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On spectral radius and Nordhaus-Gaddum type inequalities of the generalized distance matrix of graphs

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If $Tr(G)$ and $D(G)$ are respectively the diagonal matrix of vertex transmission degrees and distance matrix of a connected graph G, the generalized distance matrix $D_\alpha(G)$ is defined as $D_\alpha(G) = \alpha\ Tr(G) + (1-\alpha)\ D(G)$, where $0 \leq \alpha \leq 1$. If $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ are the eigenvalues of $D_\alpha(G)$, the largest eigenvalue ρ_1 (or $\rho_\alpha(G)$) is called the spectral radius of the generalized distance matrix $D_\alpha(G)$. The generalized distance energy is defined as $E^{D_\alpha}(G) = \sum_{i=1}^{n} \left|\rho_i - \frac{2\alpha W(G)}{n}\right|$, where $W(G)$ is the Wiener index of G. In this paper, we obtain the bounds for the spectral radius $\rho_\alpha(G)$ and the generalized distance energy of G involving Wiener index. We derive the Nordhaus-Gaddum type inequalities for the spectral radius and the generalized distance energy of G.

Key words and phrases: distance matrix, generalized distance matrix, spectral radius, generalized distance energy, Nordhaus-Gaddam type inequality.

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1 Introduction

Let $G(V(G), E(G))$ be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and order $|V(G)| = n$. The degree $d(v_i)$ or d_i of a vertex v_i is the number of edges incident on v_i. The set of vertices adjacent to $v \in V(G)$, denoted by $N(v)$, refers to the *neighborhood* of v. In G, the *distance* between two vertices $u, v \in V(G)$, denoted by d_{uv}, is defined as the length of a shortest path between u and v. The *distance matrix* of G, denoted by $D(G)$, is defined as $D(G) = (d_{uv})_{u,v\in V(G)}$. The *transmission* $t_G(v)$ of a vertex v is defined as the sum of the distances from v to all other vertices in G, that is, $t_G(v) = \sum\limits_{u\in V(G)} d_{uv}$. A graph G is said to be *k-transmission regular* if $t_G(v) = k$ for each $v \in V(G)$. For any vertex $v_i \in V(G)$, the transmission $t_G(v_i)$ is also called the *transmission degree*, shortly denoted by t_i and the sequence $\{t_1, t_2, \ldots, t_n\}$ is called the *transmission degree sequence* of the graph G. The matrix $Tr(G) = diag\ (t_1, t_2, \ldots, t_n)$ is the diagonal matrix of vertex transmissions. The *generalized distance matrix* [7] is defined as $D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G)$ for $0 \leq \alpha \leq 1$. Let $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ be the eigenvalues of $D_\alpha(G)$. We will denote the largest eigenvalue (generalized distance spectral radius) ρ_1 by $\rho_\alpha(G)$ (or simply ρ_α). As $D_\alpha(G)$ is non-negative and irreducible, by the Perron-Frobenius theorem, ρ_α is unique and there is a unique positive unit eigenvector X corresponding to ρ_α, which is called the *generalized distance Perron vector* of G. The generalized distance energy is defined as $E^{D_\alpha}(G) = \sum_{i=1}^{n} |\rho_i - \frac{2\alpha W(G)}{n}|$, where $W(G)$ is the Wiener index of G. For some recent results,

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we refer to [1,3,5,7–10,15–18] and the references therein. For standard definitions, we refer to [6,14].

The chromatic number of a graph G is the minimum number of colors required to color the vertices of G so that no two adjacent vertices share the same color. E.A. Nordhaus and J.W. Gaddum studied the chromatic number of a graph G together with its complement $\overline{G}$, see [13]. They obtained the lower and upper bounds for the sum and the product of the chromatic numbers of G and $\overline{G}$, in terms of the number of vertices of G. Since then, any relation for the sum and/or the product of an invariant in a graph G and the same variant in $\overline{G}$ is called a *Nordhaus-Gaddum type inequality*. Nordhaus-Gaddum type theorems establish bounds for $f(G) + f(\overline{G})$ for some graph invariant f. A survey of Nordhaus-Gaddum type inequalities for graph invariants can be seen in [4].

The rest of the paper is organized as follows. In Section 2, we obtain the bounds for the spectral radius of the generalized distance matrix. Further, as an application of these results, we obtain the lower bounds for the generalized distance energy. In Section 3, we obtain Nordhaus-Gaddum type inequalities for both the spectral radius and energy of generalized distance matrix of graph G.

2 Bounds for generalized distance spectral radius of graphs

Our first result is a lower bound for the spectral radius ρ_α of $D_\alpha(G)$ in terms of the transmissions and Wiener index. As usual K_n denotes the complete graph with n vertices and $\overline{K}_n$ denotes an empty graph.

Theorem 1. *Let G be a connected graph of order n and size m with Wiener index $W(G)$. If $0 \le \alpha < 1$, then*

$$\rho_\alpha \ge \frac{2W}{n} + \frac{\frac{1}{n}\sum_{i=1}^{n} Tr_i^{\,2} - \frac{4W^2}{n^2}}{2\sqrt{\alpha^2 \sum_{i=1}^{n} Tr_i^{\,2} + (1-\alpha)^2 (2W)^2}}, \tag{1}$$

and equality holds if G is a transmission regular graph.

Proof. Let $\rho_\alpha = \rho_\alpha(G)$ be the spectral radius of $D_\alpha(G)$ and $\mathbf{x}$ be the unique eigenvector of $D_\alpha(G)$ corresponding to ρ_α. For $1 \le i \le n$, assume that r_i is the ith row of $D_\alpha(G)$. For $1 \le i \le n$, we have

$$||x||^2 = 1 \quad \text{and} \quad ||r_i||^2 = \alpha^2 Tr_i^2 + (1-\alpha) \sum_{\substack{j=1 \\ j \ne i}}^{n} d_{ij}^{\,2}.$$

Since $(\rho_\alpha) x_i = r_i \mathbf{x}$, therefore

$$\rho_\alpha^{\,2} x_i^{\,2} = ||(r_i)\mathbf{x}||^2 \le ||r_i||^2 ||\mathbf{x}||^2 = ||r_i||^2 \cdot 1 = \alpha^2 Tr_i^2 + (1-\alpha)^2 \sum_{\substack{j=1 \\ j \ne i}}^{n} d_{ij}^{\,2}.$$

So

$$\begin{aligned} \rho_\alpha^2 = \sum_{i=1}^{n} \rho_\alpha^2 x_i^2 &\le \sum_{i=1}^{n} (\alpha^2 Tr_i^2 + (1-\alpha)^2 \sum_{j \ne i=1}^{n} d_{ij}^2) \\ &= \alpha^2 \sum_{i=1}^{n} Tr_i^2 + (1-\alpha)^2 \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \ne i}}^{n} d_{ij}^2 \le \alpha^2 \sum_{i=1}^{n} Tr_i^2 + (1-\alpha)^2 (2W)^2, \end{aligned}$$

where W is the Wiener index of the graph G. This implies that

$$\rho_\alpha \leq \sqrt{\alpha^2 \sum_{i=1}^{n} Tr_i^2 + (1-\alpha)^2 (2W)^2}.$$

Therefore,

$$2(\rho_\alpha - \frac{2W}{n})\sqrt{\alpha^2 \sum_{i=1}^{n} Tr_i^2 + (1-\alpha)^2 (2W)^2} \geq 2\rho_\alpha^2 - 2\rho_\alpha \frac{2W}{n} \geq \rho_\alpha^2 - \frac{4W^2}{n^2}.$$

Since $\rho_\alpha^2 \geq \frac{1}{n}\sum_{i=1}^{n} Tr_i^2$, therefore

$$\rho_\alpha \geq \frac{2W}{n} + \frac{\frac{1}{n}\sum_{i=1}^{n} Tr_i^2 - \frac{4W^2}{n^2}}{2\sqrt{\alpha^2 \sum_{i=1}^{n} Tr_i^2 (1-\alpha)^2 (2W)^2}}.$$

Equality holds in (1), when G is a transmission regular graph, proving the theorem. ☐

Let $S_i(B)$ be the ith row sum of any matrix B. For a real symmetric matrix, M. Ellingham and X. Zha [11] proved the following result.

Lemma 1. *For the real symmetric $n \times n$ matrix B, let λ be an eigenvalue of B with an eigenvector $\mathbf{x}$ all of whose entries are non-negative. Then*

$$\min_{1\leq i\leq n} S_i(B) \leq \lambda \leq \max_{1\leq i\leq n} S_i(B).$$

Moreover, if the row sums of B are not all equal and if all the entries of $\mathbf{x}$ are positive, then both inequalities above are strict.

For a connected graph G, let $\mathbf{x}$ be an eigenvector of $D_\alpha(G)$, corresponding to the spectral radius $\rho_\alpha(G)$. If $0 \leq \alpha < 1$, then $\mathbf{x}$ is positive. For any polynomial $p(.)$, it is obvious that $p(\rho_\alpha)$ is an eigenvalue of $p(D_\alpha(G))$ and $\mathbf{x}$ is also an eigenvector corresponding to $p(\rho_\alpha)$. Thus, from Lemma 1, we have the following observation.

Lemma 2. *Let G be a connected graph of order n and let $p(.)$ be any polynomial. If $0 \leq \alpha < 1$, then*

$$\min_{1\leq i\leq n} S_i(p(D_\alpha(G))) \leq p(D_\alpha(G)) \leq \max_{1\leq i\leq n} S_i(p(D_\alpha(G))).$$

Moreover, if the row sums of $p(D_\alpha(G))$ are not all equal, then both inequalities above are strict.

Now, we obtain a lower bound for the spectral radius in terms of the maximum and minimum transmission degrees and the Wiener index of the graph G.

Theorem 2. *Let G be a connected graph with order n, size m, Wiener index W and having maximum and minimum transmission degree Tr_{max} and Tr_{min} respectively. If $0 \leq \alpha < 1$, then*

$$\rho_\alpha \leq \frac{-(1-\alpha)(\frac{n}{2}-1)(n-1) + \sqrt{[(1-\alpha)^2(\frac{n}{2}-1)^2(n-1)^2 + 4(\alpha Tr_{max}^2 + (1-\alpha)2W)]}}{2}. \quad (2)$$

Equality holds when G is isomorphic to K_n.

Proof. Assume that $D_\alpha = D_\alpha(G)$, $Tr = Tr(G)$ and $D = D(G)$. From $D_\alpha = \alpha Tr + (1-\alpha)D$, we have $D_\alpha^2 = (\alpha Tr + (1-\alpha)D)^2 = \alpha^2 Tr^2 + (1-\alpha)^2 D^2 + \alpha(1-\alpha)TrD + \alpha(1-\alpha)DTr$. Note that $S_i(D)^2 = T_i$, $S_iT(rD) = Tr_i^2$ and $S_i(DTr) = T_i$. For $i \in V(G)$, the row sum of D_α^2 corresponding to the vertex i is

$$\begin{aligned} S_i(D_\alpha^2) &= \alpha^2 Tr_i^2 + (1-\alpha)^2 T_i + \alpha(1-\alpha)Tr_i^2 + \alpha(1-\alpha)T_i \\ &= (\alpha^2 + \alpha(1-\alpha))Tr_i^2 + ((1-\alpha)^2 + \alpha(1-\alpha))T_i \\ &= \alpha Tr_i^2 + (1-\alpha)T_i \le \alpha Tr_i^2 + (1-\alpha)(2W - (\frac{n}{2}-1)(n-1)Tr_iTr_{min}), \end{aligned}$$

which implies that

$$S_i\left(D_\alpha^2 + (1-\alpha)\left(\frac{n}{2}-1\right)(n-1)Tr_{min}D_\alpha\right) \le \alpha Tr_{max}^2 + (1-\alpha)2W.$$

Using Lemma 2, we have $\rho_\alpha^2(G) + (1-\alpha)(\frac{n}{2}-1)(n-1)Tr_{min}\rho_\alpha \le \alpha Tr_{max}^2 + (1-\alpha)2W$, that is,

$$\rho_\alpha \le \frac{-(1-\alpha)(\frac{n}{2}-1)(n-1) + \sqrt{[(1-\alpha)^2(\frac{n}{2}-1)^2(n-1)^2 + 4(\alpha Tr_{max}^2 + (1-\alpha)(2W))]}}{2}.$$

If equality holds in (2), then all the inequalities become equalities. This implies that $T_i = 2W - (\frac{n}{2-1})(n-1)Tr_iT_{min}$, which further implies that all the distances must be equal to 1. Thus the graph is complete. □

Now, as a consequence of Theorem 1, we obtain a lower bound for the generalized distance energy $E^{D_\alpha(G)}$ in terms of the Wiener index of G.

Theorem 3. *If G is a connected graph of order n and with Wiener index W, then*

$$E^{D_\alpha}(G) \ge 2\left(\frac{(1-\alpha)2W}{n} + \frac{\frac{1}{n}\sum_{i=1}^n Tr_i^2 - \frac{4W^2}{n^2}}{2\sqrt{\alpha^2 Tr_i^2 + (1-\alpha)^2(2W)^2}}\right). \tag{3}$$

Equality holds if G is a transmission regular graph.

Proof. The generalized distance energy of a graph G is defined as

$$E^{D_\alpha}(G) = \sum_{i=1}^n \left|\rho_\alpha - \frac{2\alpha W}{n}\right|,$$

where W is the Wiener index of G. Let s be the largest positive integer such that $\rho_s \ge \frac{2\alpha W}{n}$ and $\rho_{s+1} < \frac{2\alpha W}{n}$. Then, we have

$$\begin{aligned} E^{D_\alpha}(G) = \sum_{i=1}^n \left|\rho_i - \frac{2\alpha W}{n}\right| &= \sum_{i=1}^s |\rho_i - \frac{2\alpha W}{n}| + \sum_{i=s+1}^n |\frac{2\alpha W}{n} - \rho_i| \\ &= 2\left(\sum_{i=1}^s \rho_i - \frac{2\alpha s W}{n}\right) = 2\max_{1\le s\le n}\left(\sum_{i=1}^s \rho_i - \frac{2\alpha s W}{n}\right) \ge 2\left[\rho_i - \frac{2\alpha W}{n}\right]. \end{aligned}$$

Using Theorem 1, we obtain

$$\begin{aligned} E^{D_\alpha}(G) &\ge 2\left(\frac{2W}{n} + \frac{\frac{1}{n}\sum_{i=1}^n Tr_i^2 - \frac{4W^2}{n^2}}{2\sqrt{\alpha^2\sum_{i=1}^n Tr_i^2 + (1-\alpha)^2(2W)^2}} - \frac{2\alpha W}{n}\right) \\ &= 2\left(\frac{(1-\alpha)2W}{n} + \frac{\frac{1}{n}\sum_{i=1}^n Tr_i^2 - \frac{4W^2}{n^2}}{2\sqrt{\alpha^2\sum_{i=1}^n Tr_i^2 + (1-\alpha)^2(2W)^2}}\right). \end{aligned}$$

Therefore,

$$E^{D_\alpha}(G) \geq 2\left(\frac{(1-\alpha)2W}{n} + \frac{\frac{1}{n}\sum_{i=1}^{n} Tr_i^2 - \frac{4W^2}{n}}{2\sqrt{\alpha^2\sum_{i=1}^{n} Tr_i^2 + (1-\alpha)^2(2W)^2}}\right)$$

In (3), equality holds, if G is a transmission regular graph, completing the proof. $\square$

3 Nordhaus-Gaddum type inequalities for spectral radius and generalized distance energy

Now, we obtain Nordhaus-Gaddum type inequalities for the spectral radius of the generalized distance matrix of graphs.

Theorem 4. *Let G be a simple connected graph with n vertices and diameter $d \geq 3$ and let $\overline{G}$ be the complement of G. Then*

$$\rho_\alpha + \overline{\rho}_\alpha \geq n-1+\frac{t^2}{n^2(n-1)\sqrt{\alpha^2 n(1+d^2)+(1-\alpha)^2(\frac{n^2}{2})}},$$

where $t = \sum_{i=1}^{n} |Tr_i - \frac{2W}{n}|$. Further, equality holds if G is a transmission regular graph.

Proof. Clearly,

$$\sum_{i=1}^{n} Tr_i^2 - \frac{4W^2}{n} = \sum_{i=1}^{n}\left(Tr_i - \frac{2W}{n}\right)^2 \geq \frac{(\sum_{i=1}^{n}|Tr_i - \frac{2W}{n}|)^2}{n} = \frac{t^2}{n},$$

where $t = \sum_{i=1}^{n} |Tr_i - \frac{2W}{n}|$. Using Theorem 1, we get

$$\rho_\alpha \geq \frac{2W}{n} + \frac{t^2}{2n^2\sqrt{\alpha^2\sum_{i=1}^{n} {Tr_i}^2 + (1-\alpha)^2(2W)^2}}.$$

Let $\overline{\rho}_\alpha = \rho_\alpha(\overline{G})$, $\overline{W}$ be the Wiener index of $\overline{G}$ and $\overline{Tr}_i$ be the transmission degree of vertex i in $\overline{G}$. Also, assume that $\overline{d}$ is the diameter of $\overline{G}$. In [12], for $d \geq 3$ it has been proved that $\overline{d} \leq 3$. So $\overline{d} \leq d$. It is easy to see that $t^2(G) = t^2(\overline{G})$. Therefore,

$$\overline{\rho}_\alpha \geq \frac{2\overline{W}}{n} + \frac{t^2}{2n^2\sqrt{\alpha^2\sum_{i=1}^{n} \overline{T}r_i^2 + (1-\alpha)^2(2\overline{W})^2}}.$$

Thus, we have

$$\begin{aligned}\rho_\alpha + \overline{\rho_\alpha} \geq &\left(\frac{2W+2\overline{W}}{n}\right)\\ &+\frac{t^2}{2n^2}\left(\frac{1}{\sqrt{\alpha^2\sum_{i=1}^{n} Tr_i^2 + (1-\alpha)^2(2W)^2}} + \frac{1}{\sqrt{\alpha^2\sum_{i=1}^{n} \overline{T}r_i^2 + (1-\alpha)^2(2\overline{W})^2}}\right).\end{aligned}$$

Now, we have $2W + 2\overline{W} = n(n-1)$. Using AM-GM inequality, we get

$$\frac{1}{\sqrt{\alpha^2 \sum_{i=1}^{n} Tr_i^2 + (1-\alpha)^2 (2W)^2}} + \frac{1}{\sqrt{\alpha^2 \sum_{i=1}^{n} \overline{T}r_i^2 + (1-\alpha)^2 (2\overline{W})^2}}$$
$$\geq 2\sqrt{\frac{1}{\sqrt{\alpha^2 \sum_{i=1}^{n} Tr_i^2 + (1-\alpha)^2 (2W)^2}\sqrt{\alpha^2 \sum_{i=1}^{n} \overline{T}r_i^2 + (1-\alpha)^2 (2\overline{W})^2}}}$$
$$\geq 2\sqrt{\frac{2}{(\alpha^2 \sum_{i=1}^{n} Tr_i^2 + (1-\alpha)^2 (2W)^2) + (\alpha^2 \sum_{i=1}^{n} \overline{T}r_i^2 + (1-\alpha)^2 (2\overline{W})^2)}}$$
$$= \frac{2\sqrt{2}}{\sqrt{\alpha^2 (\sum_{i=1}^{n} Tr_i^2 + \sum_{i=1}^{n} \overline{T}r_i^2) + (1-\alpha)^2 ((2W)^2 + (2\overline{W})^2)}}.$$

Let $N(v_i)$ be the set of vertices adjacent to vertex v_i in G and let $d_i = |N(v_i)|$ be the degree of vertex v_i. The transmission degree Tr_i of vertex v_i is the sum of distances from v_i to all other vertices of $V(G)$. So the transmission degree Tr_i of vertex v_i is the sum of its degree d_i (since the distances of the vertices adjacent to v_i is 1) and the distances of v_i from the rest of the vertices that will be greater than or equal to 2 and less than or equal to the diameter d of the graph G. The maximum distance in graph G is its diameter d. Therefore, $Tr_i \leq d_i + (n - d_i - 1)d$,

$$Tr_i^2 \leq (d_i + (n - d_i - 1)d)^2 \leq 2\left(d_i^2 + (n - d_i - 1)^2 d^2\right),$$

$$\sum_{i=1}^{n} Tr_i^2 \leq 2\sum_{i=1}^{n}\left(d_i^2 + (n - d_i - 1)^2 d^2\right) = 2\left(\sum_{i=1}^{n} d_i^2 + d^2 \sum_{i=1}^{n} (n - d_i - 1)^2\right).$$

Thus, we get

$$\sum_{i=1}^{n} Tr_i^2 \leq 2\left(\sum_{i=1}^{n} d_i^2 + d^2 \sum_{i=1}^{n} (n - d_i - 1)^2\right).$$

Similarly, we have

$$\sum_{i=1}^{n} \overline{T}r_i^2 \leq 2\left(\sum_{i=1}^{n} \overline{d_i^2} + \overline{d^2} \sum_{i=1}^{n} (n - \overline{d_i} - 1)^2\right).$$

Since $\overline{d} \leq d$ [12], therefore

$$\sum_{i=1}^{n} \overline{Tr_i^2} \leq 2\left(\sum_{i=1}^{n} \overline{d_i^2} + d^2 \sum_{i=1}^{n} (n - \overline{d_i} - 1)^2\right).$$

Also, $d_i + \overline{d_i} = n - 1$, so that $\overline{d_i} = n - 1 - d_i$. Thus,

$$\sum_{i=1}^{n} \overline{T}r_i^2 \leq 2\left(\sum_{i=1}^{n} (n - d_i - 1)^2 + d^2 \sum_{i=1}^{n} d_i^2\right).$$

Therefore, we have

$$\sum_{i=1}^{n} Tr_i^2 + \sum_{i=1}^{n} \overline{T}r_i^2 \leq 2\left((1+d^2)\sum_{i=1}^{n} d_i^2 + (1+d^2)\sum_{i=1}^{n} (n - d_i - 1)^2\right)$$
$$= 2\left((1+d^2)(\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} (n - d_i - 1)^2)\right)$$
$$= 2(1+d^2)\left(\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} \overline{d_i^2}\right) \leq 2(1+d^2)\left(n(n-1)^2\right).$$

Thus, we get

$$\sum_{i=1}^{n} Tr_i^2 + \sum_{i=1}^{n} \overline{T}r_i^2 \leq 2(1+d^2)\left(n(n-1)^2\right).$$

Also, $(2W)^2 + (\overline{2W})^2 \leq (n(n-1))^2$. Therefore,

$$\begin{aligned}
\frac{1}{\sqrt{\alpha^2 \sum_{i=1}^{n} Tr_i^2 + (1-\alpha)^2 (2W)^2}} &+ \frac{1}{\sqrt{\alpha^2 \sum_{i=1}^{n} \overline{T}r_i^2 + (1-\alpha)^2 (2\overline{W})^2}} \\
&\geq \frac{2\sqrt{2}}{\sqrt{\alpha^2 (\sum_{i=1}^{n} Tr_i^2 + \sum_{i=1}^{n} \bar{Tr}_i^2) + (1-\alpha)^2((2W)^2 + (2\overline{W})^2)}} \\
&\geq \frac{2\sqrt{2}}{\sqrt{2\alpha^2(1+d^2)n(n-1)^2 + (1-\alpha)^2(n(n-1))^2}} \\
&= \frac{2}{(n-1)\sqrt{\alpha^2(1+d^2)n + (1-\alpha)^2(\frac{n^2}{2})}}.
\end{aligned}$$

Hence,

$$\rho_\alpha + \overline{\rho}_\alpha \geq (n-1) + \frac{t^2}{n^2(n-1)\sqrt{\alpha^2(1+d^2)n + (1-\alpha)^2(\frac{n^2}{2})}}.$$

Clearly, equality holds when G is a transmission regular graph. This completes the proof. □

The following theorem gives the Nordhaus-Gaddum type inequalities for the energy of the generalized distance matrix of graphs.

Theorem 5. *Let G be a simple connected graph with n vertices, diameter $d \geq 3$ and $\overline{G}$ be its complement. Then*

$$E^{D_\alpha}(G) + E^{D_\alpha}(\overline{G}) \geq 2\left((1-\alpha)(n-1)\right) + \left(\frac{2t^2}{n^2(n-1)\sqrt{\alpha^2 n(1+d^2) + (1-\alpha)^2(\frac{n^2}{2})}}\right),$$

where $t = \sum_{i=1}^{n} |Tr_i - \frac{2W}{n}|$. Moreover, equality holds if G is a transmission regular graph.

Proof. Using Theorem 3, we have

$$E^{D_\alpha}(G) \geq 2\left(\frac{(1-\alpha)2W}{n} + \frac{\frac{1}{n}\sum_{i=1}^{n} Tr_i^2 - \frac{4W^2}{n^2}}{2\sqrt{\alpha^2 Tr_i^2 + (1-\alpha)^2(2W)^2}}\right).$$

We know that

$$\sum_{i=1}^{n} Tr_i^2 - \frac{4W^2}{n} = \sum_{i=1}^{n}\left(Tr_i - \frac{2W}{n}\right)^2 \geq \frac{(\sum_{i=1}^{n} |Tr_i - \frac{2W}{n}|)^2}{n} = \frac{t^2}{n},$$

where $t = \sum_{i=1}^{n} |Tr_i - \frac{2W}{n}|$. So

$$E^{D_\alpha}(G) \geq 2\left(\frac{(1-\alpha)2W}{n} + \frac{t^2}{2n^2\sqrt{\alpha^2 Tr_i^2 + (1-\alpha)^2(2W)^2}}\right).$$

Let $E^{D_\alpha}(\overline{G})$ be the generalized distance energy of the complement graph $\overline{G}$. Also, $t^2(G) = t^2(\overline{G})$. So

$$E^{D_\alpha}(\overline{G}) \geq 2\left(\frac{(1-\alpha)2\bar{W}}{n} + \frac{t^2}{2n^2\sqrt{\alpha^2\bar{Tr}_i^2 + (1-\alpha)^2(2\bar{W})^2}}\right).$$

Now, we have

$$E^{D_\alpha}(G) + E^{D_\alpha}(\overline{G}) \geq 2\left(\frac{(1-\alpha)(2W+2\overline{W})}{n}\right.$$
$$\left. + \frac{t^2}{n^2}\left(\frac{1}{\sqrt{\alpha^2 Tr_i^2 + (1-\alpha)^2(2W)^2}} + \frac{1}{\sqrt{\alpha^2\bar{Tr}_i^2 + (1-\alpha)^2(2\overline{W})^2}}\right)\right).$$

Using the same method as in Theorem 4, it follows that

$$E^{D_\alpha}(G) + E^{D_\alpha}(\overline{G}) \geq 2\left((1-\alpha)(n-1)\right) + \left(\frac{2t^2}{n^2(n-1)\sqrt{\alpha^2(1+d^2)n + (1-\alpha)^2(\frac{n^2}{2})}}\right).$$

Equality holds if G is a transmission regular graph, completing the proof. □

Several interesting and sharp bounds for the generalized distance energy of graphs can be found in [2].

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Мерахуддін М., Бхатнагар С., Пірзада С. *Про спектральний радіус і нерівності типу Нордхауза-Ґаддума матриці узагальнених відстаней графів* // Карпатські матем. публ. — 2022. — Т.14, №1. — С. 185–193.

Якщо $Tr(G)$ і $D(G)$ є відповідно діагональною матрицею порядків передачі вершин та матрицею відстаней зв'язного графа G, матриця узагальнених відстаней $D_\alpha(G)$ визначена наступним чином $D_\alpha(G) = \alpha\ Tr(G) + (1-\alpha)\ D(G)$, де $0 \leq \alpha \leq 1$. Якщо $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ є власними значеннями $D_\alpha(G)$, то найбільше власне значення ρ_1 (або $\rho_\alpha(G)$) називають спектральним радіусом матриці узагальнених відстаней $D_\alpha(G)$. Енергія узагальнених відстаней визначена як $E^{D_\alpha}(G) = \sum_{i=1}^{n} \left|\rho_i - \frac{2\alpha W(G)}{n}\right|$, де $W(G)$ є індексом Вінера графа G. У цій статті ми отримуємо межі для спектрального радіуса $\rho_\alpha(G)$ і енергії узагальнених відстаней графа G з індексом Вінера. Ми виводимо нерівності типу Нордхауза-Ґаддума для спектрального радіуса та енергії узагальнених відстаней графа G.

Ключові слова і фрази: матриця відстаней, матриця узагальнених відстаней, спектральний радіус, енергія узагальнених відстаней, нерівність типу Нордхауза-Ґаддума.