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A Bezout ring with nonzero principal Jacobson radical

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In this paper, we study a commutative Bezout domain with nonzero Jacobson radical being a principal ideal. It has been proved that such a Bezout domain is a ring of the stable range 1. As a result, we have obtained that such a Bezout domain is a ring over which any matrix can be reduced to a canonical diagonal form by means of elementary transformations of its rows and columns.

Key words and phrases: Bezout domain, Jacobson radical, stable range.

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Introduction

All rings considered in the paper are commutative ones with $1 \neq 0$. Throughout the paper, $\mathbb{Z}$ and $\mathbb{Q}$ denote the ring of integers and rational numbers, respectively. We will denote the Jacobson radical and prime radical of a ring R by $J(R)$ and $P(R)$, respectively. Let us consider the example of M. Henriksen [3]:

$$R = \{z_0 + a_1x + a_2x^2 + \ldots \mid z_0 \in \mathbb{Z},\ a_i \in \mathbb{Q}\}.$$

It has been constructed as an example of a Bezout domain, which is an elementary divisor ring and is not an adequate ring. We should notice that the Jacobson radical in the ring R is of the following form

$$J(R) = \{a_1x + a_2x^2 + \ldots \mid a_i \in \mathbb{Q}\}$$

and it is a nonzero prime ideal, which is not a principal ideal.

A stable range of this ring is equal to 2. The ring of integers $\mathbb{Z}$ is an example of a ring of the stable range 2 with zero Jacobson radical.

There appears a question about the structure of a Bezout domain with nonzero principal Jacobson radical. A trivial example of such a ring is a local domain of principal ideals which is not a field.

Moreover, for the case of a domain of principal ideals, the following result holds.

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1 Main results

Theorem 1. *Let R be a domain of principal ideals. Then R is a ring with nonzero Jacobson radical if and only if R is a semilocal domain.*

Proof. Let $J(R) = aR$ be a principal ideal. Since a domain of principal ideals is a factorial and $a \neq 0$, then the element a is decomposed into the finite product of atoms, which are the generators of the maximal ideals of the ring R. Since the Jacobson radical coincides with the point of intersect of all maximal ideals, then we obtain that R contains only a finite number of maximal ideals, which means that R is a semilocal domain. The converse proposition is obvious. □

The main objective is to investigate the Bezout domains with nonzero principal Jacobson radical. For this purpose, let us introduce the necessary definitions and facts.

By a Bezout ring we mean a ring in which all finitely generated ideals are principal. A Bezout ring R with identity is said to be adequate if it satisfies the following conditions: for every $a, b \in R$, with $a \neq 0$, there exist $a_i, d \in R$ such that

(i) $a = a_i d$,

(ii) $(a_i, b) = (1)$,

and

(iii) for every nonunit divisor d' of d, we have $(d', b) \neq (1)$.

Definition 1. *A ring R is said to have the stable range 1 if for all $a, b \in R$ such that $aR + bR = R$, there exists $y \in R$ such that $(a + by)R = R$ [1].*

First, note the following results.

Theorem 2 ([2])**.** *Let R be a Bezout domain and for $a \in R\backslash(0)$ $J(R/aR)$ be a principal ideal. Then an element a is contained only in a finite number of maximal ideals which are principal ideals.*

Let us now prove the main result.

Theorem 3. *Let R be a Bezout domain in which the Jacobson radical $J(R)$ is a nonzero principal ideal. Then R is a ring of the stable range 1.*

Proof. Let $J(R) = aR$ be a principal ideal. Denote by $\bar{R} = R/J(R)$ the factor ring over the Jacobson radical, and $\bar{b} = b + J(R)$. Let $\bar{b} \neq \bar{0}$. Denote

$$\operatorname{Ann} \bar{b} = \{\bar{x} \in \bar{R} \mid \bar{x} \cdot \bar{b} = \bar{0}\},$$

where $\bar{x} = x + J(R)$.

Let $\bar{x} \neq \bar{0}$, then $bx = ay$ for some element $y \in R$. Since R is a Bezout ring, then $aR + bR = dR$, and therefore, $a = a_0 d$, $b = b_0 d$, and $a_0 dR + b_0 dR = dR$. Hence, $a_0 R + b_0 R = R$ for some elements $a_0, b_0 \in R$.

Thus, $a_0 u + b_0 v = 1$ for some elements $u, v \in R$. Since $bx = ay$, then $db_0 x = da_0 y$, $d \neq 0$. Therefore, $b_0 x = a_0 y$. It follows from the equality $a_0 u + b_0 v = 1$ that $a_0 ux + b_0 vx = x$, $a_0 ux + a_0 yv = x$. Thus, $x \in a_0 R$.

Since $a_0 b = ab_0$, then $\bar{a}_0 \in \operatorname{Ann} \bar{b}$, where $\bar{a}_0 = a_0 + J(R)$. Due to the arbitrariness of a nonzero element $\bar{x}$ and due to the fact that $x \in a_0 R$, $\bar{a}_0 \in \operatorname{Ann} \bar{b}$, we obtain that $\operatorname{Ann} \bar{b} = a_0 \bar{R}$. Moreover, $\operatorname{Ann} \bar{a}_0 = \bar{b}\bar{R}$. Note that $\bar{b}\bar{R} \cap \bar{a}_0 \bar{R} = \{\bar{0}\}$. Indeed, if $k \in \bar{b}\bar{R} \cap \bar{a}_0 \bar{R}$, then $\bar{k} = \bar{b}\bar{\lambda} = \bar{a}_0 \bar{\beta}$ for some elements $\bar{\lambda}, \bar{\beta} \in \bar{R}$. Then $\bar{k}^2 = \bar{b}\bar{\lambda} \cdot \bar{a}_0 \bar{\beta} = \bar{b}\bar{a}_0 \bar{\lambda}\bar{\beta} = \bar{0}$. Since $J(R/J(R)) = \{\bar{0}\}$, we have that the ring $\bar{R}$ is reduced, and since $\bar{k}^2 = \bar{0}$, then $\bar{k} = \bar{0}$. Therefore, $\bar{b}\bar{R} \cap \bar{a}_0 \bar{R} = \{\bar{0}\}$.

Since a Bezout domain, as well as its homomorphic image, are rings of Hermite, then according to [3, Theorem 1], we have $\bar{b}\bar{R} + \bar{a}_0 \bar{R} = \bar{\delta}\bar{R}$ for some element $\bar{\delta} \in \bar{R}$. Then $\bar{b} = \bar{b}_1 \bar{\delta}$, $\bar{a}_0 = \bar{a}_1 \bar{\delta}$ and $\bar{b}_1 \bar{R} + \bar{a}_1 \bar{R} = \bar{R}$.

Since $\bar{b}\bar{a}_1 \in \bar{b}\bar{R} \cap \bar{a}_0 \bar{R}$, then $\bar{b}\bar{a}_1 = \bar{b}_1 \bar{a}_0 = \bar{0}$.

Moreover, prove that $\bar{b}_1 \bar{a}_1 = \bar{0}$. Since $\bar{b}\bar{a}_1 = \bar{0}$, then $\bar{a}_1 \in Ann \bar{b} = \bar{a}_0 \bar{R}$, i.e. $\bar{a}_1 = \bar{a}_0 \bar{t}$ for some element $\bar{t} \in \bar{R}$. Hence,

$$\bar{b}_1 \bar{a}_1 = \bar{b}_1 \bar{a}_0 \bar{t} = \bar{b}_1 \bar{a}_1 \bar{t} \in \bar{b}\bar{R} \cap \bar{a}_0 \bar{R} = \{\bar{0}\}.$$

Consider $\bar{b}_1 \bar{a}_0 = \bar{0}$. Then $\bar{b}_1 \in Ann \bar{a}_0 = \bar{b}\bar{R}$ and $\bar{b}_1 \in \bar{b}\bar{R}$. Thus, we have shown that $\bar{a}_1 \bar{R} \subset \bar{a}_0 \bar{R}$, $\bar{b}_1 \bar{R} \subset \bar{b}\bar{R}$.

Since $\bar{R} = \bar{a}_1 \bar{R} + \bar{b}_1 \bar{R} \subset \bar{a}_0 \bar{R} + \bar{b}\bar{R}$, then $\bar{a}_0 \bar{R} + \bar{b}\bar{R} = \bar{R}$. Hence

$$\bar{a}_0 \bar{m} + \bar{b}\bar{m} = \bar{1}$$

for some elements $\bar{m}, \bar{n} \in \bar{R}$. Since $\bar{b}\bar{a}_0 = \bar{0}$, then $\bar{b}^2 \bar{n} = \bar{b}$, i.e. the element $\bar{b}$ is regular in the sense of von Neumann. Due to the arbitrariness of the element $\bar{b}$, we obtain that $\bar{R}$ is regular by von Neumann. Since a stable range of a regular ring is 1, the stable range of the ring $\bar{R}$ is equal to 1. And since $\bar{R} = R/J(R)$, then according to [1], the stable range of R is 1. □

According to [4], as a result we obtain the following assertion.

Theorem 4. *Let R be a Bezout domain with nonzero principal Jacobson radical. Then over the ring R, an arbitrary matrix can be reduced to a canonical diagonal form by means of elementary transformations of its rows and columns.*

Also we note that if R is a Bezout domain with nonzero principal Jacobson radical $J(R) = aR$, then the element $a \in R$ is free of squares in the sense of the following definition.

Definition 2. *Let R be a domain. An element $a \in R$ is said to be free of squares if for an arbitrary decomposition $a = b \cdot c$, where b, c are irreversible elements of the ring R, it always follows that $bR + cR = R$.*

Theorem 5. *Let R be a Bezout domain with nonzero principal Jacobson radical $J(R) = aR$. Then the element a is free of squares.*

Proof. Let $a = b \cdot c$, where b, c are irreversible elements of the ring R. Since R is a Bezout domain, then $bR + cR = dR$. Suppose that d is not an invertible element of the ring R. Let $b = db_0$, $c = dc_0$, where $b_0 R + c_0 R = R$.

Let us denote $\bar{R} = R/J(R)$. Show that $\bar{b}_0 \bar{d}\bar{c}_0 \in J(\bar{R}) = \{\bar{0}\}$. First, note that $\bar{x} = x + J(R)$ is an invertible element in $\bar{R}$ if and only if $xR + aR = R$. Indeed, if $xR + aR = R$, then $xu + av = 1$ for some elements $u, v \in R$. Hence

$$\bar{x}\bar{u} + \bar{a}\bar{v} = \bar{1},$$

and since $\bar{a} = \bar{0}$, then $\bar{x} \cdot \bar{u} = \bar{1}$, i.e. $\bar{x}$ is an invertible element of the ring $\bar{R}$. The converse proposition is analogous.

Prove that $\bar{b}_0\bar{d}\bar{c}_0 \in J(\bar{R})$. Let $\bar{x}$ be an arbitrary element of the ring $\bar{R}$. We show that $\bar{1} - \bar{b}_0\bar{d}\bar{c}_0\bar{x}$ is an invertible element in $\bar{R}$, i.e.

$$(1 - b_0dc_0x)R + aR = R.$$

Since $b_0dc_0 = bc_0 = b_0c$, then $(1 - bc_0x)R + bR = R$.

$$(1 - bc_0x) \cdot 1 + (b \cdot c_0x) = 1.$$

Then $(1 - b_0dc_0x)R + bR = R$ and similarly is proved that $(1 - b_0dc_0x)R + cR = R$. Hence

$$(1 - b_0dc_0x)R + bcR = (1 - b_0dc_0x)R + aR = R.$$

Therefore, $b_0dc_0 = at$ for some element $t \in R$. Therefore, $b_0da_0 = bct = b_0dc_0dt$, i.e. $dt = 1$. Thus, here we obtain a contradiction with the choice of the element d. □

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У роботі досліджено комутативну область Безу з ненульовим радикалом Джекобсона, який є головним ідеалом. Доведено, що така область Безу є кільцем стабільного рангу 1. Як наслідок, отримано, що така область Безу є кільцем, над яким довільна матриця зводиться до канонічного діагонального вигляду шляхом елементарних перетворень рядків і стовпців.

Ключові слова і фрази: область Безу, радикал Джекобсона, стабільний ранг.