_s(\mathcal{L}_s^2(\ell_2^n)))} \) and \( \text{sm } B_{\mathcal{L}_s(\mathcal{L}_s^2(\ell_2^n)))} = \text{sm } B_{\mathcal{L}_s(\mathcal{L}_s^2(\ell_2^n)))} \.

S.G. Kim [10] classified the extreme and exposed points of the unit ball of the space \( \mathbb{R}^n_{\mathcal{L}_s(\mathcal{L}_s^2(\ell_2^n)))} = \mathcal{L}_s(\mathcal{L}_s^2(\ell_2^n))) \). Let \( T \in \mathcal{L}_s(\mathcal{L}_s^2(\ell_2^n))) \). Then, for \((t_1, t_2, t_3), (s_1, s_2, s_3) \in \mathbb{R}^3 \),

\[
T((t_1, t_2, t_3), (s_1, s_2, s_3)) = at_{1}s_1 + bt_{2}s_2 + ct_{3}s_3 + d(t_{1}s_2 + t_{2}s_1) + e(t_{1}s_3 + t_{3}s_1) + f(t_{2}s_3 + t_{3}s_2),
\]

where

\[
a = T(x_1y_1, x_1y_1), b = T(x_2y_2, x_2y_2), c = T(x_1y_2 + x_2y_1, x_1y_2 + x_2y_1),
d = T(x_1y_1, x_2y_1), e = T(x_1y_1, x_1y_2 + x_2y_1), f = T(x_2y_2, x_1y_2 + x_2y_1).
\]

Notice that \( \{t_1s_1, t_2s_2, t_3s_3, t_1s_2 + t_2s_1, t_1s_3 + t_3s_1, t_2s_3 + t_3s_2\} \) is a basis for \( \mathcal{L}_s(\mathcal{L}_s^2(\ell_2^n))) \) and \( \dim(\mathcal{L}_s(\mathcal{L}_s^2(\ell_2^n))) = 6 \). For simplicity we denote \( T = (a, b, c, d, e, f)^t \).

Let \( S \) be a non-empty subset of a real Banach space \( E \). Let

\[
\text{conv}(S) := \left\{ \sum_{j=1}^{k} t_j a_j : 0 \leq t_j \leq 1, t_1 + \cdots + t_k = 1, a_j \in S \right\}
\]

We call \( \text{conv}(S) \) the convex hull of \( S \). Recall that the Krein-Milman Theorem [14] states that every nonempty compact convex subset of a Hausdorff locally convex space is the closed convex hull of its set of extreme points. Hence, the unit ball of \( \mathcal{L}_s(\mathcal{L}_s^2(\ell_2^n))) \) is the closed convex hull of \( \text{ext } B_{\mathcal{L}_s(\mathcal{L}_s^2(\ell_2^n)))} \).

The geometrical structures of the unit ball of the space \( \mathcal{L}_s(\mathcal{L}_s^2(\ell_2^n))) \) were investigated in [6]. In particular, it was shown that

\[
\text{ext } B_{\mathcal{L}_s(\mathcal{L}_s^2(\ell_2^n)))} = \exp B_{\mathcal{L}_s(\mathcal{L}_s^2(\ell_2^n)))} = \left\{ \pm (1, 0, 0)^t, \pm (0, 1, 0)^t, \pm \left( \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right)^t \right\}.
\]

The following presents an explicit formulae for the norm of \( T \in \mathcal{L}_s(\mathcal{L}_s^2(\ell_2^n))) \).

**Theorem 2.** Let \( T \in \mathcal{L}_s(\mathcal{L}_s^2(\ell_2^n))) \) with \( T = (a, b, c, d, e, f)^t \) for some \( a, b, c, d, e, f \in \mathbb{R} \). Then,

\[
\|T\| = \max \left\{ |a|, |b|, |d|, \frac{1}{2}(|a - d| + |e|), \frac{1}{2}(|b - d| + |f|), \frac{1}{4}|a + b - c - 2d|, \frac{1}{4}|a + b + c - 2d| + 2|e - f| \right\}.
\]

**Proof.** In [6] it was shown that

\[
\text{ext } B_{\mathcal{L}_s(\mathcal{L}_s^2(\ell_2^n)))} = \left\{ \pm (1, 0, 0)^t, \pm (0, 1, 0)^t, \pm \left( \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right)^t \right\}.
\]

Let

\[
U_1 := \left( (1, 0, 0)^t, (1, 0, 0)^t \right), U_2 := \left( (0, 1, 0)^t, (0, 1, 0)^t \right),
U_3 := \left( (1, 0, 0)^t, (0, 1, 0)^t \right), U_4 := \left( (1, 0, 0)^t, \left( \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right)^t \right),
U_5 := \left( (1, 0, 0)^t, \left( \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right)^t \right), U_6 := \left( (0, 1, 0)^t, \left( \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right)^t \right),
U_7 := \left( (0, 1, 0)^t, \left( \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right)^t \right), U_8 := \left( \left( \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right)^t, \left( \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right)^t \right),
U_9 := \left( \left( \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right)^t, \left( \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right)^t \right), U_{10} := \left( \left( \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right)^t, \left( \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right)^t \right).
\]
Notice that $T(U_1) = a, T(U_2) = b, T(U_3) = d, T(U_4) = \frac{1}{2}(a - d + e), T(U_5) = \frac{1}{2}(a - d - e)$, $T(U_6) = \frac{1}{2}(-b + d + f), T(U_7) = \frac{1}{2}(-b - d - f), T(U_8) = \frac{1}{2}(a + b + c) + \frac{1}{2}(-d + e - f)$, $T(U_9) = \frac{1}{2}(a + b - c) - \frac{1}{2}d, T(U_{10}) = \frac{1}{2}(a + b + c) + \frac{1}{2}(-d + e + f)$.

By the bilinearity of $T$ and the Krein-Milman theorem,

\[ \| T \| = \sup \{ |T(L_1, L_2)| : L_1, L_2 \in \text{ext } \mathcal{B}_{\mathcal{L}_s^{(2^2_{\ell_2})}} \} = \sup_{1 \leq j \leq 10} |T(U_j)|. \]

For $z_1, \ldots, z_6 \in \mathbb{R}$, we let

\[
\begin{align*}
Y_1(z_1, \ldots, z_6) &:= z_1, & X_1 &:= [1, 0, 0, 0, 0, 0], \\
Y_2(z_1, \ldots, z_6) &:= z_2, & X_2 &:= [0, 1, 0, 0, 0, 0], \\
Y_3(z_1, \ldots, z_6) &:= z_3, & X_3 &:= [0, 0, 0, 1, 0, 0], \\
Y_4(z_1, \ldots, z_6) &:= \frac{1}{2}(z_1 - z_4 + z_5), & X_4 &:= \left[ \frac{1}{2}, 0, 0, -\frac{1}{2}, 0 \right], \\
Y_5(z_1, \ldots, z_6) &:= \frac{1}{2}(z_1 - z_4 - z_5), & X_5 &:= \left[ \frac{1}{2}, 0, 0, -\frac{1}{2}, -\frac{1}{2} \right], \\
Y_6(z_1, \ldots, z_6) &:= \frac{1}{2}(-z_2 + z_4 + z_6), & X_6 &:= \left[ 0, -\frac{1}{2}, 0, 0, 0, \frac{1}{2} \right], \\
Y_7(z_1, \ldots, z_6) &:= \frac{1}{2}(-z_2 + z_4 - z_6), & X_7 &:= \left[ 0, -\frac{1}{2}, 0, 0, -\frac{1}{2}, 0 \right], \\
Y_8(z_1, \ldots, z_6) &:= \frac{1}{4}(z_1 + z_2 + z_3 + \frac{1}{2}(-z_4 + z_5 + z_6), & X_8 &:= \left[ \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right], \\
Y_9(z_1, \ldots, z_6) &:= \frac{1}{4}(z_1 + z_2 - z_3) - \frac{1}{2}z_4, & X_9 &:= \left[ \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, 0, 0 \right], \\
Y_{10}(z_1, \ldots, z_6) &:= \frac{1}{4}(z_1 + z_2 + z_3 + \frac{1}{2}(-z_4 - z_5 + z_6), & X_{10} &:= \left[ \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{2}, -\frac{1}{2} \right].
\end{align*}
\]

Notice that for $j = 1, \ldots, 10$,

\[ X_j \cdot T = T(U_j) = Y_j(a, b, c, d, e, f). \]

**Lemma 1.** Let $T = (a, b, c, d, e, f)^t \in \mathcal{L}_s(2^3_{\ell_2})$ and $H \in \mathcal{L}_s(2^3_{\ell_2})^*$ with $\| T \| = \| H \| = |H(T)| = 1$. Suppose that $\| T \pm (\delta_1, \ldots, \delta_6)^t \| \leq 1$ for some $\delta_1, \ldots, \delta_6 \in \mathbb{R}$. Then,

\[ \sum_{j=1}^{6} \delta_j v_j = 0, \]

where

\[
\begin{align*}
v_1 &:= H(t_1 s_1), & v_2 &:= H(t_2 s_2), & v_3 &:= H(t_3 s_3), \\
v_4 &:= H(t_1 s_2 + t_2 s_1), & v_5 &:= H(t_1 s_3 + t_3 s_1), & v_6 &:= H(t_2 s_3 + t_3 s_2).
\end{align*}
\]

**Proof.** It follows that

\[
1 \geq \max \left\{ \left| H(T + (\delta_1, \ldots, \delta_6)^t) \right|, \left| H(T - (\delta_1, \ldots, \delta_6)^t) \right| \right\}
= \max \left\{ \left| H(T) + H((\delta_1, \ldots, \delta_6)^t) \right|, \left| H(T) - H((\delta_1, \ldots, \delta_6)^t) \right| \right\}
= 1 + \left| H((\delta_1, \ldots, \delta_6)^t) \right| = 1 + \sum_{j=1}^{6} \delta_j v_j,
\]

which implies that $\sum_{j=1}^{6} \delta_j v_j = 0$. \qed
By Lemma 1, we can characterize \( \text{sm} B_{L_{s}(L_{s}(2_{\mathbb{R}_{\infty}})))} \).

**Theorem 3.** Let \( T = (a, b, c, d, e, f)^{t} \in L_{s}(\mathbb{R}_{\infty}) \) with \( \| T \| = 1 \). Then \( T \in \text{sm} B_{L_{s}(L_{s}(2_{\mathbb{R}_{\infty}})))} \) if and only if there is \( j_{0} \in \{1, \ldots, 10\} \) such that

\[
|T(U_{j_{0}})| = 1 \quad \text{and} \quad |T(U_{k})| < 1 \quad \text{for all} \quad k \neq j_{0}.
\]

**Proof.** Necessity. Suppose that \( T \in \text{sm} B_{L_{s}(L_{s}(2_{\mathbb{R}_{\infty}})))} \). We claim show that there is \( j_{0} \in \{1, \ldots, 10\} \) such that \( |T(U_{j_{0}})| = 1 \) and \( |T(U_{k})| < 1 \) for all \( k \neq j_{0} \).

Otherwise. There are \( j_{1} \neq j_{2} \in \{1, \ldots, 10\} \) such that \( |T(U_{j_{1}})| = |T(U_{j_{2}})| = 1 \). For \( k = 1, 2 \), let \( H_{k} = \text{sign}(T(U_{j_{k}})) \delta_{U_{j_{k}}} \), where \( \delta_{U_{j_{k}}} \in L_{s}(2_{\mathbb{R}_{\infty}}) \) is defined by \( \delta_{U_{j_{k}}} \in L_{s}(2_{\mathbb{R}_{\infty}}) \) for all \( \delta_{U_{j_{k}}} \).

Then,

\[
H_{1} \neq H_{2}, H_{1} = H_{k}(T), \quad \forall k = 1, 2.
\]

Hence, \( T \notin \text{sm} B_{L_{s}(L_{s}(2_{\mathbb{R}_{\infty}})))} \).

Sufficiency. Let \( H \in L_{s}(2_{\mathbb{R}_{\infty}}) \) be such that \( H(T) = 1 \) with \( v_{1} := H(t_{1}s_{1}), \ v_{2} := H(t_{2}s_{2}), \ v_{3} := H(t_{3}s_{3}), \ v_{4} := H(t_{4}s_{4}), \ v_{5} := H(t_{5}s_{5}), \ v_{6} := H(t_{6}s_{6}). \)

For simplicity we denote \( H := [v_{1}, v_{2}, v_{3}, v_{4}, v_{5}, v_{6}]. \)

**Claim.** \( H = \text{sign}(T(U_{j_{0}}))X_{j_{0}}. \)

We divide into ten cases as follows.

**Case 1.** \( j_{0} = 1. \)

It follows that

\[
1 = \text{sign}(T(U_{1}))a.
\]

By Theorem 2, there is \( N \in \mathbb{N} \) such that

\[
\left\| T \pm \left(0, \frac{1}{N}, 0, 0, 0, 0, 0\right)^{t}\right\| \leq 1, \quad \left\| T \pm \left(0, 0, \frac{1}{N}, 0, 0, 0, 0\right)^{t}\right\| \leq 1,
\]

\[
\left\| T \pm \left(0, 0, 0, \frac{1}{N}, 0, 0, 0\right)^{t}\right\| \leq 1, \quad \left\| T \pm \left(0, 0, 0, 0, \frac{1}{N}, 0, 0\right)^{t}\right\| \leq 1, \quad \left\| T \pm \left(0, 0, 0, 0, 0, \frac{1}{N}, 0\right)^{t}\right\| \leq 1.
\]

By Lemma 1, \( v_{2} = v_{3} = v_{4} = v_{5} = v_{6} = 0. \) It follows that

\[
1 = av_{1} = \text{sign}(T(U_{1}))v_{1},
\]

which shows that \( v_{1} = \text{sign}(T(U_{1})) \) and \( H = \text{sign}(T(U_{1}))X_{1}. \) Therefore, \( T \in \text{sm} B_{L_{s}(2_{\mathbb{R}_{\infty}}))}. \)

**Case 2.** \( j_{0} = 2. \)

It follows that

\[
1 = \text{sign}(T(U_{2}))b.
\]

By Theorem 2, there is \( N \in \mathbb{N} \) such that

\[
\left\| T \pm \left(\frac{1}{N}, 0, 0, 0, 0, 0, 0\right)^{t}\right\| \leq 1, \quad \left\| T \pm \left(0, \frac{1}{N}, 0, 0, 0, 0, 0\right)^{t}\right\| \leq 1,
\]

\[
\left\| T \pm \left(0, 0, \frac{1}{N}, 0, 0, 0, 0\right)^{t}\right\| \leq 1, \quad \left\| T \pm \left(0, 0, 0, \frac{1}{N}, 0, 0, 0\right)^{t}\right\| \leq 1, \quad \left\| T \pm \left(0, 0, 0, 0, \frac{1}{N}, 0, 0\right)^{t}\right\| \leq 1.
\]

By Lemma 1, \( v_{1} = v_{3} = v_{4} = v_{5} = v_{6} = 0. \) It follows that

\[
1 = bv_{2} = \text{sign}(T(U_{2}))v_{2},
\]
which shows that \( v_2 = \text{sign}(T(U_2)) \) and \( H = \text{sign}(T(U_2))X_2 \). Therefore, \( T \in \text{sm } B_{L_s(\mathbb{L}_a^2 \mathbb{L}_b^2)} \).

**Case 3.** \( j_0 = 3 \).

It follows that

\[
1 = \text{sign}(T(U_3))d.
\]

By Theorem 2, there is \( N \in \mathbb{N} \) such that

\[
\left\| T \pm \left( \frac{1}{N}, 0, 0, 0, 0 \right) \right\| \leq 1, \quad \left\| T \pm \left( 0, \frac{1}{N}, 0, 0, 0 \right) \right\| \leq 1, \\
\left\| T \pm \left( 0, 0, \frac{1}{N}, 0, 0 \right) \right\| \leq 1, \quad \left\| T \pm \left( 0, 0, 0, \frac{1}{N}, 0 \right) \right\| \leq 1, \\
\left\| T \pm \left( 0, 0, 0, 0, \frac{1}{N} \right) \right\| \leq 1.
\]

By Lemma 1, \( v_1 = v_2 = v_3 = v_5 = v_6 = 0 \). It follows that

\[
1 = dv_4 = \text{sign}(T(U_3))v_4,
\]

which shows that \( v_4 = \text{sign}(T(U_3)) \) and \( H = \text{sign}(T(U_3))X_3 \). Therefore, \( T \in \text{sm } B_{L_s(\mathbb{L}_a^2 \mathbb{L}_b^2)} \).

**Case 4.** \( j_0 = 4 \).

It follows that

\[
1 = \text{sign}(T(U_4))\frac{1}{2}(a - d + e).
\]

By Theorem 2, there is \( N \in \mathbb{N} \) such that

\[
\left\| T \pm \left( \frac{1}{N}, 0, 0, \frac{1}{N}, 0 \right) \right\| \leq 1, \quad \left\| T \pm \left( 0, \frac{1}{N}, 0, 0, \frac{1}{N} \right) \right\| \leq 1, \\
\left\| T \pm \left( 0, 0, \frac{1}{N}, 0, 0 \right) \right\| \leq 1, \quad \left\| T \pm \left( 0, 0, 0, \frac{1}{N}, 0 \right) \right\| \leq 1, \\
\left\| T \pm \left( 0, 0, 0, 0, \frac{1}{N} \right) \right\| \leq 1.
\]

By Lemma 1, \( v_1 = -v_4 = v_5, v_2 = v_3 = v_6 = 0 \). It follows that

\[
1 = av_1 + bv_2 + cv_3 + dv_4 + ev_5 + fv_6 = v_1(a - d + e) = 2\text{sign}(T(U_4))v_1,
\]

which shows that \( v_1 = \frac{\text{sign}(T(U_4))}{2} \) and \( H = \text{sign}(T(U_4))X_4 \). Therefore, \( T \in \text{sm } B_{L_s(\mathbb{L}_a^2 \mathbb{L}_b^2)} \).

**Case 5.** \( j_0 = 5 \).

It follows that

\[
1 = \text{sign}(T(U_5))\frac{1}{2}(a - d - e).
\]

By Theorem 2, there is \( N \in \mathbb{N} \) such that

\[
\left\| T \pm \left( \frac{1}{N}, 0, 0, \frac{1}{N}, 0 \right) \right\| \leq 1, \quad \left\| T \pm \left( 0, \frac{1}{N}, 0, 0, \frac{1}{N} \right) \right\| \leq 1, \\
\left\| T \pm \left( 0, 0, \frac{1}{N}, 0, 0 \right) \right\| \leq 1, \quad \left\| T \pm \left( 0, 0, 0, \frac{1}{N}, 0 \right) \right\| \leq 1, \\
\left\| T \pm \left( 0, 0, 0, 0, \frac{1}{N} \right) \right\| \leq 1.
\]

By Lemma 1, \( v_1 = -v_4 = -v_5, v_2 = v_3 = v_6 = 0 \). It follows that

\[
1 = av_1 + bv_2 + cv_3 + dv_4 + ev_5 + fv_6 = v_1(a - d - e) = 2\text{sign}(T(U_5))v_1,
\]

which shows that \( v_1 = \frac{\text{sign}(T(U_5))}{2} \) and \( H = \text{sign}(T(U_5))X_5 \). Therefore, \( T \in \text{sm } B_{L_s(\mathbb{L}_a^2 \mathbb{L}_b^2)} \).
Case 6. \( j_0 = 6 \).

It follows that
\[
1 = \text{sign}(T(U_6)) \frac{1}{2} (-b + d + f).
\]

By Theorem 2, there is \( N \in \mathbb{N} \) such that
\[
\left\| T \pm \left( 0, \frac{1}{N}, 0, 0, 0, 0 \right) \right\| \leq 1, \quad \left\| T \pm \left( 0, 0, 0, 0, 0, \frac{1}{N} \right) \right\| \leq 1,
\]
\[
\left\| T \pm \left( \frac{1}{N}, 0, 0, 0, 0, 0 \right) \right\| \leq 1, \quad \left\| T \pm \left( 0, 0, \frac{1}{N}, 0, 0, 0 \right) \right\| \leq 1, \quad \left\| T \pm \left( 0, 0, 0, 0, 0, \frac{1}{N} \right) \right\| \leq 1.
\]

By Lemma 1, \( v_2 = -v_4 = -v_6, v_1 = v_3 = v_5 = 0 \). It follows that
\[
1 = av_1 + bv_2 + cv_3 + dv_4 + ev_5 + fv_6 = v_2(b - d - f) = -2 \text{sign}(T(U_6))v_2,
\]
which shows that \( v_2 = -\frac{\text{sign}(T(U_6))}{2} \) and \( H = \text{sign}(T(U_6))X_6 \). Therefore, \( T \in \text{sm } B_L(2L_2(\mathbb{R}^6)) \).

Case 7. \( j_0 = 7 \).

It follows that
\[
1 = \text{sign}(T(U_7)) \frac{1}{2} (-b + d - f).
\]

By Theorem 2, there is \( N \in \mathbb{N} \) such that
\[
\left\| T \pm \left( 0, \frac{1}{N}, 0, 0, 0, 0 \right) \right\| \leq 1, \quad \left\| T \pm \left( 0, 0, 0, 0, 0, \frac{1}{N} \right) \right\| \leq 1,
\]
\[
\left\| T \pm \left( \frac{1}{N}, 0, 0, 0, 0, 0 \right) \right\| \leq 1, \quad \left\| T \pm \left( 0, 0, \frac{1}{N}, 0, 0, 0 \right) \right\| \leq 1, \quad \left\| T \pm \left( 0, 0, 0, 0, 0, \frac{1}{N} \right) \right\| \leq 1.
\]

By Lemma 1,
\[
v_2 = -v_4 = v_6, v_1 = v_3 = v_5 = 0.
\]

It follows that
\[
1 = av_1 + bv_2 + cv_3 + dv_4 + ev_5 + fv_6 = v_2(b - d + f) = -2 \text{sign}(T(U_7))v_2,
\]
which shows that \( v_2 = -\frac{\text{sign}(T(U_7))}{2} \) and \( H = \text{sign}(T(U_7))X_7 \).

Therefore, \( T \in \text{sm } B_L(2L_2(\mathbb{R}^6)) \).

Case 8. \( j_0 = 8 \).

It follows that
\[
1 = \text{sign}(T(U_8)) \left( \frac{1}{4} (a + b + c) + \frac{1}{2} (-d + e - f) \right).
\]

By Theorem 2, there is \( N \in \mathbb{N} \) such that
\[
\left\| T \pm \left( \frac{1}{N}, -\frac{1}{N}, 0, 0, 0, 0 \right) \right\| \leq 1, \quad \left\| T \pm \left( \frac{1}{N}, 0, -\frac{1}{N}, 0, 0, 0 \right) \right\| \leq 1,
\]
\[
\left\| T \pm \left( \frac{1}{N}, 0, \frac{1}{2N}, 0, 0, 0 \right) \right\| \leq 1, \quad \left\| T \pm \left( \frac{1}{N}, 0, 0, 0, 0, -\frac{1}{2N} \right) \right\| \leq 1,
\]
\[
\left\| T \pm \left( \frac{1}{N}, 0, 0, 0, 0, \frac{1}{2N} \right) \right\| \leq 1.
\]

By Lemma 1, \( v_2 = v_1 = v_3 = -\frac{1}{2}v_4 = \frac{1}{2}v_5 = -\frac{1}{2}v_6 \). It follows that
\[
1 = av_1 + bv_2 + cv_3 + dv_4 + ev_5 + fv_6 = v_1 \left( a + b + c + 2(-d + e - f) \right) = 4 \text{sign}(T(U_8))v_1,
\]
which shows that $v_1 = \frac{\text{sign}(T(U_9))}{4}$ and $H = \text{sign}(T(U_9))X_9$. Therefore, $T \in \text{sm } B_{L_c(2L_2^0)}$.

**Case 9.** $j_0 = 9$.

It follows that

$$1 = \text{sign}(T(U_9))\left(\frac{1}{4}(a + b - c) - \frac{1}{2}d\right).$$

By Theorem 2, there is $N \in \mathbb{N}$ such that

$$\left\| T \pm \left(\frac{1}{N}, -\frac{1}{N}, 0, 0, 0, 0\right)^t \right\| \leq 1,$$

$$\left\| T \pm \left(\frac{1}{N}, 0, 0, 0, 0, 0\right)^t \right\| \leq 1,$$

By Lemma 1, $v_2 = v_1 = -v_3 = -\frac{1}{2}v_4, v_5 = v_6 = 0$. It follows that

$$1 = av_1 + bv_2 + cv_3 + dv_4 + ev_5 + f v_6 = v_1(a + b - c - 2d) = 4 \text{sign}(T(U_9))v_1,$$

which shows that $v_1 = \frac{\text{sign}(T(U_9))}{4}$ and $H = \text{sign}(T(U_9))X_9$. Therefore, $T \in \text{sm } B_{L_c(2L_2^0)}$.

**Case 10.** $j_0 = 10$.

It follows that

$$1 = \text{sign}(T(U_{10}))\left(\frac{1}{4}(a + b + c) + \frac{1}{2}(-d - e + f)\right).$$

By Theorem 2, there is $N \in \mathbb{N}$ such that

$$\left\| T \pm \left(\frac{1}{N}, -\frac{1}{N}, 0, 0, 0, 0\right)^t \right\| \leq 1,$$

$$\left\| T \pm \left(\frac{1}{N}, 0, 0, 0, 0, 0\right)^t \right\| \leq 1,$$

By Lemma 1, $v_2 = v_1 = v_3 = -\frac{1}{2}v_4 = -\frac{1}{2}v_5 = \frac{1}{2}v_6$. It follows that

$$1 = av_1 + bv_2 + cv_3 + dv_4 + ev_5 + f v_6 = v_1(a + b + c - 2d - 2e + 2f) = 4 \text{sign}(T(U_{10}))v_1,$$

which shows that $v_1 = \frac{\text{sign}(T(U_{10}))}{4}$ and $H = \text{sign}(T(U_{10}))X_{10}$. Therefore, $T \in \text{sm } B_{L_c(2L_2^0)}$.

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**References**


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У статті [Carpathian Math. Publ. 2020, 12 (2), 340-352] автор класифікував екстремальні та виставлені точки одиничної кулі простору симетричних білінійних форм на просторі $L_s(\ell_\infty^2)$, де $L_s(\ell_\infty^2)$ — це простір симетричних білінійних форм на площині із супремум нормою. У продовження результатів згаданої статті ми класифікуємо неперервні точки одиничної кулі простору симетричних білінійних форм на $L_s(\ell_\infty^2)$.

Ключові слова і фрази: гладка точка, простір симетричних білінійних форм.