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Further investigations on a unique range set under weight 0 and 1

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In this paper, we have found the most generalized form of the famous Frank-Reinders polynomial. With the help of this, we have investigated the unique range set of a meromorphic function under two smallest possible weights namely 0 and 1. Our results extend some existing results in the literature.

Key words and phrases: meromorphic function, unique range set, weighted sharing.

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1 Definitions and terminologies

First we would like to recall some basic terminologies of value distribution theory over $\mathbb{C}$. The details are available in the book [7]. Let p be the multiplicity of z such that z is a root of $f(z) - a = 0$. Let $\mathcal{M}(\mathbb{C})$ denotes the collection of all meromorphic functions on $\mathbb{C}$. For $f \in \mathcal{M}(\mathbb{C})$ and $a \in \mathbb{C} \cup \{\infty\}$ we define

$$E_f(a) = \{(z, p) \in \mathbb{C} \times \mathbb{N} : z \text{ is root of } f(z) - a = 0\}.$$

In the case of ignoring multiplicities we denote the set by $\overline{E}_f(a)$. Let f, $g \in \mathcal{M}(\mathbb{C})$, we say f and g share the value a counting multiplicity (CM for short) if $E_f(a) = E_g(a)$ and share the value a ignoring multiplicity (IM for short) if $\overline{E}_f(a) = \overline{E}_g(a)$. Now for $f \in \mathcal{M}(\mathbb{C})$ and $S \subset \mathbb{C} \cup \{\infty\}$, we define

$$E_f(S) = \cup_{a \in S} \{(z, p) \in \mathbb{C} \times \mathbb{N} : z \text{ is root of } f(z) - a = 0\}.$$

If we do not take multiplicities into account then we denote the same set by $\overline{E}_f(S)$. Two functions f, $g \in \mathcal{M}(\mathbb{C})$ are said to share a set S CM (resp. IM), if $E_f(S) = E_g(S)$ (resp. $\overline{E}_f(S) = \overline{E}_g(S)$).

The notion of weighted sharing of sets was introduced in [8], defined as follows. The set of all a-points of f with multiplicity m is counted m times if $m \leq k$ and counted $k+1$ times if $m > k$, is denoted by $E_f(a, k)$. For two functions f, $g \in \mathcal{M}(\mathbb{C})$ if $E_f(a, k) = E_g(a, k)$, then we say f, g share the value a with weight k. We say f, g share the set S with weight k if $E_f(S, k) = E_g(S, k)$ for a set $S \subset \mathbb{C} \cup \{\infty\}$. We write f, g share (S, k) to mean that f, g share the set S with weight k. In particular, if $S = \{a\}$, then we write f, g share (a, k). In view of the above notion of weighted sharing the following definition is well known in the literature.

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Definition 1 ([2])**.** *Let f, g be two meromorphic functions over $\mathbb{C}$ and $S \subset \mathbb{C} \cup \{\infty\}$. If $E_f(S,k) = E_g(S,k)$ implies $f \equiv g$ then S is called a unique range set for meromorphic functions with weight k or in brief URSMk.*

The case $k = \infty$ and 0 corresponds to URSM and URSM-IM respectively.

Definition 2 ([10])**.** *Let $P(z)$ be a polynomial in $\mathbb{C}$. If for any two non-constant meromorphic functions f and g, the condition $P(f) \equiv P(g)$ implies $f \equiv g$, then P is called a uniqueness polynomial for meromorphic functions. We say $P(z)$ is UPM for short.*

H. Fujimoto [5] introduced the following definition, which he called as "property H". The same definition latter characterized as "critical injection property".

Definition 3 ([2])**.** *Let $P(z)$ be a polynomial such that $P'(z)$ have l distinct zeros $z_1, z_2, \ldots, z_l$. If $P(z_i) \neq P(z_j)$ for $i \neq j$, $i, j \in \{1, 2, \ldots, l\}$, then $P(z)$ is said to satisfy the critical injection property.*

Definition 4. *For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r,a;f \mid= 1)$ the counting function of simple a-points of f. For $k \in \mathbb{Z}_+$ we denote by $N(r,a;f \mid\leq k)$ (resp. $N(r,a;f \mid\geq k)$) the counting function of those a-points of f whose multiplicities are not greater (resp. less) than k, where each a-point is counted according to its multiplicity. The functions $\overline{N}(r,a;f \mid\leq k)$ and $\overline{N}(r,a;f \mid\geq k)$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.*

Definition 5. *Let $a \in \mathbb{C} \cup \{\infty\}$ and f and g be two non-constant meromorphic functions such that f and g share the value a IM. Let z_1 be an a-point of f with multiplicity s and an a-point of g with multiplicity t.*

By $N_E^{1)}(r,a;f)$ we mean the counting function of those a-points of f and g, where $s = t = 1$. Note that $N_E^{1)}(r,a;f) = \overline{N}_E^{1)}(r,a;f)$.

For $k \in \mathbb{Z}_+$, $\overline{N}_E^{(k}(r,a;f)$ denotes the reduced counting function of those a-points of f and g, where $s = t \geq k$.

By $\overline{N}_L(r,a;f)$ (resp. $\overline{N}_L(r,a;g)$) we mean the reduced counting function of those a-points of f and g, where $s > t$ (resp. $t > s$).

We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Note that $\overline{N}_*(r,a;f,g) = \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.

2 Background and motivations

To find the minimum cardinality of a URSM-IM, in 1999, S. Bartels [3] considered a set whose elements are the roots of famous [4] Frank-Reinders polynomial and obtained the following result.

Theorem A ([3])**.** *Let $n \geq 17$ be an integer and $c \neq 0, 1$ be a complex number. Then the polynomial $P_{FR}(z)$ defined by*

$$P_{FR}(z) = \frac{(n-1)(n-2)}{2} z^n - n(n-2) z^{n-1} + \frac{n(n-1)}{2} z^{n-2} - c \tag{1}$$

have only simple zeros and $S = \{z \in \mathbb{C} : P_{FR}(z) = 0\}$ is a URSM-IM.

Next an interesting question is whether the lower bound of the cardinality of a URSM-IM can further be reduced? In this respect, till date no such fruitful results were obtained and it seemed that the problem of reducing the cardinality of URSM-IM is hard nut to crack. Thus the only way to reduce the cardinality is to increase the weight gradually.

In 2016, the first author of this paper [1] proved that the set S defined in Theorem A is a URSM1 with cardinality ≥ 12 (see [1, Remark, p. 205], which follows from [1, Theorem 1.1] by putting $k = 2$). So from the above discussion it is evident that over $\mathbb{C}$, the least cardinalities of URSM-IM and URSM1 are 17 and 12 respectively and hence the only way to improve the existing results is to generalize the results so that all the results can be accommodated under a single umbrella.

3 Main results

We are going to define a new polynomial of degree $m + n + 1$ in the following manner, which accommodates all polynomial of Frank-Reinders types [4]. Set

$$\begin{aligned} P(z) = &\sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{m+n+1-j} z^{m+n+1-j} a^j \\ &+ \sum_{i=1}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} \frac{(-1)^{i+j}}{m+n+1-i-j} z^{m+n+1-i-j} a^j b^i - c = Q(z) - c, \end{aligned} \tag{2}$$

where a and b be distinct such that $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}, c \neq 0, Q(a), Q(b)$. It is easy to verify that

$$P'(z) = (z-a)^n (z-b)^m.$$

As the only two zeros of $P'(z)$ are a, b and $c \neq Q(a), Q(b)$, so $P(z)$ have only simple zeros.

Remark 1. *Notice that $P(z) - P(b) = (z-b)^{m+1} W_1(z)$, where $W_1(b) \neq 0$. Now let us assume $W_1(z)$ has a zero with multiplicity ≥ 2. As the only zeros of $P'(z)$ are a and b, so the only possible multiple zero of $W_1(z)$ is a and it should be of multiplicity $n+1$. Thus the degree of $P(z) - P(b)$ is at least $m+n+2$, which is a contradiction. Hence $W_1(z)$ has no multiple zero. Similarly, $P(z) - P(a) = (z-a)^{n+1} W_2(z)$, where $W_2(a) \neq 0$ and $W_2(z)$ has no multiple zero. If possible let $P(a) = P(b)$, then this implies $(z-a)^{n+1} W_2(z) = (z-b)^{m+1} W_1(z)$. As $a \neq b$, so $W_2(z)$ has a factor $(z-b)^{m+1}$, hence the degree of $P(z)$ is at least $m+n+2$, which is a contradiction. Thus $P(a) \neq P(b)$. Therefore $P(z)$ is a critically injective polynomial.*

Remark 2. *For $a = 1$, $b = 0$ and $n = 2$ the polynomial* (2) *reduces to*

$$\frac{z^{m+3}}{m+3} - 2\frac{z^{m+2}}{m+2} + \frac{z^{m+1}}{m+1} - c.$$

Multiplying this by $\frac{(m+1)(m+2)(m+3)}{2}$ and put $m + 3 = t$, actually we get $P_{FR}(z)$. Thus $P(z)$ is a generalization of $P_{FR}(z)$.

In our first theorem, considering the set of all zeros of $P(z)$, we investigate the sufficient conditions under which the set becomes a URSM-IM.

Theorem 1. *Let f, g be two non-constant meromorphic functions and m, n be two positive integers such that $n \geq 2$, $m \geq n+3$ and $m+n \geq 16$. Consider the polynomial* (2) *and $\widetilde{S} = \{z \in \mathbb{C} : P(z) = 0\}$. Then $\widetilde{S}$ is URSM-IM.*

Corollary 1. *(i) Let $m \geq 14$. Consider the polynomial*

$$P_1(z) = \frac{z^{m+3}}{m+3} - 2\frac{z^{m+2}}{m+2} + \frac{z^{m+1}}{m+1} - c,$$

where $c \neq 0, \frac{2}{(m+1)(m+2)(m+3)}$. Then the set $\widetilde{S}_1 = \{z \in \mathbb{C} : P_1(z) = 0\}$ is URSM-IM.

(ii) Let $t \geq 17$. Consider the polynomial

$$P_2(z) = \frac{(t-1)(t-2)}{2}z^t - t(t-2)z^{t-1} + \frac{t(t-1)}{2}z^{t-2} - d,$$

where $d \neq 0, 1$. Then the set $\widetilde{S}_2 = \{z \in \mathbb{C} : P_2(z) = 0\}$ is URSM-IM.

Note that, *(ii)* of Corollary 1 is actually Theorem A, so Theorem 1 generalizes Theorem A and the result of A. Banerjee (see [1, Remark, p. 205]) at a large extent.

Next, we increase to weight by 1 to investigate its effect on the cardinality of the range set in Theorem 1.

Theorem 2. *Let f, g be two non-constant meromorphic functions and m, n be two positive integers such that $n \geq 2$, $m \geq n+3$ and $m+n \geq 11$. Consider the polynomial* (2) *and $\widetilde{S} = \{z \in \mathbb{C} : P(z) = 0\}$. Then $\widetilde{S}$ is URSM1.*

From Theorem 2, it is seen that the cardinality of URSM1 is ≥ 12.

Corollary 2. *(i) Let $m \geq 9$. Consider the polynomial $P_1(z)$ as in Corollary 1. Then the set $\widetilde{S}_1 = \{z \in \mathbb{C} : P_1(z) = 0\}$ is URSM1.*

(ii) Let $t \geq 12$. Consider the polynomial $P_2(z)$ as in Corollary 1. Then the set $\widetilde{S}_2 = \{z \in \mathbb{C} : P_2(z) = 0\}$ is URSM1.

Thus Theorem 2 is an extension of the result of A. Banerjee (see [1, Remark, p. 205]).

4 Lemmas

Lemma 1 ([5])**.** *Let $P(z)$ be a polynomial of degree ≥ 5 without multiple zeros, whose first derivative have mutually k-distinct zeros, given by $d_1, d_2, \ldots, d_k$ with multiplicities $q_1, q_2, \ldots, q_k$, respectively. Assume that $P(z)$ satisfies the critical injection property and there are two distinct non-constant meromorphic functions f and g such that*

$$\frac{1}{P(f)} = \frac{c_0}{P(g)} + c_1,$$

for some constant $c_0 \neq 0$ and c_1. If $k \geq 3$, or if $k = 2$ and $\min\{q_1, q_2\} \geq 2$, then $c_1 = 0$.

Lemma 2 ([6])**.** *Let $P(z)$ be a monic polynomial without multiple zero whose first derivative have mutually k-distinct zeros, given by $d_1, d_2, \ldots, d_k$ with multiplicities $q_1, q_2, \ldots, q_k$, respectively. Suppose that $P(z)$ satisfy the critical injection property. Then $P(z)$ will be a UPM if and only if*

$$\sum_{1 \leq l < m \leq k} q_l q_m > \sum_{l=1}^{k} q_l.$$

In particular, the above inequality is always satisfied whenever $k \geq 4$. When $k = 3$ and $\max\{q_1, q_2, q_3\} \geq 2$ or when $k = 2$, $\min\{q_1, q_2\} \geq 2$ and $q_1 + q_2 \geq 5$ then also the above inequality holds.

Lemma 3 ([9]). *Let $N(r,0;f^{(k)} \mid f \neq 0)$ denotes the counting function of zeros of $f^{(k)}$, which are not the zeros of f, then*

$$N(r,0;f^{(k)} \mid f \neq 0) \leq k\overline{N}(r,\infty;f) + N(r,0;f \mid< k) + k\overline{N}(r,0;f \geq k) + S(r,f).$$

In particular, for $k=1$ we have $N(r,0;f' \mid f \neq 0) \leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + S(r,f)$.

Now let us consider two non-constant meromorphic functions $\mathcal{F}$ and $\mathcal{G}$ such that

$$\mathcal{F} := \frac{Q(f)}{c} = \frac{P(f)+c}{c} \quad \text{and} \quad \mathcal{G} := \frac{Q(g)}{c} = \frac{P(g)+c}{c}, \tag{3}$$

where $P(z)$ is defined as in (2). Besides this we also consider a function $\mathcal{H}$ as follows

$$\mathcal{H} = \left(\frac{\mathcal{F}''}{\mathcal{F}'} - \frac{2\mathcal{F}'}{\mathcal{F}-1}\right) - \left(\frac{\mathcal{G}''}{\mathcal{G}'} - \frac{2\mathcal{G}'}{\mathcal{G}-1}\right). \tag{4}$$

Lemma 4. *Let $\mathcal{H} \not\equiv 0$ and $\mathcal{F}$, $\mathcal{G}$ share $(1,0)$, then*

$$N_E^{1)}(r,1;\mathcal{F}) = N_E^{1)}(r,1;\mathcal{G}) \leq N(r,\infty;\mathcal{H}) + S(r),$$

where $S(r) = S(r,f) + S(r,g)$.

Proof. As $\mathcal{F}$ and $\mathcal{G}$ share $(1,0)$, so each simple 1-point of $\mathcal{F}$ is also simple 1-point of $\mathcal{G}$ and vice versa. Now each simple 1-point of $\mathcal{F}$ (i.e. simple 1-point of $\mathcal{G}$) is a zero of $\mathcal{H}$. Note that $m(r,\mathcal{H}) = S(r)$. Hence

$$N_E^{1)}(r,1;\mathcal{F}) = N_E^{1)}(r,1;\mathcal{G}) \leq N(r,0;\mathcal{H}) \leq T(r,\mathcal{H}) \leq N(r,\infty;\mathcal{H}) + S(r).$$

□

Lemma 5. *Let $\widetilde{S} = \{z \in \mathbb{C} : P(z) = 0\}$, where $P(z)$ is defined as in (2). Let $\mathcal{H} \not\equiv 0$ and f, g be any two non-constant meromorphic functions on $\mathbb{C}$ such that $E_f(\widetilde{S},0) = E_g(\widetilde{S},0)$, then*

$$\begin{aligned}\overline{N}(r,\infty;\mathcal{H}) \leq \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}(r,a;f) + \overline{N}(r,a;g) + \overline{N}(r,b;f) \\ + \overline{N}(r,b;g) + \overline{N}_*(r,1;\mathcal{F},\mathcal{G}) + \overline{N}_0\left(r,0;f'\right) + \overline{N}_0(r,0;g'),\end{aligned}$$

where $\overline{N}_0\left(r,0;f'\right)$ denotes reduced counting function of those zeros of f', which are not zeros of $(\mathcal{F}-1)(f-a)(f-b)$, and $\overline{N}_0\left(r,0;g'\right)$ denotes similar counting function.

Proof. Note that $\mathcal{F}' = \frac{P'(f)}{c} = \frac{1}{c}(f-a)^n(f-b)^m f'$. The lemma directly follows by calculating all the possible poles of $\mathcal{H}$ and observe that all poles of $\mathcal{H}$ are simple. □

5 Proofs of the theorems

Proof of Theorem 1. Let $\widetilde{S} = \{z \in \mathbb{C} : P(z) = 0\}$. Consider functions $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ as defined in (3) and (4).

Case 1. First assume $\mathcal{H} \equiv 0$. Integrating (4) two times we obtain

$$\begin{aligned}&\frac{1}{\mathcal{F}-1} \equiv \frac{A}{\mathcal{G}-1} + B \quad \text{(where } A, B \text{ are constants such that } A \neq 0) \\ &\implies \frac{c}{P(f)} \equiv \frac{cA}{P(g)} + B \implies \frac{1}{P(f)} \equiv \frac{A}{P(g)} + \frac{B}{c} \quad (\text{as } c \neq 0).\end{aligned} \tag{5}$$

As $n \geq 2$ and $m \geq n+3$, so obviously $\min\{m,n\} \geq 2$ and degree of the polynomial is ≥ 17. Now applying *Lemma 1* for the equation (5) and we get $\frac{B}{c} = 0$. Consider a constant $A_1 = \frac{1}{A}$.

Sub-case 1.1. Let us assume $A_1 \neq 1$. Now from (5) we obtain

$$\begin{aligned} &P(f) \equiv A_1 P(g) \\ &\implies P(f) + c \equiv A_1(P(g) + c) - c(A_1 - 1) \\ &\implies Q(f) \equiv A_1 Q(g) + c(1 - A_1) \\ &\implies Q(f) - Q(b) \equiv A_1 Q(g) - (Q(b) - c(1 - A_1)). \end{aligned} \tag{6}$$

Note that $T(r,f) = T(r,g) + O(1)$ since $P(f) \equiv A_1 P(g)$. Recall that the only zeros of $Q'(z)$ are a and b. So only possible multiple zeros of $\psi(z) := A_1 Q(z) - (Q(b) - c(1 - A_1))$ are a and b. First assume b is the multiple zero of $\psi(z)$. Thus $\psi(b) = 0$, i.e.

$$A_1 Q(b) = Q(b) - c(1 - A_1) \implies (1 - A_1)(Q(b) - c) = 0 \implies c = Q(b),$$

a contradiction as we have $c \neq Q(b)$. Next assume a is the multiple zero of $\psi(z)$. It is easy to see that $\psi(z) = (z-a)^{n+1} W_1(z)$, where $W_1(a) \neq 0$ and all zeros of $W_1(z)$ are simple, namely α_j, $j = 1, 2, \ldots, m$. Notice that $Q(z) - Q(b) = (z-b)^{m+1} W_2(z)$, where $W_2(b) \neq 0$ and all zeros of $W_2(z)$ are simple, namely β_j, $j = 1, 2, \ldots, n$. Hence from (6)

$$\overline{N}(r,b;f) + \sum_{j=1}^{n} \overline{N}(r,\beta_j;f) = \overline{N}(r,a;g) + \sum_{j=1}^{m} \overline{N}(r,\alpha_j;g).$$

Next using the Second Fundamental Theorem, the above equation and the fact $T(r,f) = T(r,g) + O(1)$ we get

$$\begin{aligned} (m-1)T(r,g) &\leq \overline{N}(r,a;g) + \sum_{j=1}^{m} \overline{N}(r,\alpha_j;g) + S(r,g) \\ &= \overline{N}(r,b;f) + \sum_{j=1}^{n} \overline{N}(r,\beta_j;f) + S(r,g) \leq (n+1)T(r,g) + S(r,f). \end{aligned}$$

Thus we have $(m-n-2)T(r,g) \leq S(r,g)$, this contradicts the given condition $m \geq n+3$. Hence we see neither a nor b are the multiple zeros of $\psi(z)$, and hence all the zeros of $\psi(z)$ are simple, say γ_j, $j = 1, 2, \ldots, m+n+1$. From (6) we have

$$\overline{N}(r,b;f) + \sum_{j=1}^{n} \overline{N}(r,\beta_j;f) = \sum_{j=1}^{m+n+1} \overline{N}(r,\gamma_j;g).$$

Using the Second Fundamental Theorem and the above equation, we deduce

$$\begin{aligned} (m+n-1)T(r,g) &\leq \sum_{j=1}^{m+n+1} \overline{N}(r,\gamma_j;g) + S(r,g) \\ &= \overline{N}(r,b;f) + \sum_{j=1}^{n} \overline{N}(r,\beta_j;f) + S(r,g) \leq (n+1)T(r,g) + S(r,f). \end{aligned}$$

Hence we get $(m-2)T(r,g) \leq S(r,g)$, a contradiction as $n \geq 2$ and $m \geq n+3$.

Sub-case 1.2. Next assume $A_1 = 1$. Thus $P(f) \equiv P(g)$. Now by Lemma 2 we conclude $P(z)$ is a UPM. Therefore $f \equiv g$.

Case 2. Next assume $\mathcal{H} \not\equiv 0$. As $E_f(\widetilde{S}, 0) = E_g(\widetilde{S}, 0)$, thus $\mathcal{F}$ and $\mathcal{G}$ share $(1, 0)$. By the Second Fundamental Theorem and Lemmas 4, 5, we get

$$\begin{aligned}(m+n+2)T(r,f) &\leq \overline{N}(r,\infty;f) + \overline{N}(r,1;\mathcal{F}) + \overline{N}(r,a;f) + \overline{N}(r,b;f) - N_0\left(r,0;f'\right) + S(r,f)\\ &\leq \overline{N}(r,\infty;f) + \overline{N}(r,\infty;\mathcal{H}) + \overline{N}_L(r,1;\mathcal{F}) + \overline{N}_L(r,1;\mathcal{G}) + \overline{N}_E^{(2}(r,1;\mathcal{F})\\ &\quad + \overline{N}(r,a;f) + \overline{N}(r,b;f) - N_0\left(r,0;f'\right) + S(r,f)\\ &\leq 2\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + 2\overline{N}(r,a;f) + 2\overline{N}(r,b;f)\\ &\quad + \overline{N}(r,a;g) + \overline{N}(r,b;g) + \overline{N}_0\left(r,0;g'\right) + \overline{N}_E^{(2}(r,1;\mathcal{F})\\ &\quad + \overline{N}_*(r,1;\mathcal{F},\mathcal{G}) + \overline{N}_L(r,1;\mathcal{F}) + \overline{N}_L(r,1;\mathcal{G}) + S(r,f)\\ &\leq 2\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + 2\overline{N}(r,a;f) + 2\overline{N}(r,b;f)\\ &\quad + \overline{N}(r,a;g) + \overline{N}(r,b;g) + \overline{N}_0\left(r,0;g'\right) + \overline{N}_E^{(2}(r,1;\mathcal{F})\\ &\quad + 2\overline{N}_L(r,1;\mathcal{F}) + 2\overline{N}_L(r,1;\mathcal{G}) + S(r,f).\end{aligned} \tag{7}$$

Now we deduce

$$\begin{aligned}&\overline{N}_0\left(r,0;g'\right) + \overline{N}_E^{(2}(r,1;\mathcal{F}) + 2\overline{N}_L(r,1;\mathcal{F}) + 2\overline{N}_L(r,1;\mathcal{G})\\ &\quad\leq \overline{N}_0\left(r,0;g'\right) + \overline{N}_E^{(2}(r,1;\mathcal{G}) + 2\overline{N}_L(r,1;\mathcal{F}) + 2\overline{N}_L(r,1;\mathcal{G})\\ &\quad\leq \overline{N}_0\left(r,0;g'\right) + \overline{N}(r,1;\mathcal{G}\mid\geq 2) + \overline{N}_L(r,1;\mathcal{G}) + 2\overline{N}_L(r,1;\mathcal{F})\\ &\quad\leq N(r,0;g'\mid g\neq 0) + \overline{N}_L(r,1;\mathcal{G}) + 2\overline{N}_L(r,1;\mathcal{F}).\end{aligned} \tag{8}$$

In view of Lemma 3 we get

$$\overline{N}_L(r,1;\mathcal{G}) \leq \overline{N}(r,1;\mathcal{G}\mid\geq 2) \leq N(r,0;g'\mid g\neq 0) \leq \overline{N}(r,\infty;g) + \overline{N}(r,0;g) + S(r,g),$$

and similarly $\overline{N}_L(r,1;\mathcal{F}) \leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + S(r,f)$ holds. Thus from (8) we have

$$\begin{aligned}&\overline{N}_0\left(r,0;g'\right) + \overline{N}_E^{(2}(r,1;\mathcal{F}) + 2\overline{N}_L(r,1;\mathcal{F}) + 2\overline{N}_L(r,1;\mathcal{G})\\ &\quad\leq 2\{\overline{N}(r,\infty;g) + \overline{N}(r,0;g) + \overline{N}(r,\infty;f) + \overline{N}(r,0;f)\} + S(r),\end{aligned} \tag{9}$$

where $S(r) := S(r,f) + S(r,g)$. Combining (7) and (9), we obtain

$$\begin{aligned}(m+n+2)T(r,f) &\leq 4\overline{N}(r,\infty;f) + 3\overline{N}(r,\infty;g) + 2\overline{N}(r,a;f) + 2\overline{N}(r,b;f)\\ &\quad + \overline{N}(r,a;g) + \overline{N}(r,b;g) + 2\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + S(r)\\ &\leq 10T(r,f) + 7T(r,g) + S(r).\end{aligned} \tag{10}$$

Similarly we can get

$$(m+n+2)T(r,g) \leq 10T(r,g) + 7T(r,f) + S(r). \tag{11}$$

Thus adding (10) and (11), we get

$$(m+n-15)(T(r,f) + T(r,g)) \leq S(r),$$

which is a contradiction as $m+n \geq 16$.

Therefore combining Case 1 and 2 we get, for $n \geq 2$, $m \geq n+3$ and $m+n \geq 16$ the set $\widetilde{S}$ is URSM-IM. □

Proof of Corollary 1. *(i)* Putting $a = 1$, $b = 0$ and $n = 2$, the polynomial (2) reduces to

$$P_1(z) = \frac{z^{m+3}}{m+3} - 2\frac{z^{m+2}}{m+2} + \frac{z^{m+1}}{m+1} - c,$$

where $c \neq 0, \frac{2}{(m+1)(m+2)(m+3)}$. By Theorem 1, we get that the set $\widetilde{S}_1$ with $m \geq 14$ is URSM-IM.

(ii) From *(i)*, assuming $m + 3 = t$ we get

$$\begin{aligned} P_1(z) &= \frac{2}{t(t-1)(t-2)} \left[\frac{(t-1)(t-2)}{2} z^t - t(t-2)z^{t-1} + \frac{t(t-1)}{2} z^{t-2} - d \right] \\ &= \frac{2}{t(t-1)(t-2)} P_2(z), \end{aligned}$$

where $d = c\frac{t(t-1)(t-2)}{2}$. Note that as $m \geq 14$, so $t \geq 17$. From *(i)*, we have $c \neq 0, \frac{2}{t(t-1)(t-2)}$, so $d \neq 0, 1$. Therefore for $t \geq 17$ and $d \neq 0, 1$, the set $\widetilde{S}_2$ is URSM-IM. $\square$

Proof of Theorem 2. Let $\widetilde{S} = \{z \in \mathbb{C} : P(z) = 0\}$. Consider two function $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ as in Theorem 1.

Case 1. First assume $\mathcal{H} \equiv 0$. Proceeding same steps as *Case 1* of Theorem 1 we get $f \equiv g$.

Case 2. Next assume $\mathcal{H} \not\equiv 0$. As $E_f(\widetilde{S}, 1) = E_g(\widetilde{S}, 1)$, thus $\mathcal{F}$ and $\mathcal{G}$ share $(1, 1)$, this implies $N_E^{1)}(r, 1; \mathcal{F}) = N(r, 1; \mathcal{F} \mid= 1)$. By Second Fundamental Theorem and Lemmas 4, 5, we get

$$\begin{aligned} (m+n+2)T(r,f) &\leq \overline{N}(r,\infty;f) + \overline{N}(r,1;\mathcal{F}) + \overline{N}(r,a;f) + \overline{N}(r,b;f) - N_0\left(r,0;f'\right) + S(r,f) \\ &\leq \overline{N}(r,\infty;f) + \overline{N}(r,1;\mathcal{F} \mid= 1) + \overline{N}(r,1;\mathcal{F} \mid\geq 2) \\ &\quad + \overline{N}(r,a;f) + \overline{N}(r,b;f) - N_0\left(r,0;f'\right) + S(r,f) \\ &\leq \overline{N}(r,\infty;f) + \overline{N}(r,\infty;\mathcal{H}) + \overline{N}(r,1;\mathcal{G} \mid\geq 2) \\ &\quad + \overline{N}(r,a;f) + \overline{N}(r,b;f) - N_0\left(r,0;f'\right) + S(r,f) \\ &\leq 2\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + 2\overline{N}(r,a;f) + 2\overline{N}(r,b;f) + \overline{N}(r,a;g) \\ &\quad + \overline{N}(r,b;g) + \overline{N}_0\left(r,0;g'\right) + \overline{N}(r,1;\mathcal{G} \mid\geq 2) + \overline{N}_*(r,1;\mathcal{F},\mathcal{G}) + S(r,f). \end{aligned} \tag{12}$$

Now using Lemma 3, we deduce

$$\begin{aligned} &\overline{N}_0\left(r,0;g'\right) + \overline{N}(r,1;\mathcal{G} \mid\geq 2) + \overline{N}_*(r,1;\mathcal{F},\mathcal{G}) \\ &\quad \leq \overline{N}_0\left(r,0;g'\right) + \overline{N}(r,1;\mathcal{G} \mid\geq 2) + \overline{N}(r,1;\mathcal{G} \mid\geq 3) + \overline{N}(r,1;\mathcal{F} \mid\geq 3) \\ &\quad \leq N(r,0;g' \mid g \neq 0) + \frac{1}{2}N(r,0;f' \mid f \neq 0) \\ &\quad \leq \overline{N}(r,\infty;g) + \overline{N}(r,0;g) + \frac{1}{2}\{\overline{N}(r,\infty;f) + \overline{N}(r,0;f)\} + S(r). \end{aligned} \tag{13}$$

Combining (12) and (13), we get

$$\begin{aligned} (m+n+2)T(r,f) &\leq \frac{5}{2}\overline{N}(r,\infty;f) + 2\overline{N}(r,\infty;g) + 2\overline{N}(r,a;f) + 2\overline{N}(r,b;f) \\ &\quad + \overline{N}(r,a;g) + \overline{N}(r,b;g) + \frac{1}{2}\overline{N}(r,0;f) + \overline{N}(r,0;g) + S(r) \\ &\leq 7T(r,f) + 5T(r,g) + S(r). \end{aligned} \tag{14}$$

Similarly we can obtain

$$(m+n+2)T(r,g) \leq 7T(r,g) + 5T(r,f) + S(r). \tag{15}$$

Adding (14) and (15), we obtain $(m+n-10)(T(r,f)+T(r,g)) \leq S(r)$, this is a contradiction as $m+n \geq 11$.

Therefore combining Cases 1 and 2 we get, for $n \geq 2$, $m \geq n+3$ and $m+n \geq 11$ the set $\widetilde{S}$ is URSM1. $\square$

Proof of Corollary 2. We left the proof since it is same as the proof of Corollary 1. $\square$

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Банерджі А., Майті С. *Подальші дослідження множини єдиності образу з вагою 0 і 1* // Карпатські матем. публ. — 2022. — Т.14, №2. — С. 504–512.

У цій статті ми знайшли найбільш загальну форму відомого полінома Франка-Рейндерса. За допомогою цього ми дослідили множину єдиності образу мероморфної функції при двох найменших можливих вагах, а саме 0 і 1. Наші результати узагальнюють деякі відомі результати в літературі.

Ключові слова і фрази: мероморфна функція, множина єдиності образу, зважений обмін.