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Approximation by trigonometric polynomials in the variable exponent weighted Morrey spaces

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In this paper we investigate the best approximation by trigonometric polynomials in the variable exponent weighted Morrey spaces $\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)$, where w is a weight function in the Muckenhoupt $A_{p(\cdot)}(I_0)$ class. We get a characterization of K-functionals in terms of the modulus of smoothness in the spaces $\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)$. Finally, we prove the direct and inverse theorems of approximation by trigonometric polynomials in the spaces $\widetilde{\mathcal{M}}_{p(\cdot),\lambda(\cdot)}(I_0,w)$, the closure of the set of all trigonometric polynomials in $\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)$.

Key words and phrases: variable exponent weighted Morrey space, best approximation, trigonometric polynomial, direct and inverse theorem.

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1 Introduction and preliminaries

Let $I_0=[0,2\pi]$ and $p(\cdot)$ be a measurable function on I_0 with values in $[1,\infty)$. We mainly suppose that

$$1<p_-\le p(x)\le p_+<\infty, \tag{1}$$

where $p_-:=\operatorname{ess\,inf}_{x\in I_0}p(x)>1$, $p_+:=\operatorname{ess\,sup}_{x\in I_0}p(x)<\infty$. We say that $p(x)$ satisfies the local log-condition if the following inequality

$$|p(x)-p(y)|\le\frac{A}{-\ln|x-y|},\qquad |x-y|\le\frac{1}{2},\ \ x,y\in I_0, \tag{2}$$

holds, where $A=A(p)>0$ is independent of x and y.

For intervals we write $I(x,r)=(x-r,x+r)\subset\mathbb{R}$ and $I_0(x,r)=I(x,r)\cap I_0$.

By w we always denote a weight, that is a positive, 2π-periodic and locally integrable function on I_0. We say that w is in the Muckenhoupt $A_{p(\cdot)}(I_0)$ class (see [10,19]) if the following

$$\sup_{I_0}|I_0|^{-1}\|w\|_{L_{p(\cdot)}(I_0(x,r))}\|w^{-1}\|_{L_{p'(\cdot)}(I_0(x,r))}<\infty$$

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holds, where $L_{p(\cdot)}(I_0)$ is the variable exponent Lebesgue space, the space with the following norm

$$\|f\|_{L_{p(\cdot)}I_0} := \inf_{\eta>0}\left\{\int_{I_0}\left|\frac{f(x)}{\eta}\right|^{p(x)} dx \le 1\right\}.$$

The variable exponent weighted Lebesgue space $L_{p(\cdot)}(I_0,w)$ is defined as the set of all 2π-periodic measurable functions for which $\|f\|_{L_{p(\cdot)}(I_0,w)} = \|fw\|_{L_{p(\cdot)}(I_0)}$.

Muckenhoupt's $A_{p(\cdot)}(I_0)$ class weights are important tools in mathematical analysis. Their characterizing property in $\mathbb{R}^n$ is that the Hardy-Littlewood maximal operator is bounded in the variable exponent weighted space $L_{p(\cdot)}(I_0,w)$ if and only if the weight w is in $A_{p(\cdot)}(I_0)$ class (see e.g. [9]). Due to the importance of $A_{p(\cdot)}(I_0)$ class-weighted function spaces, various norm inequalities have been established for $A_{p(\cdot)}(I_0)$ class both in Euclidean spaces and in more general settings. The results taken with Muckenhoupt's $A_{p(\cdot)}(I_0)$ class weights help us to obtain approximation results in unweighted cases, such as varible exponent Lebesgue spaces.

Let $\lambda(\cdot) : I_0 \to [0,1]$ be a measurable function. We suppose that

$$0 \le \lambda_- \equiv \inf_{x\in I_0}\lambda(x) \le \sup_{x\in I_0}\lambda(x) \equiv \lambda_+ \le 1.$$

Following the convention, we add $(\cdot)$ to indicate that the parameters are actually dependent on the position. Note that $p(\cdot)$ is required to be continuous while $\lambda(\cdot)$ is allowed to be merely measurable and bounded. This implies that the local regularity is essential in the theory of approximation. We remark that this fact is observed in [23, Theorem 4.4], [26, Theorem 4.1] and [27, Theorem 3.3].

We define the variable exponent Morrey space $\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0)$ as the space of all 2π-periodic measurable functions such that the modular

$$I_{p(\cdot),\lambda(\cdot)}(f) = \sup_{\substack{x\in I_0\\ 0<t<2\pi}} t^{-\lambda(x)}\int_{I_0(x,r)}|f(y)|^{p(y)}dy$$

is finite. The norm is defined by

$$\|f\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0)} = \|f\|_1 = \inf\left\{\eta>0 : I_{p(\cdot),\lambda(\cdot)}\Big(\frac{f}{\eta}\Big) < 1\right\}.$$

There is another plausible definition of the norm. We may define the Morrey norm by

$$\|f\|_2 = \sup_{\substack{x\in I_0\\ 0<t<2\pi}} t^{-\frac{\lambda(x)}{p(x)}}\|f\chi_{I_0(x,r)}\|_{L_{p(\cdot)}(I_0)}.$$

The following lemma shows that these norms are equivalent.

Lemma 1 ([4, Lemma 3]). *For every $f\in\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0)$ we have $\|f\|_1 = \|f\|_2$.*

The interested reader can find more information about various versions of variable exponent Morrey spaces in the recent survey papers [4,12,17,25].

The variable exponent weighted Morrey spaces $\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)$ are defined as the set of all 2π-periodic integrable functions f on I_0 with finite norm

$$\|f\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} = \sup_{\substack{x\in I_0\\ 0<r<2\pi}} r^{\frac{\lambda(x)}{p(x)}}\|w\|^{-1}_{L_{p(\cdot)}(I_0(x,r))}\|f\chi_{I_0(x,r)}\|_{L_{p(\cdot)}(I_0,w)}.$$

Hardy-Littlewood maximal function Mf of f on I_0 is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|I(x,r)|} \int_{I_0(x,r)} |f(t)|dt, \ x \in I_0.$$

Let $f \in \mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0, w)$ and $w \in A_{p(\cdot)}(I_0)$. Assume that $\lambda(\cdot)$ satisfies the condition $0 \leq \lambda_- \leq \lambda_+ < 1$ and $p(\cdot)$ satisfies the conditions (1) and (2). Using the suitable result from [11, Corollary 3.1], we obtain

$$\|Mf\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \leq C\|f\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}, \tag{3}$$

where a positive constant C does not depend on f.

We denote by $\mathcal{T}_n$ the set of trigonometric polynomials having degree not exceeding n and $C(I_0)$ the set of 2π-periodic continuous functions. The Weierstrass well-known theorem on the approximation of the continuous function by the trigonometric polynomials and its quantitative refinement represented by Jackson inequality $E_n(f)_{C(I_0)} \leq C\omega\left(f, \frac{1}{n}\right)$ are one of the basics of the Approximation Theory, where $f \in C(I_0)$ and $E_n(f)$ is the best approximation of f by the trigonometric polynomials, i.e. $E_n(f)_{C(I_0)} = \inf\limits_{T_n \in \mathcal{T}_n} \|f - T_n\|_{C(I_0)}$, and $\omega(f, \delta)$, $\delta > 0$ is the modulus of continuity of f (see [8]). The analog of Jackson inequality is correct for the mean approximation and higher order modulus of continuity as well (see [30]).

S. Bernstein [5] obtained the reversed estimations of Jackson's inequality in the space of continuous functions for some specific cases. Later E.S. Quade [24], S.B. Stechkin [29], A.F. Timan [30], A.F. Timan and M.F. Timan [31] etc. proved the reversed type inequalities of Jackson's inequality, including $L_p, 1 < p < \infty$, spaces. These type inequalities played an important role in the investigation of properties of the conjugate functions, in the study of absolute convergent Fourier series [29], and in the related problems. For the approximation in weighted and nonweighted Lebesgue spaces, and weighted and nonweighted variable exponent Lebesgue spaces the sufficiently wide presentation can be found in the works [1–3, 14–16, 18, 20–22, 28]. In [6,7], Z. Cakir, C. Aykol, D. Soylemez and A. Serbetci investigate the best approximation by trigonometric polynomials in Morrey spaces $L_{p,\lambda}(I_0)$ and weighted Morrey spaces $M_{p,\lambda}(I_0, w)$, respectively. In [13], V.S. Guliyev, A. Ghorbanalizadeh and Y. Sawano study the approximation by trigonometric polynomials in variable exponent Morrey spaces $\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0)$. In the theory of approximation, variable exponent spaces are useful to show that the approximation is essentially local.

We aim to study approximation properties of trigonometric polynomials in the variable exponent weighted Morrey spaces $\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0, w)$, where the weight function w is in the Muckenhoupt class $A_{p(\cdot)}(I_0)$. Furthermore, we prove the direct and inverse theorems of approximation by trigonometric polynomials in the spaces $\widetilde{\mathcal{M}}_{p(\cdot),\lambda(\cdot)}(I_0, w)$, the closure of the set of all trigonometric polynomials in $\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0, w)$.

We organize this paper as follows. In Section 2, we give some lemmas which will be very useful in the proofs of our main results. We define rth, $r = 1, 2, 3, \ldots$, modulus of smoothness and the Peetre K-functional. Then we give a characterization of K-functional in terms of the modulus of smoothness, a Jackson type inequality and a Bernstein type inequality in the variable exponent weighted Morrey spaces $\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0, w)$. Finally, the last section is devoted to the direct and inverse theorems in $\widetilde{\mathcal{M}}_{p(\cdot),\lambda(\cdot)}(I_0, w)$.

Constants such as C may have different values from one occurrence to the next, but they will always be irrelevant for the arguments used.

2 Some auxiliary lemmas

In this section, we will give some lemmas which we need in the proofs of our main results.

If a continuous function f on an interval I_0 satisfies $\omega(f,t)=o(t)$, then f is constant. Thus the modulus of continuity is not useful for measuring higher smoothness. For the latter, we will use the modulus of smoothness Ω which is connected with differences of higher orders. The modulus of smoothness provide us a better tool to deal with the rate of the best approximation, inverse theorems and also some other similar problems.

Definition 1. *Let $f\in\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)$, $\lambda(\cdot)$ satisfy the condition $0\le\lambda_-\le\lambda_+<1$ and $p(\cdot)$ satisfy the conditions* (1) *and* (2). *The function $\Omega^r(f,\cdot;\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)):[0,\infty]\to[0,+\infty)$, defined by*

$$\Omega^r(f,h;\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w))=\sup_{0<\delta\le\min\{2\pi,h\}}\|\sigma^r_\delta(f)\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \tag{4}$$

is called the modulus of smoothness of f of order r in $\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)$ with $r\in\mathbb{N}$ and $\delta,h>0$. Here

$$\sigma^r_\delta(f)(x):=\frac{1}{\delta}\int_0^\delta|\triangle^r_t f(x)|dt$$

with

$$\triangle^r_t f(x):=\sum_{s=0}^{r}\binom{r}{s}(-1)^{r+s+1}f(x+st). \tag{5}$$

Thus

$$\Omega^1(f,h;\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w))=\Omega(f,h;\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w))$$

is the modulus of continuity.

Let $\lambda(\cdot)$ satisfy the condition $0\le\lambda_-\le\lambda_+<1$ and $p(\cdot)$ satisfy the conditions (1) and (2). From Lemma 3 we get $\Omega^r(f,h,\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w))\le c\|f\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}$ for every $f\in\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)$ and

$$\Omega^r(f_1+f_2,\cdot,\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w))\le\Omega^r(f_1,\cdot,\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0))+\Omega^r(f_2,\cdot,\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w))$$

for $f_1,f_2\in\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)$.

For $f\in\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)$, $0\le\lambda_-\le\lambda_+<1$ and $p(\cdot)$ satisfy the conditions (1) and (2), we denote

$$E_n(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}=\inf_{T_n\in\mathcal{T}_n}\|f-T_n\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}.$$

Let $r\in\mathbb{N}\cup\{0\}$. The homogeneous Sobolev-Morrey space $\dot{W}^r_{p(\cdot),\lambda(\cdot)}(I_0,w)$ is defined as the set of all functions $f\in L_1^{loc}(I_0,w)$ for which the weak derivative $f^{(r)}$ exists on I_0 and

$$\|f\|_{\dot{W}^r_{p(\cdot),\lambda(\cdot)}(I_0,w)}=\|f^{(r)}\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}<\infty.$$

The nonhomogeneous Sobolev-Morrey space $W^r_{p(\cdot),\lambda(\cdot)}(I_0)$ is the subset of $\dot{W}^r_{p(\cdot),\lambda(\cdot)}(I_0)$, consisting of all functions $f\in\dot{W}^r_{p(\cdot),\lambda(\cdot)}(I_0,w)$ for which

$$\|f\|_{W^r_{p(\cdot),\lambda(\cdot)}(I_0,w)}:=\|f\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}+\|f^{(r)}\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}<\infty.$$

Let $f\in\dot{W}^r_{p(\cdot),\lambda(\cdot)}(I_0)$. For almost all $x\in I_0$, $r\in\mathbb{N}$ and $t>0$ we have

$$\Delta^r_t f(x)=\int_0^t\dots\int_0^t f^{(r)}(x+t_1+\dots+t_r)dt_1\dots dt_r. \tag{6}$$

We use (6) to prove the following fact.

Lemma 2. *Let* $f \in \dot{W}^r_{p(\cdot),\lambda(\cdot)}(I_0, w)$ *and* $w \in A_{p(\cdot)}(I_0)$. *Assume that* $\lambda(\cdot)$ *satisfies the condition* $0 \le \lambda_- \le \lambda_+ < 1$ *and* $p(\cdot)$ *satisfies the conditions* (1) *and* (2). *Then for* $h > 0$ *we have*

$$\Omega^r(f, h; \mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0, w)) \le C\delta^r \ \|f^{(r)}\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}, \quad \delta > 0.$$

Proof. Let $f \in \dot{W}^r_{p(\cdot),\lambda(\cdot)}(I_0, w)$. Then $f^{(r)} \in \mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0, w)$. Using (6) in definition of the function $\Omega^r(f, h; \mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0, w))$ and applying r times generalized Minkowski inequality, we obtain

$$\Big\|\frac{1}{\delta}\int_0^\delta \Big|\Delta_t^r f(\cdot)\Big|dt\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \le \Big\|\frac{1}{\delta}\int_0^\delta\int_0^t\cdots\int_0^t \Big|f^{(r)}\Big(\cdot+t_1+\ldots+t_r\Big)\Big|dt_1\ldots dt_r\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}$$
$$\le \delta^r\Big\|\frac{1}{\delta^r}\int_0^\delta\cdots\int_0^\delta \Big|f^{(r)}\Big(\cdot+t_1+\ldots+t_r\Big)\Big|dt_1\ldots dt_r\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}.$$

If we set $u = t_1 + t_2 + \ldots + t_r$, then we have

$$\Big\|\frac{1}{\delta}\int_0^\delta\Big|\Delta_t^r f(\cdot)\Big|dt\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \le \delta^r\Big\|\frac{1}{\delta^{r-1}}\int_0^\delta\cdots\int_0^\delta\Big\{\frac{1}{\delta}\int_{t_2+\ldots+t_r}^{\delta+t_2+\ldots+t_r}\Big|f^{(r)}(\cdot+u)\Big|du\Big\}dt_2\ldots dt_r\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}$$
$$\le r\delta^r\frac{1}{\delta^{r-1}}\int_0^\delta\cdots\int_0^\delta\Big\|\frac{1}{r\delta}\int_0^{r\delta}\Big|f^{(r)}\Big(\cdot+u\Big)\Big|du\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}dt_2\ldots dt_r$$
$$= \delta^r\Big\|\frac{1}{r\delta}\int_0^{r\delta}\Big|f^{(r)}\Big(\cdot+u\Big)\Big|du\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}$$
$$\le r\delta^r\Big\|Mf^{(r)}\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \le C(r)\delta^r\Big\|f^{(r)}\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}.$$

In the last line we use the boundedness of maximal operators in the variable exponent weighted Morrey space (see (3)). Thus the proof is completed. □

Lemma 3. *Let* $f \in \mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0, w)$ *and* $w \in A_{p(\cdot)}(I_0)$. *Assume that* $\lambda(\cdot)$ *satisfies the condition* $0 \le \lambda_- \le \lambda_+ < 1$ *and* $p(\cdot)$ *satisfies the conditions* (1) *and* (2). *Then for* $h > 0$ *we have*

$$\Omega^r(f, h; \mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0, w)) \le 2^r \ \|f\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}.$$

Proof. By (4) and (5) using the triangle inequality we have

$$\Omega^r(f, h; \mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0, w)) = \sup_{0<\delta\le\min(2\pi,h)}\Big\|\frac{1}{\delta}\int_0^\delta\Big|\sum_{s=0}^r\binom{r}{s}(-1)^{r+s+1}f(\cdot+st)\Big|dt\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}$$
$$\le \sup_{0<\delta\le\min(2\pi,h)}\sum_{s=0}^r\binom{r}{s}\Big\|\frac{1}{\delta}\int_0^\delta\Big|f(\cdot+st)\Big|dt\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}$$
$$\le \sup_{0<\delta\le\min(2\pi,h)}\Big\{\Big\|\frac{1}{\delta}\int_0^\delta\Big|f(\cdot)\Big|dt\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}+\sum_{s=1}^r\binom{r}{s}\Big\|\frac{1}{\delta}\int_0^\delta\Big|f(\cdot+st)\Big|dt\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\Big\}$$
$$= \sup_{0<\delta\le\min(2\pi,h)}\Big\{\Big\|f\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}+\sum_{s=1}^r\binom{r}{s}\Big\|\frac{1}{s\delta}\int_0^{s\delta}\Big|f(\cdot+u)\Big|du\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\Big\}$$
$$\le \sup_{0<\delta\le\min(2\pi,h)}\Big\{\Big\|f\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}+\sum_{s=1}^r\binom{r}{s}\Big\|\frac{1}{\delta}\int_0^{r\delta}\Big|f(\cdot+u)\Big|du\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\Big\}$$
$$\le \sup_{0<\delta\le\min(2\pi,h)}\Big\{\Big\|f\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}+r2^r\Big\|\frac{1}{r\delta}\int_0^{r\delta}\Big|f(\cdot+u)\Big|du\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\Big\}.$$

Since function f on $\mathbb{R}$ is 2π-periodic, without loss of generality, we can assume $r\delta < 2\pi$ and by boundedness of maximal operator in weighted Morrey spaces with variable exponent (3), we get

$$\begin{aligned}\Omega^r(f,h;\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)) &\le \sup_{0<\delta\le\min(2\pi,h)} \left\{\|f\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + r2^r C(p(\cdot))\|f\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\right\} \\ &= C(p(\cdot),r)\|f\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}.\end{aligned}$$

□

For $f \in \mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)$ and $r \ge 1$ the K-functional is defined as follows

$$\mathcal{K}_r\Big(f,t\Big)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} := \inf_{g\in W^r_{p(\cdot),\lambda(\cdot)}(I_0,w)} \left\{\|f-g\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + t^r\|g^{(r)}\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\right\}.$$

For $h > 0$, the K-functional $\mathcal{K}_r\Big(f,t\Big)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}$ and $\Omega^r(f,h;\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w))$ are equivalent as the following lemma shows.

Lemma 4. *Let $f \in \mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)$ and $w \in A_{p(\cdot)}(I_0)$. Assume that $\lambda(\cdot)$ satisfies the condition $0 \le \lambda_- \le \lambda_+ < 1$ and $p(\cdot)$ satisfies the conditions* (1) *and* (2). *Then for $h > 0$ and for every $r \in \mathbb{N}$ we have*

$$c\,\Omega^r(f,h;\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)) \le \mathcal{K}_r\Big(f,h\Big)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \le C\Omega^r(f,h;\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)),$$

where the constants $c, C > 0$ are independent of f and h.

Proof. Let $g \in W^r_{p(\cdot),\lambda(\cdot)}(I_0,w)$, taking into account the definition of $\mathcal{K}_r\Big(f,t\Big)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}$, Lemma 2 and Lemma 3, we obtain

$$\begin{aligned}&\Omega^r(f,h;\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)) \le \Omega^r(f-g,h;\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)) + \Omega^r(g,h;\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)) \\ &\qquad \le C(p(\cdot))\Big(\|f-g\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + \delta^r\|g^{(r)}\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\Big) \le 2C(p(\cdot))\mathcal{K}_r\Big(f,h\Big)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\end{aligned}$$

for any $f \in \mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)$.

In order to prove the converse inequality, we introduce a Steklov-type transform for $f \in \mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)$ and $h > 0$:

$$f_{r,h}(x) := \frac{2}{h}\int_{\frac{h}{2}}^{h}\frac{1}{\delta^r}\underbrace{\int_0^\delta\dots\int_0^\delta}_{r\text{ times}}\sum_{s=0}^{r-1}\binom{r}{s}(-1)^{r+s+1}f\Big(x+\frac{r-s}{r}(t_1+\dots+t_r)\Big)dt_1\dots dt_r\,d\delta.$$

Then

$$\begin{aligned}&\Big\|f_{r,h}-f\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \\ &= \Big\|\frac{2}{h}\int_{\frac{h}{2}}^{h}\frac{1}{\delta^r}\underbrace{\int_0^\delta\dots\int_0^\delta}_{r\text{ times}}\sum_{s=0}^{r}\binom{r}{s}(-1)^{r+s+1}f\Big(\cdot+\frac{r-s}{r}(t_1+\dots+t_r)\Big)dt_1\dots dt_r d\delta\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \\ &= \Big\|\frac{2}{h}\int_{\frac{h}{2}}^{h}\frac{1}{\delta^r}\underbrace{\int_0^\delta\dots\int_0^\delta}_{r\text{ times}}\Delta^r_{\frac{t_1+\dots+t_r}{r}}f(\cdot)\;\;dt_1\dots dt_r d\delta\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \\ &\le \sup_{\frac{h}{2}\le\delta\le h}\Big\|\frac{1}{\delta^r}\underbrace{\int_0^\delta\dots\int_0^\delta}_{r\text{ times}}\Delta^r_{\frac{t_1+\dots+t_r}{r}}f(\cdot)\;\;dt_1\dots dt_r\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}.\end{aligned}$$

Therefore substituting $t = t_1 + \ldots + t_r$ we get

$$\begin{aligned}
\|f_{r,h} - f\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} &\le \sup_{\frac{h}{2}\le\delta\le h} \Big\| \frac{1}{\delta^r} \underbrace{\int_0^{\delta} \ldots \int_{t_1+\ldots+t_{r-1}}^{\delta+t_1+\ldots+t_{r-1}}}_{r \text{ times}} \Delta^r_{\frac{t}{r}} f(\cdot)dt \;\; dt_1 \ldots dt_{r-1} \Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \\
&\le \sup_{\frac{h}{2}\le\delta\le h} \Big\| \frac{1}{\delta^r} \underbrace{\int_0^{\delta} \ldots \int_0^{r\delta}}_{r \text{ times}} \Delta^r_{\frac{t}{r}} f(\cdot)dt \;\; dt_1 \ldots dt_{r-1} \Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \\
&= r \sup_{\frac{h}{2}\le\delta\le h} \Big\| \frac{1}{\delta^{r-1}} \int_0^{\delta} \ldots \int_0^{\delta} \Big\{ \frac{1}{\delta} \int_0^{\delta} \Delta^r_t f(\cdot)dt \Big\} \;\; dt_1 \ldots dt_{r-1} \Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}.
\end{aligned}$$

Now, by Minkowski's integral inequality for variable exponent Lebesgue space, we get

$$\begin{aligned}
\|f_{r,h} - f\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} &\le rC(p(\cdot)) \sup_{\frac{h}{2}\le\delta\le h} \frac{1}{\delta^{r-1}} \int_0^{\delta} \cdots \int_0^{\delta} \Big\| \frac{1}{\delta} \int_0^{\delta} \Delta^r_t f(\cdot)dt \Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} dt_1 \ldots dt_{r-1} \\
&\le C(r, p(\cdot)) \sup_{0<\delta<\min(2\pi,h)} \Big\| \frac{1}{\delta} \int_0^{\delta} \Delta^r_t f(\cdot)dt \Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \\
&\le C(r, p(\cdot))\Omega^r(f, h; \mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0, w)).
\end{aligned}$$

Meanwhile, by differentiating $f_{r,h}(x)$ in x, we have

$$\begin{aligned}
f^{(r-1)}_{r,h}(x) &= \frac{2}{h} \int_{\frac{h}{2}}^{h} \frac{1}{\delta^r} \sum_{s=0}^{r-1} \binom{r}{s} (-1)^{r+s+1} \int_0^{\delta} \Big(\frac{r}{r-s}\Big)^{r-1} \Delta^{r-1}_{\frac{r-s}{r}\delta} f\Big(x + \frac{r-s}{r} t_r\Big) dt_r d\delta \\
&= \frac{2}{h} \int_{\frac{h}{2}}^{h} \frac{1}{\delta^r} \sum_{s=0}^{r-1} \binom{r}{s} (-1)^{r+s+1} \int_0^{\frac{r-s}{r}\delta} \Big(\frac{r}{r-s}\Big)^{r} \Delta^{r-1}_{\frac{r-s}{r}\delta} f\Big(x + t\Big) dt d\delta \\
&= \frac{2}{h} \int_{\frac{h}{2}}^{h} \frac{1}{\delta^r} \sum_{s=0}^{r-1} \binom{r}{s} (-1)^{r+s+1} \int_x^{x+\frac{r-s}{r}\delta} \Big(\frac{r}{r-s}\Big)^{r} \Delta^{r-1}_{\frac{r-s}{r}\delta} f\Big(u\Big) du d\delta.
\end{aligned}$$

Therefore,

$$f^{(r)}_{r,h}(x) = \frac{2}{h} \int_{\frac{h}{2}}^{h} \frac{1}{\delta^r} \sum_{s=0}^{r-1} \binom{r}{s} (-1)^{r+s+1} \Big(\frac{r}{r-s}\Big)^{r} \Delta^{r}_{\frac{r-s}{r}\delta} f\Big(x\Big) d\delta.$$

Hence,

$$\begin{aligned}
\Big| f^{(r)}_{r,h}(x) \Big| &\le \frac{2}{h} \int_{\frac{h}{2}}^{h} \frac{1}{\delta^r} \sum_{s=0}^{r-1} \binom{r}{s} (-1)^{r+s+1} \Big(\frac{r}{r-s}\Big)^{r} \Big| \Delta^{r}_{\frac{r-s}{r}\delta} f\Big(x\Big) \Big| d\delta \\
&\le \frac{2^{r+1}}{h^r} \sum_{s=0}^{r-1} \binom{r}{s} \Big(\frac{r}{r-s}\Big)^{r} \frac{1}{h} \int_0^{h} \Big| \Delta^{r}_{\frac{r-s}{r}\delta} f\Big(x\Big) \Big| d\delta \\
&\le \frac{2^{r+1}}{h^r} \sum_{s=0}^{r-1} \binom{r}{s} \Big(\frac{r}{r-s}\Big)^{r} \frac{1}{\frac{r-s}{r}h} \int_0^{\frac{r-s}{r}h} \Big| \Delta^{r}_{t} f\Big(x\Big) \Big| dt.
\end{aligned}$$

From the definition of $\Omega^r(f, h; \mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0, w))$, we have

$$\|f^{(r)}_{r,h}(\cdot)\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \le Ch^{-r}\Omega^r(f, h; \mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0, w)).$$

Consequently, we deduce

$$\begin{aligned}\mathcal{K}_r\Big(f,h\Big)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} &\le \|f-f_{r,h}\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + h^r\|f_{r,h}^{(r)}\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\\ &\le C\,\Omega^r(f,h;\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)).\end{aligned}$$

□

The functions in Morrey spaces are not easy to approximate with polynomials since the set of the all trigonometric polynomials are not dense in $\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)$ (see [13,32]). If we take $f\in\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)$ instead of $f\in\widetilde{\mathcal{M}}_{p(\cdot),\lambda(\cdot)}(I_0,w)$, then it is not guarenteed that there exists $T_n\in\mathcal{T}_n$, the trigonometric polynomial of the best approximation to f in $\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)$. For this reason, in the rest of paper we will use $\widetilde{\mathcal{M}}_{p(\cdot),\lambda(\cdot)}(I_0,w)$, the closure of the set of all trigonometric polynomials in $\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)$, for stating our results.

Now let

$$f(x)\sim\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos kx+b_k\sin kx$$

be the Fourier series of $f\in\widetilde{\mathcal{M}}_{p(\cdot),\lambda(\cdot)}(I_0,w)$. Assume that $\lambda(\cdot)$ satisfies the condition $0\le\lambda_-\le\lambda_+<1$ and $p(\cdot)$ satisfies the conditions (1) and (2) and $S_n(x,f)$ be its nth partial sum. Under the condition $w\in A_{p(\cdot)}(I_0)$, using the method of proof of Lemma 2 and applying the appropriate results in variable exponent weighted Lebesgue spaces given in [16], we see that

$$\begin{aligned}\|f-S_n(\cdot,f)\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} &\le cE_n(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)},\\ E_n(\tilde f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} &\le cE_n(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)},\end{aligned}\tag{7}$$

where $\tilde f$ is the conjugate function of f.

Lemma 5. *Let $f\in\dot W^r_{p(\cdot),\lambda(\cdot)}(I_0,w)$, $w\in A_{p(\cdot)}(I_0)$ and $r\ge1$. Assume that $\lambda(\cdot)$ satisfies the condition $0\le\lambda_-\le\lambda_+<1$ and $p(\cdot)$ satisfies the conditions (1) and (2). Then we have*

$$E_n(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\le\frac{C}{n^r}\|f^{(r)}\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)},\quad n\in\mathbb{N},$$

where the constant C is independent of f and n.

Proof. Let $f(x)\sim\sum_{k=0}^{\infty}a_k\cos kx+b_k\sin kx$ be the Fourier series of $f\in\widetilde{\mathcal{M}}_{p(\cdot),\lambda(\cdot)}(I_0,w)$ and $S_n(x,f)$ be its nth partial sum. Then

$$\tilde f(x)\sim\sum_{k=0}^{\infty}b_k\cos kx-a_k\sin kx.$$

Setting

$$A_k(x,f):=a_k\cos kx+b_k\sin kx,\quad k\in\mathbb{N},$$

we have $f(x)=\sum_{k=0}^{\infty}A_k(x,f)$ in the norm of $\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)$. Since

$$\begin{aligned}A_k(x,f)&=a_k\cos kx+b_k\sin kx=a_k\cos\Big(kx+\frac{r\pi}{2}-\frac{r\pi}{2}\Big)+b_k\sin\Big(kx+\frac{r\pi}{2}-\frac{r\pi}{2}\Big)\\ &=\cos\frac{r\pi}{2}\Big[a_k\cos k\Big(x+\frac{r\pi}{2k}\Big)+b_k\sin k\Big(x+\frac{r\pi}{2k}\Big)\Big]\\ &\quad+\sin\frac{r\pi}{2}\Big[a_k\sin k\Big(x+\frac{r\pi}{2k}\Big)-b_k\cos k\Big(x+\frac{r\pi}{2k}\Big)\Big]\\ &=A_k\Big(x+\frac{r\pi}{2k},f\Big)\cos\frac{r\pi}{2}+A_k\Big(x+\frac{r\pi}{2k},\tilde f\Big)\sin\frac{r\pi}{2}\end{aligned}$$

and

$$A_k\Big(x, f^{(r)}\Big) = k^r A_k\Big(x + \frac{r\pi}{2k}, f\Big),$$

we get

$$\begin{aligned}\sum_{k=0}^{\infty} A_k(x,f) &= A_0(x,f) + \cos\frac{r\pi}{2}\sum_{k=1}^{\infty} A_k\Big(x+\frac{r\pi}{2k}, f\Big) + \sin\frac{r\pi}{2}\sum_{k=1}^{\infty} A_k\Big(x+\frac{r\pi}{2k}, \tilde{f}\Big)\\ &= A_0(x,f) + \cos\frac{r\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k^r} A_k\Big(x, f^{(r)}\Big) + \sin\frac{r\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k^r} A_k\Big(x, \tilde{f}^{(r)}\Big).\end{aligned}$$

Then

$$f(x) - S_n(x,f) = \sum_{k=n+1}^{\infty} A_k(x,f) = \cos\frac{r\pi}{2}\sum_{k=n+1}^{\infty}\frac{1}{k^r}A_k(x,f^{(r)}) + \sin\frac{r\pi}{2}\sum_{k=n+1}^{\infty}\frac{1}{k^r}A_k(x,\tilde{f}^{(r)}).$$

Taking into account that

$$\begin{aligned}\sum_{k=n+1}^{\infty}\frac{1}{k^r}A_k(x,f^{(r)}) &= \sum_{k=n+1}^{\infty}\frac{1}{k^r}\Big[S_k(x,f^{(r)}) - S_{k-1}(x,f^{(r)})\Big]\\ &= \sum_{k=n+1}^{\infty}\frac{1}{k^r}\Big\{\Big[S_k(x,f^{(r)}) - f^{(r)}(x)\Big] - \Big[S_{k-1}(x,f^{(r)}) - f^{(r)}(x)\Big]\Big\}\\ &= \sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r} - \frac{1}{(k+1)^r}\Big)\Big[S_k(x,f^{(r)}) - f^{(r)}(x)\Big] - \frac{1}{(n+1)^r}\Big[S_n(x,f^{(r)}) - f^{(r)}(x)\Big]\end{aligned}$$

and

$$\sum_{k=n+1}^{\infty}\frac{1}{k^r}A_k(x,\tilde{f}^{(r)}) = \sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r} - \frac{1}{(k+1)^r}\Big)\Big[S_k(x,\tilde{f}^{(r)}) - \tilde{f}^{(r)}(x)\Big] - \frac{1}{(n+1)^r}\Big[S_n(x,\tilde{f}^{(r)}) - \tilde{f}^{(r)}(x)\Big],$$

by (7), we have

$$\begin{aligned}\|f - S_n(\cdot,f)\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} &\le \sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r} - \frac{1}{(k+1)^r}\Big)\Big\|S_k(\,\cdot,f^{(r)}) - f^{(r)}\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\\ &\quad + \frac{1}{(n+1)^r}\Big\|S_n(\cdot,f^{(r)}) - f^{(r)}\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\\ &\quad + \sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r} - \frac{1}{(k+1)^r}\Big)\Big\|S_k(\,\cdot,\tilde{f}^{(r)}) - \tilde{f}^{(r)}\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\\ &\quad + \frac{1}{(n+1)^r}\Big\|S_n(\,\cdot,\tilde{f}^{(r)}) - \tilde{f}^{(r)}\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\\ &\le C\Big\{\sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r} - \frac{1}{(k+1)^r}\Big)E_k(f^{(r)})_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + \frac{1}{(n+1)^r}E_n(f^{(r)})_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\Big\}\\ &\quad + C\Big\{\sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r} - \frac{1}{(k+1)^r}\Big)E_k(\tilde{f}^{(r)})_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + \frac{1}{(n+1)^r}E_n(\tilde{f}^{(r)})_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\Big\}.\end{aligned}$$

After simple calculations and using second relation of (7), we get

$$\begin{aligned}\Big\|f - S_n(\cdot,f)\Big\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} &\le CE_n(f^{(r)})_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\Big\{\sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r} - \frac{1}{(k+1)^r}\Big) + \frac{1}{(n+1)^r}\Big\}\\ &\quad + CE_n(\tilde{f}^{(r)})_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\Big\{\sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r} - \frac{1}{(k+1)^r}\Big) + \frac{1}{(n+1)^r}\Big\}\\ &\le \frac{C}{(n+1)^r}E_n(f^{(r)})_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}.\end{aligned}$$

Hence,

$$\begin{aligned} E_n(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} &\le \|f - S_n(\,\cdot, f)\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \\ &\le \frac{C}{n^r} E_n(f^{(r)})_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \le \frac{C}{n^r}\|f^{(r)}\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}. \end{aligned}$$

□

Now we will give the Bernstein inequality in variable exponent weighted Morrey spaces. Bernstein inequalities date back to 1912, when S.N. Bernstein proved the first inequality of this type for L_∞ norms of trigonometric polynomials. A generalization can be found in [2]; this result states that any trigonometric polynomial T_n in $\mathcal{T}_n$ satisfies

$$\|T_n^{(k)}\|_{L_{p(\cdot)}(I_0,w)} \le Cn^k \|T_n\|_{L_{p(\cdot)}(I_0,w)}, \quad k \in \mathbb{N},$$

where $p(\cdot)$ satisfies the conditions (1) and (2).

Lemma 6 (Bernstein inequality for variable exponent weighted Morrey spaces)**.** *Assume that $\lambda(\cdot)$ satisfies the condition $0 \le \lambda_- \le \lambda_+ < 1$ and $p(\cdot)$ satisfies the conditions* (1) *and* (2)*, and $w \in A_{p(\cdot)}(I_0)$. Then for every trigonometric polynomial T_n in $\mathcal{T}_n$ and $k \in \mathbb{N}$*

$$\|T_n^{(k)}\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \le Cn^k \|T_n\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}, \quad n \in \mathbb{N} \cup \{0\},$$

where the constant C is independent of n.

Proof. The proof is obtained similarly to that of Lemma 2 by using [16, Lemma 5], where the Bernstein inequality was proved in $L_{p(\cdot)}(I_0, w)$. □

3 Main results

Now, we present the direct and inverse theorems in the variable exponent weighted Morrey spaces $\widetilde{\mathcal{M}}_{p(\cdot),\lambda(\cdot)}(I_0, w)$.

Theorem 1 (Direct Theorem)**.** *Let $f \in \widetilde{\mathcal{M}}_{p(\cdot),\lambda(\cdot)}(I_0, w)$ and $w \in A_{p(\cdot)}(I_0)$. Assume that $\lambda(\cdot)$ satisfies the condition $0 \le \lambda_- \le \lambda_+ < 1$ and $p(\cdot)$ satisfies the conditions* (1) *and* (2)*. Then we have*

$$E_n(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \le C\,\Omega^r(f, \frac{1}{n}; \mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0, w)), \quad n \in \mathbb{N},$$

where the constant $C > 0$ is independent of f and n.

Proof. Let $g \in \dot{W}^r_{p(\cdot),\lambda(\cdot)}(I_0, w)$. By Lemma 5 we have

$$\begin{aligned} E_n(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} &\le E_n(f-g)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + E_n(g)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \\ &\le \|f-g\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + \frac{C}{n^r}\|g^{(r)}\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}. \end{aligned}$$

Since this inequality holds for every $g \in \dot{W}^r_{p(\cdot),\lambda(\cdot)}(I_0, w)$ by the definition of the K-functional and by Lemma 4, we get

$$E_n(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \le C\mathcal{K}_r\Big(f, \frac{1}{n}\Big)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \le C\Omega^r(f, \frac{1}{n}; \mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0, w)).$$

Thus, the proof of Theorem 1 is complete. □

Corollary 1 ([13]). *Let $f \in \widetilde{\mathcal{M}}_{p(\cdot),\lambda(\cdot)}(I_0)$, $\lambda(\cdot)$ satisfy the condition $0 \le \lambda_- \le \lambda_+ < 1$ and $p(\cdot)$ satisfy the conditions* (1) *and* (2). *Then we have*

$$E_n(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0)} \le C\Omega^r(f, \frac{1}{n}; \mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0)), \quad n \in \mathbb{N},$$

where the constant $C > 0$ independent of f and n.

Theorem 2 (Inverse Theorem). *Let $f \in \widetilde{\mathcal{M}}_{p(\cdot),\lambda(\cdot)}(I_0, w)$ and $w \in A_{p(\cdot)}(I_0)$. Assume that $\lambda(\cdot)$ satisfies the condition $0 \le \lambda_- \le \lambda_+ < 1$ and $p(\cdot)$ satisfies the conditions* (1) *and* (2). *Then we have*

$$\Omega^r(f, \frac{1}{n}; \mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0, w)) \le \frac{C}{n^r}\Big\{E_0(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + \sum_{m=1}^{n} m^{r-1}E_m(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\Big\},$$

where $n \in \mathbb{N}$ and the constant $C > 0$ is independent of f and n.

Proof. Let $T_n \in \mathcal{T}_n$ be the polynomial of the best approximation to f in $\widetilde{\mathcal{M}}_{p(\cdot),\lambda(\cdot)}(I_0, w)$. For any integer $j = 1, 2, \dots$, from the definition of K-functional we obtain

$$\begin{aligned}\mathcal{K}_r\Big(f, \frac{1}{n}\Big)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} &= \inf_{g \in W^r_{p(\cdot),\lambda(\cdot)}(I_0,w)} \Big\{\|f-g\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + \frac{1}{n^r}\|g^{(r)}\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\Big\} \\ &\le \|f - T_{2^{j+1}}\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + \frac{1}{n^r}\|T^{(r)}_{2^{j+1}}\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}.\end{aligned}$$

Using Lemma 6, we get

$$\begin{aligned}\|T^{(r)}_{2^{j+1}}\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} &\le \|T^{(r)}_1 - T^{(r)}_0\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + \sum_{i=0}^{j}\|T^{(r)}_{2^{j+1}} - T^{(r)}_{2^i}\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \\ &\le C\Big\{\|T_1 - T_0\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + \sum_{i=0}^{j} 2^{(i+1)r}\|T_{2^{i+1}} - T_{2^i}\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\Big\} \\ &\le C\Big\{E_1(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + E_0(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \\ &\qquad + \sum_{i=0}^{j} 2^{(i+1)r}\Big\{E_{2^{i+1}}(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + E_{2^i}(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\Big\}\Big\} \\ &\le C\Big\{E_0(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + \sum_{i=0}^{j} 2^{(i+1)r}E_{2^i}(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\Big\} \\ &= C\Big\{E_0(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + 2^rE_1(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + \sum_{i=1}^{j} 2^{(i+1)r}E_{2^i}(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\Big\}.\end{aligned}$$

Since

$$2^{(i+1)r}E_{2^i}(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \le 2^{2r}\sum_{m=2^{i-1}+1}^{2^i} m^{r-1}E_m(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \tag{8}$$

for $i \ge 1$, we have

$$\begin{aligned}&\|T^{(r)}_{2^{j+1}}\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} \\ &\quad\le C\Big\{E_0(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + 2^rE_1(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + 2^{2r}\sum_{m=2}^{2^j} m^{r-1}E_m(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\Big\} \\ &\quad\le C\Big\{E_0(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + \sum_{m=1}^{2^j} m^{r-1}E_m(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\Big\}.\end{aligned}$$

Selecting j such that $2^j \le n < 2^{j+1}$, from (8) we get

$$E_{2^{j+1}}(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} = \frac{2^{(j+1)r}E_{2^{j+1}}(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}}{2^{(j+1)r}} \le \frac{1}{n^r}2^{(j+1)r}E_{2^{j+1}}(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}$$
$$\le \frac{1}{n^r}\sum_{m=2^{j-1}+1}^{2j} m^{r-1}E_m(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}.$$

Now by Lemma 4, we conclude that

$$\begin{aligned}\Omega^r(f,\frac{1}{n};\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)) &\le C\mathcal{K}_r\Big(f,\frac{1}{n}\Big)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\\
&\le CE_{2^{j+1}}(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + \frac{1}{n^r}\|T_{2^{j+1}}\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\\
&\le \frac{C}{n^r}\sum_{2^{j-1}+1}^{2^j} m^{r-1}E_m(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\\
&\quad + \frac{C}{n^r}\Big\{E_0(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + \sum_{m=1}^{2^j} m^{r-1}E_m(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\Big\}\\
&\le \frac{C}{n^r}\Big\{E_0(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)} + \sum_{m=1}^{n} m^{r-1}E_m(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0,w)}\Big\}.\end{aligned}$$

Thus the proof of Theorem 2 is completed. □

Corollary 2 ([13]). *Let $f \in \widetilde{\mathcal{M}}_{p(\cdot),\lambda(\cdot)}(I_0)$, $\lambda(\cdot)$ satisfy the condition $0 \le \lambda_- \le \lambda_+ < 1$, and $p(\cdot)$ satisfy the conditions* (1) *and* (2). *Then for every $r \in \mathbb{N}$ we have*

$$\Omega^r\Big(f,\frac{1}{n},\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0)\Big) \le \frac{C}{n^r}\Big\{E_0(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0)} + \sum_{m=1}^{n} m^{r-1}E_m(f)_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0)}\Big\},$$

where $n \in \mathbb{N}$ and the constant $C > 0$ is independent of f and n.

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Сакір З., Айкол С., Гулієв В.С., Сербетсі А. *Апроксимація тригонометричними поліномами у зважених просторах Моррі зі змінною експонентою* // Карпатські матем. публ. — 2021. — Т.13, №3. — С. 750–763.

У цій роботі ми досліджуємо найкраще наближення тригонометричними поліномами у зважених просторах Моррі зі змінною експонентою $\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0, w)$, де w — це вагова функція в класі Мухенгупта $A_{p(\cdot)}(I_0)$. Доведено пряму та обернену теореми апроксимації тригонометричними поліномами в просторах $\widetilde{\mathcal{M}}_{p(\cdot),\lambda(\cdot)}(I_0, w)$, що є замиканням множини всіх тригонометричних поліномів у $\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0, w)$.

Ключові слова і фрази: зважений простір Моррі зі змінним показником, найкраще наближення, тригонометричний поліном, пряма та обернена теорема.