ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2022, **14** (2), 513–528
doi:10.15330/cmp.14.2.513-528

https://journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2022, Т.14, №2, С.513–528

On a nonlocal problem for the first-order differential-operator equations

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In this work, we study the spaces of generalized elements identified with formal Fourier series and constructed via a non-negative self-adjoint operator in Hilbert space. The spectrum of this operator is purely discrete. For a differential-operator equation of the first order, we formulate a nonlocal multipoint by time problem if the corresponding condition is satisfied in a positive or negative space that is constructed via such operator; such problem can be treated as a generalization of an abstract Cauchy problem for the specified differential-operator equation. The correct solvability of the aforementioned problem is proven, a fundamental solution is constructed, and its structure and properties are studied. The solution is represented as an abstract convolution of a fundamental solution with a boundary element. This boundary element is used to formulate a multipoint condition, and it is a linear continuous functional defined in the space of main elements. Furthermore, this solution satisfies multipoint condition in a negative space that is adjoint with a corresponding positive space of elements.

Key words and phrases: nonlocal multipoint problem, differential-operator equation, self-adjoint operator, Hilbert space, correct solvability.

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1 Introduction

In the study of many problems of analysis and mathematical physics, instead of a pair of spaces, the main and conjugate spaces, we use the triplet of Hilbert spaces of type $H_+ \subset H \subset H_-$, where H_+ is a positive space , H_- is a negative space, H_+, H_- are spaces of the main and generalized elements (continuous linear functionals on H_+). The role of H in such a chain is that the scalar product in it can be extended to a continuous bilinear form on $H_- \times H_+$, which sets the action of the generalized element on the main one.

Various functional spaces (e.g., Sobolev, analytic functions, infinitely differentiable, and Schwartz distributions) can be interpreted as positive and negative spaces with respect to L_2, constructed by functions from a differentiation operator or multiplication by an independent variable, or as projective or inductive limits. In [1], the authors discuss the spaces of the main and generalized elements that are built by functions of an arbitrary self-adjoint operator.

This paper considers spaces of generalized elements identified with formal Fourier series and constructed by an integral self-adjoint operator A in Hilbert space; the spectrum of this operator is purely discrete. For the differential-operator equation $u'(t) + \varphi(A)u(t) = 0$, $t \in (0, T]$ ($\varphi(A)$ is a function of the operator A) is a nonlocal multipoint by time problem if

УДК 517.98
2020 *Mathematics Subject Classification:* 35Sxx, 47B39.

the corresponding condition is satisfied in a positive or negative space that is constructed by the operator A (such a problem can be understood as a certain generalization of the Cauchy abstract problem for the specified differential operator equation). A nonlocal multipoint by time problem belongs to nonlocal problems for differential-operator equations. Such problems arise in: *(i)* modeling many processes and applied tasks with boundary value problems for differential-operator equations with nonlocal conditions, *(ii)* description of all correct problems for a particular operator, *(iii)* construction of a general theory of boundary value problems for differential-operator equations. Many mathematicians have been involved in the study of such nonlocal problems (see, for example, [2–11]). Important results were obtained concerning the formulation, correct solvability and solution construction, and conditions of regularity and irregularity of boundary conditions were formulated for important cases of differential-operator equations.

In this paper we establish the correct solvability of a nonlocal multipoint by time problem for the differential-operator equation $u'(t) + \varphi(A)u(t) = 0$, $t \in (0, T]$; at the same time we build a fundamental solution of $G(t)$, $t \in (0, T]$ of such problem, and investigate its structure and properties. The solution $u(t)$ is given in the form of an abstract convolution $G(t) * g$, where the boundary element g is a continuous linear functional defined on a certain space of main elements (positive space H_+, constructed by the operator A), with $\{G(t), u(t)\} \subset H_+$ at each $T \in (0, T]$, but the convolution of $G(t) * g$ satisfies multipoint condition in the negative space H_-.

2 Spaces of main and generalized elements. The formal Fourier series

Let H be a separable Hilbert space with a scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$, $\{e_k : k \in \mathbb{N}\}$ is orthonormal basis in H. Let us denote

$$\Phi_m = \Big\{\varphi \in H \,|\, \varphi = \sum_{k=1}^{m} c_{k,\varphi} e_k,\ c_{k,\varphi} \in \mathbb{C}\Big\}, \quad \Phi = \bigcup_{m=1}^{\infty} \Phi_m,$$

and Φ' is the space of all antilinear continuous functionals on Φ with weak convergence. Matching

$$H \ni \varphi \longrightarrow f_\varphi \in \Phi' : \ \langle f_\varphi, \psi\rangle = (\varphi, \psi), \quad \forall \psi \in \Phi,$$

($\langle f_\varphi, \psi\rangle$ denotes the action of f_φ on ψ) defines the $H \subset \Phi'$ embedding. So, $\Phi \subset H \subset \Phi'$, and these embeddings are dense and continuous [12]. Elements from Φ' are called generalized elements.

Let s be the space of all numerical sequences $\{c_k : k \in \mathbb{N}\}$, $c_k \in \mathbb{C}$, with coordinate convergence. Let us define the mapping

$$F : \ \Phi' \ni f \to \{c_k(f) = \langle f, e_k\rangle : k \in \mathbb{N}\} \in s,$$

which is isomorphism [12].

The series $\sum\limits_{k=1}^{\infty} c_k(f) e_k$, where $c_k(f) = \langle f, e_k\rangle$, is called the Fourier series of the element $f \in \Phi'$, and the numbers $c_k(f)$ are its Fourier coefficients. For an arbitrary element $f \in \Phi'$ its Fourier series converges in Φ' to f; on the contrary, an arbitrary series $\sum\limits_{k=1}^{\infty} c_k e_k$ weakly converges in Φ' to some element $f \in \Phi'$, and this series is a Fourier series for f [12]. So, Φ' can be understood as the space of formal series of the form $\sum_{k=1}^{\infty} c_k e_k$.

Let G be a continuous monotonically increasing on $[0,\infty)$ function such that

$$\exists\, c > 0\ \forall\, x \in [0,\infty): \quad G(x) \geq c,\ \lim_{x\to+\infty} G(x) = +\infty,\ \sum_{k=1}^{\infty} G^{-2}(k) < +\infty.$$

Let us construct an operator by function G in the space Φ'

$$\hat{A}:\ \Phi' \ni f = \sum_{k=1}^{\infty} c_k(f) e_k \longrightarrow \sum_{k=1}^{\infty} G(k) c_k(f) e_k = \hat{A} f \in \Phi',\ \ c_k(f) = \langle f, e_k\rangle.$$

Obviously, the operator $\hat{A}$ is linear and continuous in Φ'. If A is a restriction of the operator $\hat{A}$ to H, then A is an integral self-adjoint operator in H with a dense domain $\mathcal{D}(A) = \{\varphi \in H : \sum_{k=1}^{\infty} G^2(k)|c_k(\varphi)|^2 < \infty, c_k(\varphi) = (\varphi, e_k), k \in \mathbb{N}\}$, with $\Phi \subset \mathcal{D}(A)$. The spectrum of A is purely discrete with a single infinite boundary point $\sigma(A) = \{\lambda_k : k \in \mathbb{N}\}$, where $\lambda_k = G(k)$, $k \in \mathbb{N}$ (see [13, p. 14]).

Now let us introduce some classes of elements associated with the operator A. Suppose that a monotonically increasing sequence $\{m_n : n \in \mathbb{Z}_+\}$ of positive numbers has the properties:

1) $\exists\, M > 0\ \ \exists\, h > 0$: $m_{n+1} \leq M h^n m_n,\ n \in \mathbb{Z}_+,\ m_0 = 1$;

2) $\forall\, \alpha > 0\ \exists\, c_\alpha > 0$: $m_n \geq c_\alpha \alpha^n,\ \ n \in \mathbb{Z}_+$.

Let us denote by $H_\infty(A) = \lim\limits_{\alpha\to\infty} \operatorname{pr} H_\alpha(A)$ the inverse (projective) limit, $H_\alpha(A) = \mathcal{D}(A^\alpha)$, $\mathcal{D}(A^\alpha)$ is domain of the operator

$$A^\alpha:\ \mathcal{D}(A^\alpha) = \Big\{\varphi \in H:\ \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |c_k(\varphi)|^2 < \infty,\ \lambda_k = G(k)\Big\},$$

$$(\varphi,\psi)_{H_\alpha} := (\varphi,\psi) + (A^\alpha\varphi, A^\alpha\psi), \quad \forall\{\varphi,\psi\} \subset \mathcal{D}(A^\alpha),$$

$$H_\alpha\langle m_n\rangle := \{\varphi \in H_\infty(A) \,\big|\, \exists\, c > 0:\ \ \|A^n\varphi\| \leq c\alpha^n m_n,\ \alpha > 0,\ n \in \mathbb{Z}_+\}.$$

The space $H_\alpha\langle m_n\rangle \supset \Phi$ is a Banach space with respect to the norm

$$\|\varphi\|_{H_\alpha\langle m_n\rangle} = \sup_{n\in\mathbb{Z}_+} \frac{\|A^n\varphi\|}{\alpha^n m_n}.$$

Let us denote by $H_\infty\langle m_n\rangle := \bigcup\limits_{\alpha>0} \operatorname{ind} H_\alpha\langle m_n\rangle$ the direct (inductive) limit. It is obvious that $\Phi \subset H_\infty\langle m_n\rangle \subset H_\infty(A) \subset H$, and all the embeddings are dense and continuous. If $H'_\infty(A)$, $H'_\alpha\langle m_n\rangle$ are the spaces of antilinear continuous functionals defined respectively on $H_\infty(A)$, $H_\alpha\langle m_n\rangle$ with the weak convergence, then, according to [12], we obtain a chain of dense and continuous embeddings:

$$H \subset H'_\infty(A) \subset H'_\infty\langle m_n\rangle \subset \Phi', \quad \text{and at the same time} \quad H'_\infty\langle m_n\rangle = \lim_{\alpha\to\infty} \operatorname{pr} H'_\alpha\langle m_n\rangle.$$

The spaces $G_{\{\beta\}}(A) := H_\infty\langle n^{n\beta}\rangle$, $\beta > 0$, are called the Gevrey spaces generated by the operator A; $G_{\{1\}}(A)$ matches with the set of analytic vectors of the operator A [12].

Let $\rho(\lambda) = \sup\limits_{n\in\mathbb{Z}_+} \dfrac{\lambda^n}{m_n}$, $\lambda \in [1,+\infty)$. It follows from the properties of the sequence $\{m_n : n \in \mathbb{Z}_+\}$ that the function $\rho(\lambda)$ is continuous, monotonically increasing on $[1,+\infty)$ (faster than λ^n, $\forall n \in \mathbb{N}$), $\rho(\lambda) \geq 1$, $\forall\lambda \in [1,+\infty)$. The space $H_\infty\langle m_n\rangle$ is the inductive limit of Hilbert spaces (see [12])

$$H_{\{\alpha\}} = \Big\{ f \in \Phi' \;:\; \sum_{k=1}^{\infty} |c_k(f)|^2 \rho^2\Big(\frac{\lambda_k}{\alpha}\Big) < \infty, \quad \lambda_k = G(k),\ c_k(f) = \langle f, e_k\rangle \Big\},$$

the scalar product in $H_{\{\alpha\}}$ is defined by the formula

$$(f,g)_{H_{\{\alpha\}}} = \sum_{k=1}^{\infty} c_k(f)\overline{c_k(g)}\rho^2\Big(\frac{\lambda_k}{\alpha}\Big), \quad \{f,g\} \subset H_{\{\alpha\}}.$$

In terms of the behavior of the Fourier coefficients of their elements, the spaces $H_\infty\langle m_n\rangle$ and $H'_\infty\langle m_n\rangle$ are described in [12], also known as "conditions A–B":

$$(f \in H_\infty\langle m_n\rangle) \Leftrightarrow (\exists\,\mu > 0\,\exists\, c > 0\,\forall\, k \in \mathbb{N}: \quad |c_k(f)| \leq c\rho^{-1}(\mu\lambda_k)); \qquad (A)$$

$$(f \in H'_\infty\langle m_n\rangle) \Leftrightarrow (\forall\mu > 0\,\exists\, c = c(\mu) > 0\,\forall\, k \in \mathbb{N}: \quad |c_k(f)| \leq c\rho(\mu\lambda_k)). \qquad (B)$$

If $m_n = n^{n\beta}$, $\beta > 0$, then $\rho(\lambda) \sim \exp\{\lambda^{1/\beta}\}$, that is, in this case for $f \in \Phi'$, the following equivalence relations are correct:

$$(f \in G_{\{\beta\}}(A)) \Leftrightarrow (\exists\,\mu > 0\,\exists\, c > 0\ \forall\, k \in \mathbb{N}: \quad |c_k(f)| \leq c\exp(-\mu\lambda_k^{1/\beta})),$$

$$(f \in G'_{\{\beta\}}(A)) \Leftrightarrow (\forall\,\mu > 0\ \exists\, c = c(\mu) > 0\ \forall\, k \in \mathbb{N}: \quad |c_k(f)| \leq c\exp(\mu\lambda_k^{1/\beta})).$$

As an example, consider the Hilbert space $H = L_2(\mathbb{R})$. In this space, Hermite functions

$$h_k(x) = (-1)^k \pi^{-1/4}(2^k k!)^{-1/2} e^{x^2/2}(e^{-x^2})^{(k)}, \quad k \in \mathbb{Z}_+,\ x \in \mathbb{R}.$$

form the orthonormal basis. By function $G(x) = 2x + 1$, $x \in [0,\infty)$, and the corresponding scheme, we construct in H a non-negative self-adjoint operator A, which coincides with a harmonic oscillator which is a non-negative self-adjoint operator, generated in $L_2(\mathbb{R})$ by the differential expression $-d^2/dx^2 + x^2$. The Hermite functions h_k, $k \in \mathbb{Z}_+$, are the eigenfunctions of this operator, and $\lambda_k = G(k) = 2k + 1$, $k \in \mathbb{Z}_+$, are its eigenvalues. In [1, p.145] it is proved that $G_{\{\beta\}}(A) \equiv H_\infty\langle n^{n\beta}\rangle = S_{\beta/2}^{\beta/2}$, $\beta \geq 1$, where $S_{\beta/2}^{\beta/2}$ are S spaces defined in [14]. The elements of these spaces are infinitely differentiable functions on $\mathbb{R}$ that satisfy the condition

$$|x^k\varphi^{(m)}(x)| \leq cA^k k^{k\beta/2}\exp(-a|x|^{2/\beta}), \ \ x \in \mathbb{R}, \ \ \{k,m\} \subset \mathbb{Z}_+,$$

with some constants c, $A, a > 0$ (dependent on φ).

Therefore, the spaces S_β^β and $(S_\beta^\beta)'$ can be characterized as follows with the Fourier coefficients of their elements (see conditions (A), (B) above):

a) $f \in S_\beta^\beta \Leftrightarrow \exists\,\mu > 0\ \exists\, c > 0\ \forall\, k \in \mathbb{Z}_+$: $|c_k(f)| \leq c\exp(-\mu(2k+1)^{\frac{1}{2\beta}})$;

b) $f \in (S_\beta^\beta)' \Leftrightarrow \forall\,\mu > 0\ \exists\, c = c(\mu) > 0\ \forall\, k \in \mathbb{Z}_+$: $|c_k(f)| \leq c\exp(\mu(2k+1)^{\frac{1}{2\beta}})$, $c_k(f) = \langle f, h_k\rangle$.

3 Non-negative self-adjoint operators as convolution operators

Let $\{f_1, f_2\} \subset \Phi'$, $f_1 = \sum_{k=1}^{\infty} c_k(f_1)e_k$, $f_2 = \sum_{k=1}^{\infty} c_k(f_2)e_k$.

In the space Φ' we define the operation $*$, which we call the "abstract convolution" (or simply convolution), by

$$f_1 * f_2 := \sum_{k=1}^{\infty} c_k(f_1)c_k(f_2)e_k \equiv \sum_{k=1}^{\infty} c_k(f_1 * f_2)e_k.$$

Hence the properties of the commutativity and associativity of the convolution in the space Φ' hold. Therefore, Φ' is a ring (relative to the convolution) with the unit element being the generalized element $e = \sum_{k=1}^{\infty} e_k \in \Phi'$.

The convolution has the following properties:

1) if $\{f_1, f_2\} \subset H'_\infty\langle m_n\rangle$, then $f_1 * f_2 \in H'_\infty\langle m_n\rangle$,

2) if $f_1 \in H'_\infty\langle m_n\rangle$, $f_2 \in H_\infty\langle m_n\rangle$, then $f_1 * f_2$ is an element of the space $H_\infty\langle m_n\rangle$.

Proof of these properties uses the statements (A), (B), and the convexity property of $\ln\rho$ [15, p. 89], namely

$$\forall\{\lambda_1, \lambda_2\} \subset [1, +\infty) : \ln\rho(\lambda_1) + \ln\rho(\lambda_2) \leq \ln\rho(\lambda_1 + \lambda_2). \tag{1}$$

Formula (1) corresponds to the definition of the convexity of the function from [16, p. 8].

Let F: $[0, \infty) \to [0, \infty)$ be some continuous function. Let us construct the operator B by the function F:

$$B\varphi := \sum_{k=1}^{\infty} F(\lambda_k)c_k(\varphi)e_k, \;\; \lambda_k = G(k),$$

$$\varphi \in \mathcal{D}(B) = \Big\{\varphi \in H : \sum_{k=1}^{\infty} F^2(\lambda_k)|c_k(\varphi)|^2 \equiv \sum_{k=1}^{\infty} |c_k(B\varphi)|^2 < \infty\Big\}.$$

Note that B is a non-negative self-adjoint operator in H with a dense domain, and $\Phi \subset \mathcal{D}(B)$, $\sigma(B) = \{F(\lambda_k) : \lambda_k = G(k), k \in \mathbb{N}\}$.

Theorem 1. *If continuous on* $[0, \infty)$ *function* F *satisfies the condition*

$$\forall\,\varepsilon > 0\;\exists\,c_\varepsilon > 0\;\forall\,x \in [0, \infty) : 0 \leq F(x) \leq c_\varepsilon\rho(\varepsilon x), \tag{2}$$

then the operator B *is continuous in space* $H_\infty\langle m_n\rangle \subset H$ *and maps this space onto itself.*

Proof. First let us prove that $B\varphi \in H_\infty\langle m_n\rangle$ if $\varphi = \sum_{k=1}^{\infty} c_k(\varphi)e_k \in H_\infty\langle m_n\rangle$. Since

$$c_k(B\varphi) = (B\varphi, e_k) = (\varphi, Be_k) = F(\lambda_k)(\varphi, e_k) = F(\lambda_k)c_k(\varphi), \quad k \in \mathbb{N},$$

then due to the condition (A) it suffices to prove that

$$\exists\,\mu_0 > 0\;\exists\,c_0 > 0\;\forall\,k \in \mathbb{N} : \;\; F(\lambda_k)|c_k(\varphi)| \leq c_0\rho^{-1}(\mu_0\lambda_k).$$

Given that $\varphi \in H_\infty\langle m_n\rangle$, it follows that

$$\exists\, \mu_1 > 0\ \exists\, c_1 > 0\ \forall\, k \in \mathbb{N}: \quad |c_k(\varphi)| \le c_1\rho^{-1}(\mu_1\lambda_k).$$

So,

$$F(\lambda_k)|c_k(\varphi)| \le c_\varepsilon c_1 \rho(\varepsilon\lambda_k)\rho^{-1}(\mu_1\lambda_k) = c_\varepsilon c_1 e^{\ln\rho(\varepsilon\lambda_k)-\ln\rho(\mu_1\lambda_k)}.$$

Take ε from $(0,\mu_1)$. Given the inequality of the convexity (1) for $\ln\rho$, we find that

$$\ln\rho(\varepsilon\lambda_k) - \ln\rho(\mu_1\lambda_k) \le -\ln\rho((\mu_1-\varepsilon)\lambda_k) \equiv -\ln\rho(\mu_0\lambda_k),$$

where $\mu_0 = \mu_1 - \varepsilon$. Then

$$F(\lambda_k)|c_k(\varphi)| \le c_0 e^{-\ln\rho(\mu_0\lambda_k)} = c_0\rho^{-1}(\mu_0\lambda_k),$$

it follows that $B\varphi \in H_\infty\langle m_n\rangle$.

Let us prove that B is a continuous operator in the space $H_\infty\langle m_n\rangle$, that is, B maps each bounded set of this space to a bounded set of the same space. Let L be a bounded set in the space $H_\infty\langle m_n\rangle$. Since $H_\infty\langle m_n\rangle = \bigcup\limits_{\alpha>0} H_{\{\alpha\}}$, then L is a bounded set in some Hilbert space $H_{\{\alpha_0\}}$, i.e.

$$\exists\, b > 0\ \forall\, \varphi \in L: \quad \|\varphi\|_{H_{\{\alpha_0\}}} = \sum_{k=1}^{\infty} |c_k(\varphi)|^2\rho^2\Big(\frac{\lambda_k}{\alpha_0}\Big) \le b.$$

So,

$$\forall\, \varphi \in L: |c_k(\varphi)| \le b_1\rho^{-1}\Big(\frac{\lambda_k}{\alpha}\Big),\ b_1 = \sqrt{b}, \quad k \in \mathbb{N}.$$

In the inequality (2) let us assume that $\varepsilon = (2\alpha_0)^{-1}$. Then, using the inequality of the convexity (1), we find that

$$|c_k(B\varphi)| = F(\lambda_k)|c_k(\varphi)| \le c_\varepsilon b_1\rho(\varepsilon\lambda_k)\rho^{-1}\Big(\frac{\lambda_k}{\alpha_0}\Big) = b_1c_\varepsilon e^{\ln\rho(\varepsilon\lambda_k)-\ln\rho\left(\frac{\lambda_k}{\alpha_0}\right)}$$

$$\le b_1c_\varepsilon e^{-\ln\rho\left(\left(\frac{1}{\alpha_0}-\varepsilon\right)\lambda_k\right)} = b_1c_\varepsilon\rho^{-1}\Big(\Big(\frac{1}{\alpha_0}-\varepsilon\Big)\lambda_k\Big) = b_2\rho^{-1}\Big(\frac{\lambda_k}{2\alpha_0}\Big),\ b_2 = b_1c_\varepsilon, \quad k \in \mathbb{N}.$$

Therefore, the set BL is bounded in the space $H_{\{2\alpha_0\}} \subset H_\infty\langle m_n\rangle$, that is, in the space $H_\infty\langle m_n\rangle$. The theorem is proved. □

Remark 1. *The condition* (2) *on F is equivalent to the fact that $f = \sum\limits_{k=1}^{\infty} F(\lambda_k)e_k$ is an element of $H'_\infty\langle m_n\rangle$.*

Considering Remark 1 and property 2) of the convolution, operator B can be interpreted as a convolution operator in the space $H_\infty\langle m_n\rangle$:

$$B\varphi = f * \varphi, \forall\varphi \in H_\infty\langle m_n\rangle, B\varphi \in H_\infty\langle m_n\rangle, \quad f = \sum_{k=1}^{\infty} F(\lambda_k)e_k \in H'_\infty\langle m_n\rangle.$$

Further we assume that F additionally satisfies the condition

$$\exists\, c_0 > 0\ \exists\, d_0 > 0\ \forall\, x \in [0,+\infty): \quad F(x) \ge d_0\ln\rho(c_0x). \tag{3}$$

4 Nonlocal multipoint by time problem

Let us consider the differential-operator equation

$$u'(t) + Bu(t) = 0, \quad t \in (0, T], \tag{4}$$

where B is the operator built in Section 3, which is linear and continuous in $H_\infty\langle m_n\rangle$. As the solution of the equation (4) we understand the function u: $(0, T] \to H_\infty\langle m_n\rangle$, strongly differential in H, which satisfies the equation (4).

Let us consider the following problem: find a function u that is a solution of the equation (4) and satisfies the condition

$$\mu u(0) - \sum_{k=1}^{m} \mu_k B_k u(t_k) = g, \quad g \in H, \tag{5}$$

where $m \in \mathbb{N}$, $\{\mu, \mu_1, \dots, \mu_m\} \subset (0, \infty)$, $\{t_1, \dots, t_m\} \subset (0, T]$ are fixed numbers, and at the same time $\mu > \sum\limits_{k=1}^{m} \mu_k$, $t_1 < t_2 < \dots < t_m \le T$; $B_1, \dots, B_m$ are operators in H constructed by functions $g_1, \dots, g_m$ respectively (see Section 3). Here g_k: $[0, \infty) \to [0, \infty)$, $k \in \{1, \dots, m\}$, are continuous functions that satisfy the condition

$$\forall\, \varepsilon > 0\ \forall\, x \in [0, +\infty): \ \ 0 \le g_k(x) \le e^{\varepsilon F(x)}, \quad k \in \{1, \dots, m\}, \tag{6}$$

the operators $B_1, \dots, B_m$ are non-negative self-adjoint in H with the dense domains, $\sigma(B_i) = \{g_i(\lambda_k) : \lambda_k = G(k), k \in \mathbb{N})\}$, $i \in \{1, \dots, m\}$, with $u(0)$ being understood as $\lim\limits_{t \to +0} u(t)$, where the limit is considered in the space H. The problem (4), (5) is hereinafter called the nonlocal multipoint by time problem for the equation (5).

Let $u(t)$ be a solution of the equation (5). Since $u(t) \in H_\infty\langle m_n\rangle$ for every $t \in (0, T]$, then

$$u(t) = \sum_{k=1}^{\infty} c_k(u(t)) e_k, \quad c_k(u(t)) = (u(t), e_k), \quad t \in (0, T],$$

while also

$$\|u(t)\|^2 = \sum_{k=1}^{\infty} |c_k(u(t))|^2.$$

To find $c_k(u(t))$, we take scalar product of (4) and e_k, $k \in \mathbb{N}$:

$$(u'(t), e_k) + (Bu(t), e_k) = 0.$$

When $k \in \mathbb{N}$ is fixed, we have

$$(Bu(t), e_k) = (u(t), Be_k) = (u(t), F(\lambda_k) e_k) = F(\lambda_k)(u(t), e_k) = F(\lambda_k) c_k(u(t)).$$

The strong differentiability of $u(t)$ on $(0, T]$ implies the differentiability of the function $c_k(u(t)) \equiv \tilde{c}_k(t) = (u(t), e_k)$, $k \in \mathbb{N}$. So,

$$\Big(\frac{du(t)}{dt}, e_k\Big) = \frac{d}{dt}(u(t), e_k) = \frac{d}{dt} c_k(u(t)), \quad k \in \mathbb{N}.$$

Also, note that there exists a $\lim\limits_{t\to+0}\tilde{c}_k(t)=\tilde{c}_k(0)=c_k(u(0))$. Indeed,

$$\tilde{c}_k(t)=(u(t),e_k),\quad \tilde{c}_k(0)=(u(0),e_k),$$
$$|\tilde{c}_k(t)-\tilde{c}_k(0)|=|(u(t)-u(0),e_k)|\le\|u(t)-u(0)\|_H\to 0,\quad t\to+0.$$

The function $\tilde{c}_k(t)$ satisfies the differential equation $\tilde{c}\,'_k(t)+F(\lambda_k)\tilde{c}_k(t)=0$, a general solution of which is $\tilde{c}_k(t)=c_k\exp\big\{-tF(\lambda_k)\big\}$, $c_k=\text{const}$, $k\in\mathbb{N}$. Then

$$u(t)=\sum_{k=1}^{\infty}c_k\exp\{-tF(\lambda_k)\}e_k. \tag{7}$$

To find c_k, $k\in\mathbb{N}$, we take a scalar product of (5) and e_k, $k\in\mathbb{N}$. As a result, we come to the relation

$$\mu\tilde{c}_k(0)-\sum_{n=1}^{m}\mu_n g_n(\lambda_k)\tilde{c}_k(t_n)=c_k(g),\quad c_k(g)=(g,e_k),\quad k\in\mathbb{N},$$

(given that $e_k\in\mathcal{D}(B_n)$ for every $k\in\mathbb{N}$ and $n\in\{1,\dots,m\}$, e_k is an eigenvector of B_n and $g_n(\lambda_k)$ is its eigenvalue).

Given the form of $\tilde{c}_k(t)$, we find that

$$c_k\Big(\mu-\sum_{n=1}^{m}\mu_n g_n(\lambda_k)\exp\{-t_nF(\lambda_k)\}\Big)=c_k(g).$$

So,

$$c_k=c_k(g)\Big(\mu-\sum_{n=1}^{m}\mu_n g_n(\lambda_k)\exp\{-t_nF(\lambda_k)\}\Big)^{-1},\quad k\in\mathbb{N}.$$

We introduce the notations:

$$Q_1(t,\lambda_k):=\exp\{-tF(\lambda_k)\},$$
$$Q_2(\lambda_k):=\Big(\mu-\sum_{n=1}^{m}\mu_n g_n(\lambda_k)\exp\{-t_nF(\lambda_k)\}\Big)^{-1}=\Big(\mu-\sum_{n=1}^{m}\mu_n g_n(\lambda_k)Q_1(t_n,\lambda_k)\Big)^{-1}.$$

Then

$$\tilde{c}_k(t)=c_k(u(t))=Q_1(t,\lambda_k)Q_2(\lambda_k)c_k(g),\quad k\in\mathbb{N},$$
$$u(t)=\sum_{k=1}^{\infty}c_k(u(t))e_k=\sum_{k=1}^{\infty}Q_1(t,\lambda_k)Q_2(\lambda_k)c_k(g)e_k=G(t)*g,$$

where

$$G(t)=\sum_{k=1}^{\infty}Q_1(t,\lambda_k)Q_2(\lambda_k)e_k,\quad g=\sum_{k=1}^{\infty}c_k(g)e_k\in H.$$

From the constraints imposed on the functions F, $g_1,\dots,g_m$ and parameters of the problem (4), (5) the inequalities

$$Q_1(t,\lambda_k)\le e^{-d_0t\ln\rho(c_0\lambda_k)},\quad Q_2(\lambda_k)\le\Big(\mu-\sum_{n=1}^{m}\mu_n e^{\varepsilon F(\lambda_k)-t_1F(\lambda_k)}\Big)^{-1}$$

follow (it is taken into account that $t_1<t_2<\dots<t_m$). Putting $\varepsilon=t_1$, we arrive at the estimate $Q_2(\lambda_k)\le\Big(\mu-\sum\limits_{n=1}^{m}\mu_n\Big)^{-1}$, provided $\mu>\sum\limits_{n=1}^{m}\mu_n$. Therefore, for each $t\in(0,T]$ we have

$$|c_k(G)|=|Q_1(t,\lambda_k)||Q_2(\lambda_k)|\le\gamma e^{-d_0t\ln\rho(c_0\lambda_k)},\quad \gamma=\Big(\mu-\sum_{n=1}^{m}\mu_n\Big)^{-1}.$$

Based upon aforementioned statements and the characteristics of the class $H_\infty\langle m_n\rangle$, we conclude that $G(t) \in H_\infty\langle m_n\rangle$ for every $t \in (0, T]$. Indeed, if φ is a convex function on $[0, +\infty)$, then in addition to the inequality (1) we have: a) $\forall\, \alpha \in (0, 1)\quad \forall\, x \in [0, +\infty)$: $\varphi(\alpha x) \le \alpha\varphi(x)$; b) $\forall\, \alpha \ge 1\quad \forall\, x \in [0, +\infty)$: $\varphi(\alpha x) \ge \alpha\varphi(x)$. So, if $d_0 t < 1$, then taking into account a) we get such inequalities:

$$|c_k(G)| \le \gamma e^{-d_0 t \ln\rho(c_0\lambda_k)} \le \gamma e^{-\ln\rho(a_1\lambda_k)} = \gamma\rho^{-1}(a_1\lambda_k), a_1 = d_0 t, \quad k \in \mathbb{N}.$$

If $d_0 t > 1$ and $d_0 t$ is non-integer, then $d_0 t = [d_0 t] + \{d_0 t\}$. Then

$$\begin{aligned} e^{-d_0 t \ln\rho(c_0\lambda_k)} &= e^{-[d_0 t]\ln\rho(c_0\lambda_k)} \cdot e^{-\{d_0 t\}\ln\rho(c_0\lambda_k)} \\ &\le e^{-\{d_0 t\}\ln\rho(c_0\lambda_k)} \le e^{-\ln\rho(a_2\lambda_k)} = \rho^{-1}(a_2\lambda_k), \quad a_2 = \{d_0 t\}. \end{aligned}$$

If $d_0 t = n$, $n \in \{2, 3, \ldots,\}$, then $d_0 t = 1 + n - 1$ and

$$e^{-d_0 t \ln\rho(c_0\lambda_k)} = e^{-\ln\rho(c_0\lambda_k)} e^{-(n-1)\ln\rho(c_0\lambda_k)} \le e^{-\ln\rho(c_0\lambda_k)} = \rho^{-1}(c_0\lambda_k), \quad k \in \mathbb{N}.$$

Let $a = \min\{a_1, a_2, c_0\}$. Then, at fixed $t \in (0, T]$, the following is fulfilled: $|c_k(G)| \le \gamma\rho^{-1}(a\lambda_k)$, $k \in \mathbb{N}$. Based on this (and on condition (A)), it follows that $G(t, \cdot) \in H_\infty\langle m_n\rangle$ for every $t \in (0, T]$. Since $u(t) = G(t) * g$, where $g \in H \subset H'_\infty\langle m_n\rangle$, $G(t) \in H_\infty\langle m_n\rangle$ for every $t \in (0, T]$, then based on Lemma 1 we claim that $u(t) \in H_\infty\langle m_n\rangle$ for every $t \in (0, T]$.

Remark 2. *If $u(t)$ is represented by the formula* (7)*, then it is a solution of the equation* (4)*.*

In fact, according to the definition of B, we have that $Bu(t) = \sum_{k=1}^{\infty} c_k F(\lambda_k) e^{-tF(\lambda_k)} e_k$.

Further, we will now prove that $u(t)$ is a strongly differentiable function on $(0, T]$, and $u'(t) = -\sum_{k=1}^{\infty} c_k F(\lambda_k) e^{-tF(\lambda_k)} e_k$.

It follows that the function $u(t)$, $t \in (0, T]$, is a solution of the equation (4). Therefore, the formula (7) describes all solutions of the equation (4), that is, $u(t)$ is the solution of the equation (4) if and only if it is represented by the formula (7).

Show the uniqueness of the solution of the problem (4), (5). In fact, the solution of the problem (4), (5) is given as

$$u(t) = \sum_{k=1}^{\infty} c_k e^{-tF(\lambda_k)} e_k,$$

where $c_k = c_k(g) Q_2(\lambda_k)$. If $g = 0$, then $c_k(g) = (g, e_k) = 0$, $\forall\, k \in \mathbb{N}$. Therefore, $c_k = 0$ for every $k \in \mathbb{N}$, that is, $u(t) = 0$ for every $t \in (0, T]$. This implies the property of the uniqueness of the solution of the problem (4), (5). The solution of the specified problem also depends continuously on the element g in the condition (5).

Theorem 2. *The problem* (4)*,* (5) *is correctly solvable, its solution is given by the formula*

$$u(t) = G(t) * g, \quad t \in (0, T],\ g \in H,\ u(t) \in H_\infty\langle m_n\rangle \quad \textit{for every} \quad t \in (0, T].$$

From property 2) of the convolution (see Section 3) it follows $u(t) = G(t) * g \in H_\infty\langle m_n\rangle$ if $g \in H'_\infty\langle m_n\rangle$ (for each $t \in (0, T]$). Let us prove that $u(t)$ is a solution of the equation (4), which satisfies the condition (5), where $g \in H'_\infty\langle m_n\rangle$, in the sense that

$$\mu \lim_{t \to +0} u(t) - \sum_{n=1}^{m} \mu_n \lim_{t \to t_n} B_n u(t) = g, \quad g \in H'_\infty\langle m_n\rangle, \tag{8}$$

here limits are considered in $H'_\infty\langle m_n\rangle$.

Lemma 1. *The function $u(t) = G(t) * g$, $g \in H'_\infty\langle m_n\rangle$ is strongly differentiable in H on $(0, T]$, with $u'(t) = G'(t) * g$.*

Proof. We need to make sure that for every $t \in (0, T]$ there exists an element $\gamma(t) \in H$ such that the boundary relation holds in H

$$\Psi_{\Delta t} := \frac{1}{\Delta t}[u(t+\Delta t) - u(t)] - \gamma(t) \to 0, \quad \Delta t \to 0. \tag{9}$$

Let us prove

$$\gamma(t) = \sum_{k=1}^{\infty} \frac{d}{dt} Q_1(t, \lambda_k) Q_2(\lambda_k) c_k(g) e_k = -\sum_{k=1}^{\infty} F(\lambda_k) Q_1(t, \lambda_k) Q_2(\lambda_k) c_k(g) e_k.$$

First, let us establish that $\gamma(t) \in H_\infty\langle m_n\rangle \subset H$ for every $t \in (0, T]$.

Note that $c_k(\gamma(t)) = -F(\lambda_k) c_k(G) c_k(g)$ and, as shown previously, $|c_k(G)| \le \gamma\rho^{-1}(a\lambda_k)$. Since $g \in H'_\infty\langle m_n\rangle$, we can see that

$$\forall\, \mu > 0\ \exists\, c = c(\mu) > 0\ \forall\, k \in \mathbb{N}: \quad |c_k(g)| \le c\rho(\mu\lambda_k).$$

Given this inequality, the inequality (2), and the convexity property of $\ln \rho$, we find that

$$|c_k(\gamma(t))| \le c\gamma\rho^{-1}(a\lambda_k)\rho(\varepsilon\lambda_k)\rho(\mu\lambda_k),$$

where $\varepsilon > 0$ is an arbitrary parameter.

Let us fix $\varepsilon \in (0, a)$. Then $|c_k(\gamma(t))| \le \beta\rho^{-1}((a-\varepsilon)\lambda_k)\rho(\mu\lambda_k)$, $k \in \mathbb{N}$. Take $\mu < a - \varepsilon$. Then

$$|c_k(\gamma(t))| \le \beta\rho^{-1}((a-\varepsilon-\mu)\lambda_k) \equiv \beta\rho^{-1}(\mu_1\lambda_k), \quad \mu_1 = a - \varepsilon - \mu. \tag{10}$$

From (A) and (10), it follows that $\gamma(t) \in H_\infty\langle m_n\rangle \subset H$ for every $t \in (0, T]$.

Now let us prove that the boundary relation (9) holds. For this we note that

$$\begin{aligned}\frac{1}{\Delta t}[u(t+\Delta t) - u(t)] &= \frac{1}{\Delta t}\sum_{k=1}^{\infty}[Q_1(t+\Delta t, \lambda_k) - Q_1(t, \lambda_k)] Q_2(\lambda_k) c_k(g) e_k \\ &= \sum_{k=1}^{\infty} \frac{d}{dt} Q_1(t+\theta\Delta t, \lambda_k) Q_2(\lambda_k) c_k(g) e_k, \quad 0 < \theta < 1.\end{aligned}$$

So,

$$\begin{aligned}\Psi_{\Delta t} &= \sum_{k=1}^{\infty}\Big(\frac{d}{dt} Q_1(t+\theta\Delta t, \lambda_k) Q_2(\lambda_k) c_k(g) e_k + F(\lambda_k) Q_1(t, \lambda_k) Q_2(\lambda_k) c_k(g) e_k\Big) \\ &= -\sum_{k=1}^{\infty}[Q_1(t+\theta\Delta t, \lambda_k) - Q_1(t, \lambda_k)] F(\lambda_k) Q_2(\lambda_k) c_k(g) e_k \\ &= -\sum_{k=1}^{\infty} F^2(\lambda_k) Q_1(t+\theta_1\Delta t, \lambda_k) Q_2(\lambda_k) c_k(g) e_k \cdot \theta_1 \Delta t, \quad 0 < \theta_1 < 1.\end{aligned}$$

Using the inequality $\rho^2(\varepsilon\lambda_k) \le \rho(2\varepsilon\lambda_k)$, which follows from the convexity inequality for $\ln \rho$, we find that $F^2(\lambda_k) \le c_\varepsilon \rho^2(2\varepsilon\lambda_k)$. Next, let us estimate

$$|c_k(\Psi_{\Delta t})| = F^2(\lambda_k) Q_1(t+\theta_1\Delta t, \lambda_k) Q_2(\lambda_k) |c_k(g)| \theta_1 |\Delta t|$$

similar to how it was done when estimating $c_k(\gamma(t))$. As a result, we come to inequalities

$$|c_k(\Psi_{\Delta t})| \leq \beta \rho^{-1}(a_0\lambda_k) \leq \beta(a_0\lambda_k)^{-1}|\Delta t|, \quad a_0 > 0.$$

Then,

$$\|\Psi_{\Delta t}\|_H^2 = \sum_{k=1}^{\infty} |c_k(\Psi_{\Delta t})|^2 \leq \frac{\beta}{a_0^2}\sum_{k=1}^{\infty} \lambda_k^{-2} \cdot |\Delta t|^2 = \beta_1 \sum_{k=1}^{\infty} G^{-2}(k)|\Delta t|^2 = \beta_2|\Delta t|^2 \to 0, \quad \Delta t \to 0.$$

It follows that the limit relation (9) is satisfied in Hilbert space H, and $u(t)$, $t \in (0, T]$, is a strongly differentiable function, with $u'(t) = G'(t) * g \equiv \gamma(t)$, $g \in H'_\infty\langle m_n\rangle$, $u'(t) \in H_\infty\langle m_n\rangle$ for every $t \in (0, T]$. The lemma is proved. □

From the form of $G(t)$ and the definition of B, it follows that

$$\begin{aligned} B(G(t) * g) &= B\Big(\sum_{k=1}^{\infty} Q_1(t, \lambda_k)Q_2(\lambda_k)c_k(g)e_k\Big) \\ &= \sum_{k=1}^{\infty} F(\lambda_k)Q_1(t, \lambda_k)Q_2(\lambda_k)c_k(g)e_k = -\gamma(t) * g = -G'(t) * g. \end{aligned}$$

As proven in Lemma 1, we have $u'(t) = G'(t) * g$, we see that the function $u(t) = G(t) * g$, $g \in H'_\infty\langle m_n\rangle$ is the solution of the equation (4). Let us prove that this function satisfies the condition (8).

Lemma 2. *Let $u(t) = G(t) * g$, $g \in H'_\infty\langle m_n\rangle$, $t \in (0, T]$. Then in the space $H'_\infty\langle m_n\rangle$ the following limit relation holds*

$$\mu \lim_{t\to+0} u(t) - \sum_{n=1}^{m} \mu_n \lim_{t\to t_n} B_n u(t) = g. \tag{11}$$

Proof. To prove (11) we take an arbitrary element $\psi = \sum_{k=1}^{\infty} c_k(\psi)e_k \in H_\infty\langle m_n\rangle$ and note that due to the continuity of embedding $H_\infty\langle m_n\rangle$ in the space $H'_\infty\langle m_n\rangle$ and the orthonormality of the basis $\{e_k : k \in \mathbb{N}\}$ we have

$$\langle u(t), \psi\rangle = (u(t), \psi)_H = \sum_{k=1}^{\infty} c_k(u(t))c_k(\psi) = \sum_{k=1}^{\infty} Q_1(t, \lambda_k)Q_2(\lambda_k)c_k(g)c_k(\psi),$$

$$\langle B_n u(t), \psi\rangle = (B_n u(t), \psi)_H = \sum_{k=1}^{\infty} g_n(\lambda_k)c_k(u(t))c_k(\psi).$$

Then

$$\begin{aligned} &\mu \lim_{t\to+0} \langle u(t), \psi\rangle - \sum_{n=1}^{m} \mu_n \lim_{t\to t_n} \langle B_n u(t), \psi\rangle \\ &\qquad = \mu \lim_{t\to+0} \sum_{k=1}^{\infty} c_k(u(t))c_k(\psi) - \sum_{n=1}^{m} \mu_n \lim_{t\to t_n} \sum_{k=1}^{\infty} g_n(\lambda_k)c_k(u(t))c_k(\psi). \end{aligned}$$

Note that the series $\sum_{k=1}^{\infty} c_k(u(t))c_k(\psi)$ converges uniformly on $[0, T]$. This fact follows from the definition of the coefficients $c_k(u(t))$, $k \in \mathbb{N}$, and the estimates

$$|c_k(u(t))| \cdot |c_k(\psi)| \leq \tilde{c}|c_k(g)| \cdot |c_k(\psi)|, \quad t \in [0, T], \quad k \in \mathbb{N}.$$

In fact, $g \in H'_\infty\langle m_n\rangle$, that is,

$$\forall\, \mu > 0\ \exists\, c = c(\mu) > 0\ \forall\, k \in \mathbb{N}: \quad |c_k(g)| \le c\rho(\mu\lambda_k).$$

Then $\psi \in H_\infty\langle m_n\rangle$, therefore, due to the condition (A),

$$\exists\, \mu_0 > 0\ \exists\, c_0 > 0\ \forall\, k \in \mathbb{N}: \quad |c_k(\psi)| \le c_0\rho^{-1}(\mu_0\lambda_k).$$

Put $\mu = \mu_0/2$. Then, given the inequality of the convexity (1), we find that

$$|c_k(g)| \cdot |c_k(\psi)| \le cc_0\rho^{-1}(\mu_0\lambda_k)\rho\Big(\frac{\mu_0}{2}\lambda_k\Big) \le cc_0\rho^{-1}\Big(\frac{\mu_0}{2}\lambda_k\Big) \le \tilde{c}\lambda_k^{-2}, \quad \lambda_k = G(k).$$

The formulated property follows from the last inequality. Similarly, we prove that the series

$$\sum_{k=1}^{\infty} g_n(\lambda_k)Q_1(t,\lambda_k)Q_2(\lambda_k)c_k(g)c_k(\psi) \equiv \sum_{k=1}^{\infty} g_n(\lambda_k)c_k(u(t))c_k(\psi), \quad n \in \{1,\dots,m\},$$

converges uniformly on $[t_1, T]$ (using the properties (6) of $g_1,\dots,g_m$). So,

$$\lim_{t\to t_n}\sum_{k=1}^{\infty} c_k(B_nu(t))c_k(\psi) = \sum_{k=1}^{\infty} c_k(B_nu(t_n))c_k(\psi) = \sum_{k=1}^{\infty} g_n(\lambda_k)Q_1(t_n,\lambda_k)Q_2(\lambda_k)c_k(g)c_k(\psi),$$

$$\lim_{t\to+0}\sum_{k=1}^{\infty} c_k(u(t))c_k(\psi) = \sum_{k=1}^{\infty} c_k(u(0))c_k(\psi) = \sum_{k=1}^{\infty} Q_2(\lambda_k)c_k(g)c_k(\psi).$$

Taking into account the last relations, we obtain

$$\begin{aligned}\mu\lim_{t\to+0}\langle u(t),\psi\rangle - \sum_{n=1}^{m}\mu_n\lim_{t\to t_n}\langle B_nu(t),\psi\rangle &= \sum_{k=1}^{\infty}\Big[\mu - \sum_{n=1}^{m}\mu_ng_n(\lambda_k)Q_1(t_n,\lambda_k)\Big]Q_2(\lambda_k)c_k(g)c_k(\psi)\\ &= \sum_{k=1}^{\infty}\frac{\mu - \sum\limits_{n=1}^{m}\mu_ng_n(\lambda_k)Q_1(t_n,\lambda_k)}{\mu - \sum\limits_{n=1}^{m}\mu_ng_n(\lambda_k)Q_1(t_n,\lambda_k)}c_k(g)c_k(\psi)\\ &= \sum_{k=1}^{\infty}c_k(g)c_k(\psi) = \langle g,\psi\rangle, \quad \psi \in H_\infty\langle m_n\rangle.\end{aligned}$$

□

Lemma 2 allows us to formulate a multipoint problem for the equation (4) in the sense of (8). The following statement is correct.

Theorem 3. *The problem* (4), (8) *is correctly solvable, its solution is given by the formula* $u(t) = G(t) * g$, $g \in H'_\infty\langle m_n\rangle$, $t \in (0,T]$, $u(t) \in H_\infty\langle m_n\rangle$ *for every* $t \in (0,T]$.

Proof. From the above it follows that the proof requires the property of the uniqueness of the solution of the problem (4), (8) and its continuity with respect to the element $g \in H'_\infty\langle m_n\rangle$.

We obtain the uniqueness of the solution for the following reasons: if $g = 0$, then $c_k(g) = \langle g, e_k\rangle = 0$, $\forall\, k \in \mathbb{N}$, that is, $u(t) = 0$, $t \in (0,T]$.

Now let us prove that the solution of this problem is continuous with respect to the boundary condition. Let $\{g, g_n : n \in \mathbb{N}\} \subset H_\infty\langle m_n\rangle$, with $g_n \to g$ at $n \to \infty$ in the space $H'_\infty\langle m_n\rangle$. It follows that

$$c_k(g_n) = \langle g_n, e_k\rangle \underset{n\to\infty}{\longrightarrow} \langle g, e_k\rangle = c_k(g)$$

for each $k \in \mathbb{N}$. In addition, $\{u, u_n : n \in \mathbb{N}\} \subset H_\infty\langle m_n\rangle$, where u_n is the solution to the problem (4), (8), which corresponds to the boundary element $g_n \in H'_\infty\langle m_n\rangle$. Then

$$\langle u_n, \varphi\rangle = (u_n, \varphi) = \sum_{k=1}^{\infty} c_k(G(t))c_k(g_n)c_k(\varphi) \underset{n\to+\infty}{\longrightarrow} \sum_{k=1}^{\infty} c_k(G(t))c_k(g)c_k(\varphi) = (u, \varphi) = \langle u, \varphi\rangle.$$

for all $\varphi \in H_\infty\langle m_n\rangle$. Therefore, $u_n \to u$ for $n \to \infty$ in the space $H'_\infty\langle m_n\rangle$. □

Let us look at the boundary properties of the solution of multipoint problem for the evolutionary equation (4) at $t \to +0$. Based on the formulation of the solution of the problem (4), (8), it follows that it can be represented as $u(t) = \tilde{G}(t) * \tilde{f} = \sum\limits_{k=1}^{\infty} Q_1(t, \lambda_k)c_k(\tilde{f})$, where $\tilde{f} = f * \alpha$,

$$\tilde{G}(t) = \sum_{k=1}^{\infty} Q_1(t, \lambda_k)e_k \in H_\infty\langle m_n\rangle, \quad t \in (0, T], \quad \alpha = \sum_{k=1}^{\infty} Q_2(\lambda_k)e_k,$$

$$f = \sum_{k=1}^{\infty} c_k(f)e_k \in H'_\infty\langle m_n\rangle, \quad c_k(\tilde{f}) = c_k(f)Q_2(\lambda_k), \quad k \in \mathbb{N}.$$

Therefore, the solution of problem (4), (8) coincides with the solution of the Cauchy problem for equation (4), which corresponds to the generalized element $\tilde{f}$. Also, note that α is an element of $H'_\infty\langle m_n\rangle$. Given $f \in H'_\infty\langle m_n\rangle$, we have that $\tilde{f} = f * \alpha \in H'_\infty\langle m_n\rangle$, $u(t) \in H_\infty\langle m_n\rangle$ for every $t \in (0, T]$ and $u(t) \to \tilde{f}$ for $t \to +0$ in the space $H'_\infty\langle m_n\rangle$. If f belongs to narrower than $H'_\infty\langle m_n\rangle$ space, then $\tilde{f}$ may also belong to space contained in $H'_\infty\langle m_n\rangle$; the behavior of $u(t)$ in a neighborhood of the point $t = 0$ essentially depends on which space $\tilde{f} = f * \alpha$ belongs to.

Theorem 4. *The equivalence relation*

$$(\tilde{f} \in H) \Leftrightarrow (\exists c > 0\ \forall t \in (0, T]: \ \ \|u(t)\| \le c)$$

is correct, at the same time $u(t) \to \tilde{f}$ for $t \to +0$ in the space H.

Proof. Let

$$\exists c > 0\, \forall t \in (0, T]: \ \ \|u(t)\| \le c.$$

Then

$$\|u(t)\|^2 = \sum_{k=1}^{\infty} Q_1^2(t, \lambda_k)|c_k(f)|^2 \equiv \sum_{k=1}^{\infty} e^{-2tF(\lambda_k)}|c_k(f)|^2 \le c^2.$$

Making a limiting transition as $t \to +0$, we find that $\|\tilde{f}\|^2 = \sum\limits_{k=1}^{\infty} |c_k(\tilde{f})|^2 \le c^2$, that is, $\tilde{f} \in H$. Vice versa, if $\tilde{f} \in H$, then

$$\|u(t)\|^2 = \sum_{k=1}^{\infty} e^{-2tF(\lambda_k)}|c_k(\tilde{f})|^2 \le \sum_{k=1}^{\infty} |c_k(\tilde{f})|^2 = \|\tilde{f}\|^2, \quad \forall\, t \in (0, T].$$

Since

$$\|u(t) - \tilde{f}\|^2 = \sum_{k=1}^{\infty} (e^{-tF(\lambda_k)} - 1)^2|c_k(\tilde{f})|^2 \le \sum_{k=1}^{\infty} |c_k(\tilde{f})|^2 = \|f\|^2, \quad \forall\, t \in (0, T],$$

and $\exp\{-tF(\lambda_k)\} \to 1$ when $t \to +0$, we can see that

$$\lim_{t\to+0} \|u(t) - \tilde{f}\|^2 = \sum_{k=1}^{\infty} \lim_{t\to+0} (e^{-tF(\lambda_k)} - 1)^2|c_k(\tilde{f})|^2 = 0,$$

that is, $\|u(t) - \tilde{f}\| \to 0$ at $t \to +0$. The theorem is proved. □

Now, let γ be a continuous, positive and integrable function on $(0, T]$, $\gamma(0) = 0$, K_γ is a set of solutions of multipoint problems for the equation (4), each of which is constructed over a certain boundary element $g \in H'_\infty\langle m_n\rangle$, for which $\int_0^T \gamma(t)\|u(t)\|^2 dt < \infty$. Since $\gamma(t) \geq \gamma_\varepsilon > 0$ on an arbitrary segment $[\varepsilon, T]$, we can see that this estimate characterizes the behavior of $u(t)$ when $t \to +0$. Let

$$L(\lambda) := \Big(\int_0^T \gamma(t) e^{-2tF(\lambda)} dt \Big)^{-1/2}.$$

Function L is positive, continuous on $[0, \infty)$ and

$$\lim_{\lambda \to +\infty} L(\lambda) = +\infty, \quad L(\lambda) \geq c > 0,\ \forall \lambda \in [0, \infty), \quad c = \left(\int_0^T \gamma(t) dt \right)^{-1/2}.$$

We also assume that the following condition is fulfilled

$$\forall\, \varepsilon > 0\ \exists\, c = c_\varepsilon > 0: \ \ L(\lambda) \leq c_\varepsilon \rho(\varepsilon\lambda), \quad \lambda \geq 1.$$

We introduce the following scalar product in the domain $\mathcal{D}(L(A))$ of the operator $L(A)$:

$$(\varphi, \psi)_{H_L} := (L(A)\varphi, L(A)\psi), \quad \{\varphi, \psi\} \subset \mathcal{D}(L(A)).$$

Then $\mathcal{D}(L(A))$ is transformed into Hilbert space H_L (see the building schema of the chain of spaces by a non-negative self-adjoint operator [1, pp. 59–61]). If we take H_L as a positive space and identify through H'_L a negative space, constructed by H_L and H, then we get a chain

$$H_\infty\langle m_n\rangle \subset H_L \subset H \subset H'_L \subset H'_\infty\langle m_n\rangle$$

of continuous and densely embedded spaces; at the same time

$$(\varphi \in H_L) \Leftrightarrow \Big(\|\varphi\|^2_{H_L} = \sum_{k=1}^\infty L^2(\lambda_k)|c_k(\varphi)|^2 < \infty,\ c_k(\varphi) = (\varphi, e_k) \Big),$$

$$(g \in H'_L) \Leftrightarrow \Big(\|g\|^2_{H'_L} = \sum_{k=1}^\infty L^{-2}(\lambda_k)|c_k(g)|^2 < \infty,\ c_k(g) = \langle g, e_k\rangle \Big).$$

Theorem 5. *Let $u(t)$ be a solution of the multipoint problem* (4), (8), *constructed by the boundary element $f \in H'_\infty\langle m_n\rangle$. Then*

$$(\tilde f \in H'_L) \Leftrightarrow (u(t) \in K_\gamma,\ t \in (0, T]),$$

at the same time $u(t) \to \tilde f$ as $t \to +0$ in the space H'_L.

Proof. The theorem follows from the relations

$$\begin{aligned}\int_0^T \gamma(t)\|u(t)\|^2 dt &= \int_0^T \gamma(t) \sum_{k=1}^\infty \exp\{-2tF(\lambda_k)\}|c_k(\tilde f)|^2 dt \\ &= \sum_{k=1}^\infty |c_k(\tilde f)|^2 \int_0^T \gamma(t) e^{-2tF(\lambda_k)} dt = \sum_{k=1}^\infty L^{-2}(\lambda_k)|c_k(\tilde f)|^2.\end{aligned}$$

Besides,

$$\|u(t) - \tilde f\|^2_{H'_L} = \sum_{k=1}^\infty L^{-2}(\lambda_k)(e^{-tF(\lambda_k)} - 1)^2 |c_k(\tilde f)|^2 \leq \sum_{k=1}^\infty L^{-2}(\lambda_k)|c_k(\tilde f)|^2 = \|\tilde f\|^2_{H'_L} < \infty$$

for all $t \in (0, T]$. Since $\exp\{-tF(\lambda_k)\} \to 1$ as $t \to +0$, $\forall k \in \mathbb{N}$, we see that $\|u(t) - \tilde f\|_{H'_L} \to 0$ as $t \to +0$. The theorem is proved. ☐

Using the representation of generalized elements in the form of formal Fourier series and correspondingly picking the function γ, the spaces $H'_\infty(A)$, $H'_\infty\langle m_n\rangle$ can be described by characterizing the behavior of the solution of the multipoint problem (4), (8) $u(t)$ in the neighborhood of zero.

For example, if $\gamma(t) = \exp\{-t^{-q}\}$, $q > 0$, $t \in (0, T]$, then as follows from the results given in [1, page 87], the equivalence relations are correct

$$(\tilde{f} \in G'_{\{\beta\}}(A) \equiv H'_\infty\langle n^{n\beta}\rangle, \beta > 1) \Leftrightarrow \Big(\forall\, \tilde{\alpha} > 0\ \exists c = c(\tilde{\alpha}) > 0 : \|u(t)\| \leq c \exp\{\tilde{\alpha} t^{-q}\}, \\ t \in (0, T],\ q = (\beta - 1)^{-1}\Big),$$

or

$$(\tilde{f} \in G'_{\{\beta\}}(A) \equiv H'_\infty\langle n^{n\beta}\rangle,\ \beta > 1) \Leftrightarrow \Big(u(t) \in K_\gamma,\ \gamma(t) = \exp\{-t^{-q}\}, \\ t \in (0, T],\ q = (\beta - 1)^{-1}\Big).$$

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Received 01.03.2021
Revised 12.11.2022

Городецький В.В., Мартинюк О.В., Колісник Р.С. *Про одну нелокальну задачу для диференціально-операторних рівнянь першого порядку* // Карпатські матем. публ. — 2022. — Т.14, №2. — С. 513–528.

Різні функціональні простори (наприклад, соболєвські, аналітичних функцій, нескінченно диференційовних та розподілів Л. Шварца) можна трактувати як позитивні та негативні відносно L_2, побудовані за функціями від оператора диференціювання або множення на незалежну змінну, або як проективні чи індуктивні границі таких просторів. М.Л. Горбачук та В.І. Горбачук розвинули теорію просторів основних та узагальнених елементів, які будуються за функціями від довільного самоспряженого оператора.

У цій роботі розглядаються простори узагальнених елементів, які ототожнюються з формальними рядами Фур'є і будуються за невід'ємним самоспряженим оператором у гільбертовому просторі, спектр якого є суто дискретним. Для диференціально-операторного рівняння першого порядку ставиться нелокальна багатоточкова за часом задача у випадку, коли відповідна умова задовольняється в позитивному або негативному просторах, які побудовані за таким оператором (таку задачу можна розуміти як певне узагальнення абстрактної задачі Коші для зазначеного диференціально-операторного рівняння). Встановлюється коректна розв'язність зазначеної задачі, при цьому будується фундаментальний розв'язок, досліджується його структура та властивості. Розв'язок дається у вигляді абстрактної згортки фундаментального розв'язку з граничним елементом, за допомогою якого ставиться багатоточкова умова і який є лінійним неперервним функціоналом, заданим на просторі основних елементів, при цьому розв'язок задовольняє багатоточкову умову в негативному просторі, який є спряженим з відповідним позитивним простором елементів.

Ключові слова і фрази: нелокальна багатоточкова задача, диференціально-операторне рівняння, самоспряжений оператор, гільбертовий простір, коректна розв'язність.