ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2025, **17** (1), 171–186
doi:10.15330/cmp.17.1.171-186

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2025, Т.17, №1, С.171–186

$p\iota$-Continuity and weak $p\iota$-continuity

Boonpok C.

In this paper, we introduce the concepts of $p\iota$-continuous multifunctions and weakly $p\iota$-continuous multifunctions by utilizing the notion of pre-$\mathscr{I}$-open sets in ideal topological spaces. We also investigate some characterizations of $p\iota$-continuous multifunctions and weakly $p\iota$-continuous multifunctions. Furthermore, the relationships between $p\iota$-continuous multifunctions and weakly $p\iota$-continuous multifunctions are discussed.

Key words and phrases: pre-$\mathscr{I}$-open set, $p\iota$-continuous multifunction, weakly $p\iota$-continuous multifunction.

Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand
E-mail: chawalit.b@msu.ac.th

Introduction

The field of the mathematical science which goes under the name of topology is concerned with all questions directly or indirectly related to continuity. Preopen sets [23], semi-open sets [22], α-open sets [24] and β-open sets [2] play an important role in the researching of generalizations of continuity in topological spaces. Using different forms of open sets, many authors have introduced and studied various types of weak forms of continuity for functions and multifunctions. A.S. Mashhour et. al. [23] introduced and studied the notions of preopen sets and precontinuity in topological spaces. Precontinuity was also called almost-continuity in the sense of T. Husain [19]. M. Przemski [28], V. Popa and T. Noiri [27] have independently defined the notion of precontinuity in the setting of multifunctions. Moreover, V. Popa and T. Noiri [26] showed that these notions are equivalent of each other and obtained several characterizations of precontinuous multifunctions. V. Popa and T. Noiri [25] investigated some characterizations of weakly precontinuous multifunctions. The concept of ideals in topological spaces has been introduced and studied by K. Kuratowski [21] and R. Vaidyanathaswamy [29] which is one of the important areas of research in the branch of mathematics. Every topological space is an ideal topological space and all the results of ideal topological spaces are generalizations of the results established in topological spaces. D. Janković and T.R. Hamlett [20] introduced the notion of $\mathscr{I}$-open sets in ideal topological spaces. M.E. Abd El-Monsef et. al. [1] further investigated $\mathscr{I}$-open sets and $\mathscr{I}$-continuous functions. A. Açikgöz et. al. [3] introduced and investigated the notions of weakly-$\mathscr{I}$-continuous and weak*-$\mathscr{I}$-continuous functions in ideal topological spaces. J. Dontchev [14] introduced the notion of pre-$\mathscr{I}$-open sets and obtained a decomposition of $\mathscr{I}$-continuity.

УДК 517.5
2020 *Mathematics Subject Classification:* 54C08, 54C60.

This research project was financially supported by Mahasarakham University.

In 2002, E. Hatir and T. Noiri [18] introduced the notions of semi-$\mathscr{I}$-open sets, α-$\mathscr{I}$-open sets and β-$\mathscr{I}$-open sets via idealization and using these sets obtained new decompositions of continuity. In 2005, E. Hatir and T. Noiri [17] investigated further properties of semi-$\mathscr{I}$-open sets and semi-$\mathscr{I}$-continuity. Moreover, E. Hatir et. al. [16] introduced and investigated the notions of strong β-$\mathscr{I}$-open sets and strongly β-$\mathscr{I}$-continuous functions. In 2014, W. Al-Omeri et. al. [9] presented and studied new classes of functions called contra e-$\mathscr{I}$-continuous functions, almost-e-$\mathscr{I}$-continuous functions, almost contra-e-$\mathscr{I}$-continuous functions and almost weakly-e-$\mathscr{I}$-continuous functions. In 2016, W. Al-Omeri et. al. [7] introduced and investigated new classes of continuous multifunctions called upper/lower e-$\mathscr{I}$-continuous multifunctions and upper/lower $\delta\beta_I$-$\mathscr{I}$-continuous multifunctions by using the concepts of e-$\mathscr{I}$-open sets and $\delta\beta_I$-open sets. In 2018, W. Al-Omeri et. al. [6] presented the degree of semi-preopenness, semi-precontinuity and semi-preirresoluteness for functions in (L, M)-fuzzy pretopological spaces by using the implication operations and Ghareeb's operators. In 2020, W. Al-Omeri [5] introduced and investigated mixed b-fuzzy topological spaces and a fuzzy completely weak b-irresolute function over an initial universe with a fixed set of parameters. In 2021, W. Al-Omeri and T. Noiri [4] introduced a new class of functions, namely, almost e-$\mathscr{I}$-continuous functions containing the class of almost e-continuous functions by utilizing the notion of e-$\mathscr{I}$-open sets due to W. Al-Omeri et. al. [8].

The purpose of the present paper is to introduce the notions of $p\imath$-continuous multifunctions and weakly $p\imath$-continuous multifunctions. Furthermore, several interesting characterizations of $p\imath$-continuous multifunctions and weakly $p\imath$-continuous multifunctions are investigated.

1 Preliminaries

Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a topological space (X, τ), the closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties:

(1) $A \in \mathscr{I}$ and $B \subset A$ imply $B \in \mathscr{I}$;

(2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$.

A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows: $A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I}$ for every open neighbourhood U of $x\}$. In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [21], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ. Observe additionally that $\mathrm{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ, generated by the base $\mathscr{B}(\mathscr{I}, \tau) = \{U - I' \mid U \in \tau$ and $I' \in \mathscr{I}\}$. However, $\mathscr{B}(\mathscr{I}, \tau)$ is not always a topology [29]. A subset A is said to be $\star$-closed [20] if $A^\star \subset A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\mathrm{Int}^\star(A)$.

A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *pre-$\mathscr{I}$-open* [14] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(A))$. The complement of a pre-$\mathscr{I}$-open set is called *pre-$\mathscr{I}$-closed* [14]. For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all pre-$\mathscr{I}$-closed sets of X

containing A is called [15, 31] the pre-$\mathscr{I}$-closure of A and is denoted by $p\iota\mathrm{Cl}(A)$ ($p\mathrm{Cl}_{\mathscr{I}}(A)$). The union of all pre-$\mathscr{I}$-open sets of X contained in A is called [31] the pre-$\mathscr{I}$-interior of A and is denoted by $p\iota\mathrm{Int}(A)$.

Lemma 1 **([31]).** *Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$ and $x \in X$. Then the following properties hold:*

(1) $x \in p\iota Cl(A)$ if and only if $U \cap A \neq \varnothing$ for every pre-$\mathscr{I}$-open set U of X containing x;

(2) A is pre-$\mathscr{I}$-closed if and only if $A = p\iota Cl(A)$;

(3) $p\iota Int(X - A) = X - p\iota Cl(A)$;

(4) $p\iota Cl(X - A) = X - p\iota Int(X - A)$.

Lemma 2. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) if $V \in \tau$, then $V \cap Cl^\star(A) \subset Cl^\star(V \cap A)$ (see [16]);

(2) if F is closed in X, then $Int^\star(A \cup F) \subset Int^\star(A) \cup F$ (see [32]).

Lemma 3. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) $p\iota Cl(A) = A \cup Cl(Int^\star(A))$ (see [30]);

(2) $p\iota Int(A) = A \cap Int(Cl^\star(A))$.

The following notions are taken from [8, 33]. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *R-$\mathscr{I}$-open* (respectively *R-$\mathscr{I}$-closed*) if $A = \mathrm{Int}(\mathrm{Cl}^\star(A))$ (respectively $A = \mathrm{Cl}(\mathrm{Int}^\star(A))$). A point $x \in X$ is called a *δ-$\mathscr{I}$-cluster point* of A if $\mathrm{Int}(\mathrm{Cl}^\star(V)) \cap A \neq \varnothing$ for each open set V of X containing x. The set of all δ-$\mathscr{I}$-cluster points of A is called the *δ-$\mathscr{I}$-closure* of A and is denoted by $\delta\mathrm{Cl}_{\mathscr{I}}(A)$. The union of all R-$\mathscr{I}$-open sets of X contained in A is called the *δ-$\mathscr{I}$-interior* of A and is denoted by $\delta\mathrm{Int}_{\mathscr{I}}(A)$. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *δ-$\mathscr{I}$-closed* (respectively δ-$\mathscr{I}$-open) if $\delta\mathrm{Cl}_{\mathscr{I}}(A) = A$ (respectively $\delta\mathrm{Int}_{\mathscr{I}}(A) = A$). A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *e-$\mathscr{I}$-open* if $A \subset \mathrm{Cl}(\delta\mathrm{Int}_{\mathscr{I}}(A)) \cup \mathrm{Int}(\delta\mathrm{Cl}_{\mathscr{I}}(A))$. The complement of an e-$\mathscr{I}$-open set is called *e-$\mathscr{I}$-closed*.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \varnothing$ for all $x \in X$. For a multifunction $F : X \to Y$, following [10] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \varnothing\}$. In particular, $F^-(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$. For each $A \subset X$, $F(A) = \cup_{x \in A} F(x)$. Then F is said to be surjection if $F(X) = Y$, or equivalent, if for each $y \in Y$ there exists $x \in X$ such that $y \in F(x)$ and F is called injection if $x \neq y$ implies $F(x) \cap F(y) = \varnothing$. Let $\mathscr{P}(X)$ be the collection of all nonempty subsets of X. For any $\star$-open set V of an ideal topological space $(X, \tau, \mathscr{I})$, we denote $V^+ = \{G \in \mathscr{P}(X) : G \subset V\}$ and $V^- = \{G \in \mathscr{P}(X) : G \cap V \neq \varnothing\}$.

2 On $p\iota$-continuous multifunctions

In this section, we introduce the notion of $p\iota$-continuous multifunctions and investigate several characterizations of such multifunctions.

Definition 1. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be $p\iota$-continuous if for each $x \in X$ and each $\star$-open sets V_1, V_2 of Y such that $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \varnothing$, there exists a pre-$\mathscr{I}$-open set U of X containing x such that $F(U) \subset V_1$ and $F(z) \cap V_2 \neq \varnothing$ for every $z \in U$.*

Theorem 1. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) F is $p\iota$-continuous;

(2) $x \in p\iota\mathrm{Int}(F^+(V_1) \cap F^-(V_2))$ for every $\star$-open sets V_1, V_2 of Y such that $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \varnothing$;

(3) $F^+(V_1) \cap F^-(V_2)$ is pre-$\mathscr{I}$-open in X for every $\star$-open sets V_1, V_2 of Y;

(4) $F^-(K_1) \cup F^+(K_2)$ is pre-$\mathscr{I}$-closed in X for every $\star$-closed sets K_1, K_2 of Y;

(5) $\mathrm{Cl}\big(\mathrm{Int}^\star(F^-(B_1) \cup F^+(B_2))\big) \subset F^-(\mathrm{Cl}^\star(B_1)) \cup F^+(\mathrm{Cl}^\star(B_2))$ for every subsets B_1, B_2 of Y;

(6) $p\iota\mathrm{Cl}(F^-(B_1) \cup F^+(B_2)) \subset F^-(\mathrm{Cl}^\star(B_1)) \cup F^+(\mathrm{Cl}^\star(B_2))$ for every subsets B_1, B_2 of Y;

(7) $F^-(\mathrm{Int}^\star(B_1)) \cap F^+(\mathrm{Int}^\star(B_2)) \subset p\iota\mathrm{Int}(F^-(B_1) \cap F^+(B_2))$ for every subsets B_1, B_2 of Y.

Proof. *(1) $\Rightarrow$ (2)* Let V_1, V_2 be any $\star$-open sets of Y such that $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \varnothing$. Then there exists a pre-$\mathscr{I}$-open set U of X containing x such that $F(z) \subset V_1$ and $F(z) \cap V_2 \neq \varnothing$ for every $z \in U$. Thus, $U \subset F^+(V_1) \cap F^-(V_2)$ and hence $x \in p\iota\mathrm{Int}(F^+(V_1) \cap F^-(V_2))$.

(2) $\Rightarrow$ (3) Let V_1, V_2 be any $\star$-open sets of Y and $x \in F^+(V_1) \cap F^-(V_2)$. Then we get $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \varnothing$. By *(2)*, we have $x \in p\iota\mathrm{Int}(F^+(V_1) \cap F^-(V_2))$ and hence

$$F^+(V_1) \cap F^-(V_2) \subset p\iota\mathrm{Int}\left(F^+(V_1) \cap F^-(V_2)\right).$$

This shows that $F^+(V_1) \cap F^-(V_2)$ is pre-$\mathscr{I}$-open in X.

(3) $\Rightarrow$ (4) From the fact that $F^-(Y - B) = X - F^+(B)$ and $F^+(Y - B) = X - F^-(B)$ for every subset B of Y, the desired result follows.

(4) $\Rightarrow$ (5) Let B_1, B_2 be any subsets of Y. Then $\mathrm{Cl}^\star(B_1)$ and $\mathrm{Cl}^\star(B_2)$ are $\star$-closed in Y. By *(4)*, we have $F^-(\mathrm{Cl}^\star(B_1)) \cup F^+(\mathrm{Cl}^\star(B_2))$ is pre-$\mathscr{I}$-closed in X and by Lemma 3, we get

$$\begin{aligned}\mathrm{Cl}\big(\mathrm{Int}^\star(F^-(B_1) \cup F^+(B_2))\big) &\subset \mathrm{Cl}\big(\mathrm{Int}^\star\big(F^-(\mathrm{Cl}^\star(B_1)) \cup F^+(\mathrm{Cl}^\star(B_2))\big)\big) \\ &\subset p\iota\mathrm{Cl}\big(F^-(\mathrm{Cl}^\star(B_1)) \cup F^+(\mathrm{Cl}^\star(B_2))\big) = F^-(\mathrm{Cl}^\star(B_1)) \cup F^+(\mathrm{Cl}^\star(B_2)).\end{aligned}$$

(5) $\Rightarrow$ (6) Let B_1, B_2 be any subsets of Y. It follows from Lemma 3, that

$$\begin{aligned}p\iota\mathrm{Cl}\big(F^-(B_1) \cup F^+(B_2)\big) &= \big(F^-(B_1) \cup F^+(B_2)\big) \cup \mathrm{Cl}\big(\mathrm{Int}^\star\big(F^-(B_1) \cup F^+(B_2)\big)\big) \\ &\subset F^-\big(\mathrm{Cl}^\star(B_1)\big) \cup F^+\big(\mathrm{Cl}^\star(B_2)\big).\end{aligned}$$

(6) ⇒ (7) Let B_1, B_2 be any subsets of Y. By *(6)*, we have

$$\begin{aligned} X - p\iota\mathrm{Int}\big(F^-(B_1)\cap F^+(B_2)\big) &\\ &= p\iota\mathrm{Cl}\big(X - \big(F^-(B_1)\cap F^+(B_2)\big)\big) \\ &= p\iota\mathrm{Cl}\big(\big(X - F^-(B_1)\big)\cup\big(X - F^+(B_2)\big)\big) \\ &= p\iota\mathrm{Cl}\big(F^+(Y - B_1)\cup F^-(Y - B_2)\big) \\ &\subset F^+\big(\mathrm{Cl}^\star(Y - B_1)\big)\cup F^-\big(\mathrm{Cl}^\star(Y - B_2)\big) \\ &= F^+\big(Y - \mathrm{Int}^\star(B_1)\big)\cup F^-\big(Y - \mathrm{Int}^\star(B_2)\big) \\ &= \big(X - F^-\big(\mathrm{Int}^\star(B_1)\big)\big)\cup\big(X - F^+\big(\mathrm{Int}^\star(B_2)\big)\big) \\ &= X - \big(F^-\big(\mathrm{Int}^\star(B_1)\big)\cap F^+\big(\mathrm{Int}^\star(B_2)\big)\big) \end{aligned}$$

and hence

$$F^-\left(\mathrm{Int}^\star(B_1)\right)\cap F^+\left(\mathrm{Int}^\star(B_2)\right)\subset p\iota\mathrm{Int}\left(F^-(B_1)\cap F^+(B_2)\right).$$

(7) ⇒ (1) Let $x \in X$ and V_1, V_2 be any $\star$-open sets of Y such that $F(x) \subset V_1$ and $F(x)\cap V_2 \neq \varnothing$. By *(7)*, we have $F^+(V_1)\cap F^-(V_2)\subset p\iota\mathrm{Int}\,(F^+(V_1)\cap F^-(V_2))$.

Now, put $U = F^+(V_1)\cap F^-(V_2)$. Then U is a pre-$\mathscr{I}$-open set of X containing x such that $F(z)\subset V_1$ and $F(z)\cap V_2 \neq \varnothing$ for each $z \in U$. This shows that F is *pι*-continuous. □

Definition 2. *A function $f : (X,\tau,\mathscr{I}) \to (Y,\sigma,\mathscr{J})$ is said to be pι-continuous if $f^{-1}(V)$ is pre-$\mathscr{I}$-open in X for every $\star$-open set V of Y.*

Corollary 1. *For a function $f : (X,\tau,\mathscr{I}) \to (Y,\sigma,\mathscr{J})$, the following properties are equivalent:*

(1) f is pι-continuous;

(2) for each $x \in X$ and each $\star$-open set V of Y containing $f(x)$, there exists a pre-$\mathscr{I}$-open set U of X containing x such that $f(U)\subset V$;

(3) $f^{-1}(F)$ is pre-$\mathscr{I}$-closed in X for every $\star$-closed set F of Y;

(4) $Cl\big(Int^\star\big(f^{-1}(B)\big)\big)\subset f^{-1}\left(Cl^\star(B)\right)$ for every subset B of Y;

(5) $p\iota Cl\left(f^{-1}(B)\right)\subset f^{-1}\left(Cl^\star(B)\right)$ for every subset B of Y;

(6) $f^{-1}\left(Int^\star(B)\right)\subset p\iota Int\left(f^{-1}(B)\right)$ for every subset B of Y;

(7) $f\big(p\iota Cl(A)\big)\subset Cl^\star\big(f(A)\big)$ for every subset A of X.

Definition 3 ([4])**.** *A function $f : (X,\tau,\mathscr{I}) \to (Y,\sigma,\mathscr{J})$ is said to be almost e-$\mathscr{I}$-continuous if for each R-$\mathscr{I}$-open set V of Y containing $f(x)$, there exists an e-$\mathscr{I}$-open set U of X containing x such that $f(U)\subset V$.*

Remark 1. *For a function $f : (X,\tau,\mathscr{I}) \to (Y,\sigma,\mathscr{J})$, the following implication holds:*

pι-continuity ⇒ almost e-$\mathscr{I}$-continuity.

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 1. *Let* $X = \{a, b, c, d\}$ *with a topology* $\tau = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$ *and an ideal* $\mathscr{I} = \{\varnothing, \{a\}\}$. *Let* $Y = \{1, 2, 3, 4\}$ *with a topology* $\sigma = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, Y\}$ *and an ideal* $\mathscr{J} = \{\varnothing, \{1\}\}$. *Let* $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ *be a function defined as follows:* $f(a) = f(c) = 2$, $f(b) = 1$ *and* $f(d) = 4$. *Then* f *is almost* e-$\mathscr{I}$*-continuous but* f *is not* $p\imath$*-continuous.*

Recall (see [11]) that a family $\mathscr{U}$ of subsets of an ideal topological space $(X, \tau, \mathscr{I})$ is called $\star$*-locally finite* if every $x \in X$ has a $\star$-neighbourhood which intersects only finitely many elements of $\mathscr{U}$. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be $\star$*-paracompact* if every cover of A by $\star$-open sets of X is refined by a cover of A which consists of $\star$-open sets of X and is $\star$-locally finite in X. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be $\star$*-regular* if for each $x \in A$ and each $\star$-open set U of X containing x, there exists a $\star$-open set V of X such that $x \in V \subseteq \mathrm{Cl}(V) \subseteq U$.

Lemma 4 ([11]). *Let* A *be a subset of an ideal topological space* $(X, \tau, \mathscr{I})$. *If* A *is a* $\star$*-regular* $\star$*-paracompact set of* X *and each* $\star$*-open set* U *contains* A*, then there exists a* $\star$*-open set* V *such that* $A \subset V \subset Cl(V) \subset U$.

A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is called *punctually* $\star$*-paracompact* (respectively *punctually* $\star$*-regular*) if for each $x \in X$, $F(x)$ is $\star$-paracompact (respectively $\star$-regular). For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, a multifunction $p\imath\mathrm{Cl}F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is defined as follows: $[p\imath\mathrm{Cl}F](x) = p\imath\mathrm{Cl}(F(x))$ for each $x \in X$.

Lemma 5. *If* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ *is punctually* $\star$*-regular, punctually* $\star$*-paracompact, then* $[p\imath ClF]^+(V) = F^+(V)$ *for every* $\star$*-open set* V *of* Y.

Proof. Let V be any $\star$-open set of Y and $x \in [p\imath\mathrm{Cl}F]^+(V)$. Then $p\imath\mathrm{Cl}(F(x)) \subset V$ and hence $F(x) \subset V$. Therefore, we have $x \in F^+(V)$ and so $[p\imath\mathrm{Cl}F]^+(V) \subset F^+(V)$.

On the other hand, let V be any $\star$-open set of Y and $x \in F^+(V)$. Then $F(x) \subset V$. Since $F(x)$ is punctually $\star$-regular and punctually $\star$-paracompact, by Lemma 4, there exists a $\star$-open set G such that $F(x) \subset G \subset \mathrm{Cl}(G) \subset V$, hence $p\imath\mathrm{Cl}(F(x)) \subset \mathrm{Cl}(G) \subset V$. Therefore, $x \in [p\imath\mathrm{Cl}F]^+(V)$ and so $F^+(V) \subset [p\imath\mathrm{Cl}F]^+(V)$. Thus, $[p\imath\mathrm{Cl}F]^+(V) = F^+(V)$. □

Lemma 6. *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$*, it follows that for each* $\star$*-open set* V *of* Y *we have* $[p\imath ClF]^-(V) = F^-(V)$.

Proof. Let V be any $\star$-open set of Y and $x \in [p\imath\mathrm{Cl}F]^-(V)$. Thus, $p\imath\mathrm{Cl}(F(x)) \cap V \neq \varnothing$ and hence $F(x) \cap V \neq \varnothing$. Therefore, $x \in F^-(V)$ and so $[p\imath\mathrm{Cl}F]^-(V) \subset F^-(V)$.

On the other hand, let $x \in F^-(V)$. Then we have $\varnothing \neq F(x) \cap V \subset p\imath\mathrm{Cl}(F(x)) \cap V$ and so $x \in [p\imath\mathrm{Cl}F]^-(V)$. Thus, $F^-(V) \subset [p\imath\mathrm{Cl}F]^-(V)$. This shows that $[p\imath\mathrm{Cl}F]^-(V) = F^-(V)$. □

Theorem 2. *Let* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ *be punctually* $\star$*-regular and punctually* $\star$*-paracompact. Then* F *is* $p\imath$*-continuous if and only if* $p\imath ClF : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ *is* $p\imath$*-continuous.*

Proof. Suppose that F is $p\iota$-continuous. Let $x \in X$ and V_1, V_2 be any $\star$-open sets of Y such that $(p\iota\text{Cl}F)(x) \subset V_1$ and $(p\iota\text{Cl}F)(x) \cap V_2 \neq \varnothing$. By Lemma 5 and Lemma 6, we have $x \in [p\iota\text{Cl}F]^+(V_1) = F^+(V_1)$ and $x \in [p\iota\text{Cl}F]^-(V_2) = F^-(V_2)$. Therefore, $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \varnothing$. Since F is $p\iota$-continuous, by Theorem 1, $x \in p\iota\text{Int}(F^+(V_1) \cap F^-(V_2))$ and hence $x \in p\iota\text{Int}\big([p\iota\text{Cl}F]^+(V_1) \cap [p\iota\text{Cl}F]^-(V_2)\big)$. This shows that $p\iota\text{Cl}F$ is $p\iota$-continuous.

Conversely, suppose that $p\iota\text{Cl}F$ is $p\iota$-continuous. Let $x \in X$ and V_1, V_2 be any $\star$-open sets of Y such that $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \varnothing$. By Lemma 5 and Lemma 6, we have $x \in F^+(V_1) = [p\iota\text{Cl}F]^+(V_1)$ and $x \in F^-(V_2) = [p\iota\text{Cl}F]^-(V_2)$. By the $p\iota$-continuity of $p\iota\text{Cl}F$, $x \in p\iota\text{Int}\big([p\iota\text{Cl}F]^+(V_1) \cap [p\iota\text{Cl}F]^-(V_2)\big)$ and hence $x \in p\iota\text{Int}(F^+(V_1) \cap F^-(V_2))$. Therefore, by Theorem 1, F is $p\iota$-continuous. □

3 On weakly $p\iota$-continuous multifunctions

In this section, we introduce the notion of weakly $p\iota$-continuous multifunctions. Moreover, some characterizations of weakly $p\iota$-continuous multifunctions are investigated.

Definition 4. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be:*

(i) weakly $p\iota$-continuous at a point $x \in X$ if for each $\star$-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists a pre-$\mathscr{I}$-open set U of X containing x such that $F(U) \subset Cl^\star(V_1)$ and $F(z) \cap Cl^\star(V_2) \neq \varnothing$ for every $z \in U$;

(ii) weakly $p\iota$-continuous if F has this property at each point of X.

Theorem 3. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) F is weakly $p\iota$-continuous;

(2) $x \in p\iota Cl\big(F^+(Cl^\star(V_1)) \cap F^-(Cl^\star(V_2))\big)$ for every $\star$-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$;

(3) $x \in Int\big(Cl^\star\big(F^+(Cl^\star(V_1)) \cap F^-(Cl^\star(V_2))\big)\big)$ for every $\star$-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$.

Proof. (1) ⇒ (2) Let V_1, V_2 be any $\star$-open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. Then there exists a pre-$\mathscr{I}$-open set U of X containing x such that $F(z) \subset \text{Cl}^\star(V_1)$ and $F(z) \cap \text{Cl}^\star(V_2) \neq \varnothing$ for every $z \in U$. Thus, we have $x \in U \subset F^+(\text{Cl}^\star(V_1)) \cap F^-(\text{Cl}^\star(V_2))$. Since U is pre-$\mathscr{I}$-open, we have

$$x \in U = p\iota\text{Int}(U) \subset p\iota\text{Int}\big(F^+(\text{Cl}^\star(V_1)) \cap F^-(\text{Cl}^\star(V_2))\big).$$

(2) ⇒ (3) Let V_1, V_2 be any $\star$-open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. By *(2)*, we have $x \in p\iota\text{Int}\big(F^+(\text{Cl}^\star(V_1)) \cap F^-(\text{Cl}^\star(V_2))\big)$ and by Lemma 3, we get

$$x \in \text{Int}\big(\text{Cl}^\star\big(F^+(\text{Cl}^\star(V_1)) \cap F^-(\text{Cl}^\star(V_2))\big)\big).$$

(3) ⇒ (1) Let V_1, V_2 be any $\star$-open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. Then

$$x \in F^+(V_1) \cap F^-(V_2) \subset F^+(\text{Cl}^\star(V_1)) \cap F^-(\text{Cl}^\star(V_2)).$$

By (3), we have $x \in \mathrm{Int}\big(\mathrm{Cl}^\star\big(F^+(\mathrm{Cl}^\star(V_1)) \cap F^-(\mathrm{Cl}^\star(V_2))\big)\big)$ and by Lemma 3, we obtain
$$x \in p\imath\mathrm{Int}\big(F^+(\mathrm{Cl}^\star(V_1)) \cap F^-\big(\mathrm{Cl}^\star(V_2)\big)\big).$$
Put
$$U = p\imath\mathrm{Int}\big(F^+\big(\mathrm{Cl}^\star(V_1)\big) \cap F^-\big(\mathrm{Cl}^\star(V_2)\big)\big),$$
then U is a pre-$\mathscr{I}$-open set containing x such that $F(z) \subset \mathrm{Cl}^\star(V_1)$ and $F(z) \cap \mathrm{Cl}^\star(V_2) \neq \varnothing$ for every $z \in U$. This shows that F is weakly $p\imath$-continuous. □

Theorem 4. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) F is weakly $p\imath$-continuous;

(2) $F^+(V_1) \cap F^-(V_2) \subset \mathrm{Int}\big(\mathrm{Cl}^\star\big(F^+(\mathrm{Cl}^\star(V_1)) \cap F^-(\mathrm{Cl}^\star(V_2))\big)\big)$ for every $\star$-open sets V_1, V_2 of Y;

(3) $\mathrm{Cl}\big(\mathrm{Int}^\star\big(F^+(V_1) \cup F^-(V_2)\big)\big) \subset F^+(\mathrm{Cl}^\star(V_1)) \cup F^-(\mathrm{Cl}^\star(V_2))$ for every $\star$-open sets V_1, V_2 of Y;

(4) $\mathrm{Cl}\big(\mathrm{Int}^\star\big(F^-\big(\mathrm{Int}^\star(K_1)\big) \cup F^+\big(\mathrm{Int}^\star(K_2)\big)\big)\big) \subset F^+(K_1) \cup F^-(K_2)$ for every $\star$-closed sets K_1, K_2 of Y;

(5) $p\imath\mathrm{Cl}\big(F^-\big(\mathrm{Int}^\star(K_1)\big) \cup F^+\big(\mathrm{Int}^\star(K_2)\big)\big) \subset F^+(K_1) \cup F^-(K_2)$ for every $\star$-closed sets K_1, K_2 of Y;

(6) $p\imath\mathrm{Cl}\big(F^-\big(\mathrm{Int}^\star(\mathrm{Cl}^\star(B_1))\big) \cup F^+\big(\mathrm{Int}^\star\big(\mathrm{Cl}^\star(B_2)\big)\big)\big) \subset F^+\big(\mathrm{Cl}^\star(B_1)\big) \cup F^-\big(\mathrm{Cl}^\star(B_2)\big)$ for every subsets B_1, B_2 of Y;

(7) $F^+\big(\mathrm{Int}^\star(B_1)\big) \cap F^-\big(\mathrm{Int}^\star(B_2)\big) \subset p\imath\mathrm{Int}\big(F^+\big(\mathrm{Cl}^\star\big(\mathrm{Int}^\star(B_1)\big)\big) \cap F^-\big(\mathrm{Cl}^\star\big(\mathrm{Int}^\star(B_2)\big)\big)\big)$ for every subsets B_1, B_2 of Y;

(8) $F^+(V_1) \cap F^-(V_2) \subset p\imath\mathrm{Int}\big(F^+\big(\mathrm{Cl}^\star(V_1)\big) \cap F^-\big(\mathrm{Cl}^\star(V_2)\big)\big)$ for every $\star$-open sets V_1, V_2 of Y;

(9) $p\imath\mathrm{Cl}\big(F^-(V_1) \cup F^+(V_2)\big) \subset F^-\big(\mathrm{Cl}^\star(V_1)\big) \cup F^+\big(\mathrm{Cl}^\star(V_2)\big)$ for every $\star$-open sets V_1, V_2 of Y.

Proof. *(1) ⇒ (2)* Let V_1, V_2 be any $\star$-open sets of Y and $x \in F^+(V_1) \cap F^-(V_2)$. Then we have $F(x) \in V_1^+ \cap V_2^-$. By *(1)*, there exists a pre-$\mathscr{I}$-open set U of X containing x such that $F(z) \subset \mathrm{Cl}^\star(V_1)$ and $F(z) \cap \mathrm{Cl}^\star(V_2) \neq \varnothing$ for every $z \in U$. Thus, $U \subset F^+\big(\mathrm{Cl}^\star(V_1)\big) \cap F^-\big(\mathrm{Cl}^\star(V_2)\big)$. Since U is pre-$\mathscr{I}$-open, we get
$$x \in U \subset \mathrm{Int}\big(\mathrm{Cl}^\star(U)\big) \subset \mathrm{Int}\big(\mathrm{Cl}^\star\big(F^+\big(\mathrm{Cl}^\star(V_1)\big) \cap F^-\big(\mathrm{Cl}^\star(V_2)\big)\big)\big)$$
and hence $F^+(V_1) \cap F^-(V_2) \subset \mathrm{Int}\big(\mathrm{Cl}^\star\big(F^+\big(\mathrm{Cl}^\star(V_1)\big) \cap F^-\big(\mathrm{Cl}^\star(V_2)\big)\big)\big)$.

(2) ⇒ (3) Let V_1, V_2 be any $\star$-open sets of Y. Then we have
$$\begin{aligned}
&X - \big(F^+\big(\mathrm{Cl}^\star(V_1)\big) \cup F^-\big(\mathrm{Cl}^\star(V_2)\big)\big)\\
&= \big(X - F^+\big(\mathrm{Cl}^\star(V_1)\big)\big) \cap \big(X - F^-\big(\mathrm{Cl}^\star(V_2)\big)\big)\\
&= F^-\big(Y - \mathrm{Cl}^\star(V_1)\big) \cap F^+\big(Y - \mathrm{Cl}^\star(V_2)\big)\\
&\subset \mathrm{Int}\big(\mathrm{Cl}^\star\big(F^-\big(\mathrm{Cl}^\star\big(Y - \mathrm{Cl}^\star(V_1)\big)\big) \cap F^+\big(\mathrm{Cl}^\star\big(Y - \mathrm{Cl}^\star(V_2)\big)\big)\big)\big)\\
&= \mathrm{Int}\big(\mathrm{Cl}^\star\big(F^-\big(Y - \mathrm{Int}^\star\big(\mathrm{Cl}^\star(V_1)\big)\big) \cap F^+\big(Y - \mathrm{Int}^\star\big(\mathrm{Cl}^\star(V_2)\big)\big)\big)\big)\\
&\subset \mathrm{Int}\big(\mathrm{Cl}^\star\big(F^-(Y - V_1) \cap F^+(Y - V_2)\big)\big)\\
&= \mathrm{Int}\big(\mathrm{Cl}^\star\big(X - \big(F^+(V_1) \cup F^-(V_2)\big)\big)\big)\\
&= X - \mathrm{Cl}\big(\mathrm{Int}^\star\big(F^+(V_1) \cup F^-(V_2)\big)\big)
\end{aligned}$$

and hence

$$\mathrm{Cl}\big(\mathrm{Int}^\star\big(F^+(V_1)\cup F^-(V_2)\big)\big)\subset F^+\big(\mathrm{Cl}^\star(V_1)\big)\cup F^-\big(\mathrm{Cl}^\star(V_2)\big).$$

(3) ⇒ (4) Let K_1, K_2 be any $\star$-closed sets of Y. Then $\mathrm{Int}^\star(K_1)$ and $\mathrm{Int}^\star(K_2)$ are $\star$-open sets of Y. By *(3)*, we get

$$\begin{aligned}\mathrm{Cl}\big(\mathrm{Int}^\star\big(F^+\big(\mathrm{Int}^\star(K_1)\big)\cup F^-\big(\mathrm{Int}^\star(K_2)\big)\big)\big)&\subset F^+\big(\mathrm{Cl}^\star\big(\mathrm{Int}^\star(K_1)\big)\big)\cup F^-\big(\mathrm{Cl}^\star\big(\mathrm{Int}^\star(K_2)\big)\big)\\&\subset F^+(K_1)\cup F^-(K_2).\end{aligned}$$

(4) ⇒ (5) Let K_1, K_2 be any $\star$-closed sets of Y. Then by *(4)*, we have

$$\mathrm{Cl}\big(\mathrm{Int}^\star\big(F^+\big(\mathrm{Int}^\star(K_1)\big)\cup F^-\big(\mathrm{Int}^\star(K_2)\big)\big)\big)\subset F^+(K_1)\cup F^-(K_2).$$

Since $F^+\big(\mathrm{Int}^\star(K_1)\big)\cup F^-\big(\mathrm{Int}^\star(K_2)\big)\subset F^+(K_1)\cup F^-(K_2)$ and by Lemma 3, we obtain

$$p\iota\mathrm{Cl}\big(F^+\big(\mathrm{Int}^\star(K_1)\big)\cup F^-\big(\mathrm{Int}^\star(K_2)\big)\big)\subset F^+(K_1)\cup F^-(K_2).$$

(5) ⇒ (6) Let B_1, B_2 be any subsets of Y. Then we have $\mathrm{Cl}^\star(B_1)$ and $\mathrm{Cl}^\star(B_2)$ are $\star$-closed sets in Y. By *(5)*, we have

$$p\iota\mathrm{Cl}\big(F^+\big(\mathrm{Int}^\star\big(\mathrm{Cl}^\star(B_1)\big)\big)\cup F^-\big(\mathrm{Int}^\star\big(\mathrm{Cl}^\star(B_2)\big)\big)\big)\subset F^+\big(\mathrm{Cl}^\star(B_1)\big)\cup F^-\big(\mathrm{Cl}^\star(B_2)\big).$$

(6) ⇒ (7) Let B_1, B_2 be any subsets of Y. By *(6)*, we have

$$\begin{aligned}&F^+\big(\mathrm{Int}^\star(B_1)\big)\cap F^-\big(\mathrm{Int}^\star(B_2)\big)\\&\qquad=X-\big(F^-\big(\mathrm{Cl}^\star(Y-B_1)\big)\cup F^+\big(\mathrm{Cl}^\star(Y-B_2)\big)\big)\\&\qquad\subset X-p\iota\mathrm{Cl}\big(F^-\big(\mathrm{Int}^\star\big(\mathrm{Cl}^\star(Y-B_1)\big)\big)\cup F^+\big(\mathrm{Int}^\star\big(\mathrm{Cl}^\star(Y-B_2)\big)\big)\big)\\&\qquad=p\iota\mathrm{Int}\big(F^+\big(\mathrm{Cl}^\star\big(\mathrm{Int}^\star(B_1)\big)\big)\cap F^-\big(\mathrm{Cl}^\star\big(\mathrm{Int}^\star(B_2)\big)\big)\big).\end{aligned}$$

(7) ⇒ (8) This is obvious.

(8) ⇒ (1) Let V_1, V_2 be any $\star$-open sets of Y such that $F(x)\in V_1^+\cap V_2^-$. By *(8)*, we obtain $x\in F^+(V_1)\cap F^-(V_2)\subset p\iota\mathrm{Int}\big(F^+\big(\mathrm{Cl}^\star(V_1)\big)\cap F^-\big(\mathrm{Cl}^\star(V_2)\big)\big)$. Put

$$U=p\iota\mathrm{Int}\big(F^+\big(\mathrm{Cl}^\star(V_1)\big)\cap F^-\big(\mathrm{Cl}^\star(V_2)\big)\big),$$

then U is a pre-$\mathscr{I}$-open set of X containing x such that $F(z)\subset\mathrm{Cl}^\star(V_1)$ and $F(z)\cap\mathrm{Cl}^\star(V_2)\neq\varnothing$ for every $z\in U$. This shows that F is weakly $p\iota$-continuous.

(6) ⇒ (9) Let V_1, V_2 be any $\star$-open sets of Y. By *(6)*, we have

$$\begin{aligned}p\iota\mathrm{Cl}\big(F^+(V_1)\cup F^-(V_2)\big)&\subset p\iota\mathrm{Cl}\big(F^+\big(\mathrm{Int}^\star\big(\mathrm{Cl}^\star(V_1)\big)\big)\cup F^-\big(\mathrm{Int}^\star\big(\mathrm{Cl}^\star(V_2)\big)\big)\big)\\&\subset F^+\big(\mathrm{Cl}^\star(V_1)\big)\cup F^-\big(\mathrm{Cl}^\star(V_2)\big).\end{aligned}$$

(9) ⇒ (8) Let V_1, V_2 be any $\star$-open sets of Y. By *(9)*, we have

$$\begin{aligned}F^+(V_1)\cap F^-(V_2)&\subset F^+\big(\mathrm{Int}^\star\big(\mathrm{Cl}^\star(V_1)\big)\big)\cap F^-\big(\mathrm{Int}^\star\big(\mathrm{Cl}^\star(V_2)\big)\big)\\&=X-\big(F^-\big(\mathrm{Cl}^\star\big(Y-\mathrm{Cl}^\star(V_1)\big)\big)\cup F^+\big(\mathrm{Cl}^\star\big(Y-\mathrm{Cl}^\star(V_2)\big)\big)\big)\\&\subset X-p\iota\mathrm{Cl}\big(F^-\big(Y-\mathrm{Cl}^\star(V_1)\big)\cup F^+\big(Y-\mathrm{Cl}^\star(V_2)\big)\big)\\&=p\iota\mathrm{Int}\big(F^+\big(\mathrm{Cl}^\star(V_1)\big)\cap F^-\big(\mathrm{Cl}^\star(V_2)\big)\big).\end{aligned}$$

□

Remark 2. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following implication holds:*

$$p\iota\text{-continuity} \Rightarrow \text{weak } p\iota\text{-continuity.}$$

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 2. *Let $X = \{a, b, c\}$ with a topology $\tau = \{\varnothing, \{c\}, X\}$ and an ideal $\mathscr{I} = \{\varnothing\}$. Let $Y = \{1, 2, 3\}$ with a topology $\sigma = \{\varnothing, Y\}$ and an ideal $\mathscr{J} = \{\varnothing, \{1\}\}$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is defined as follows: $F(a) = F(b) = \{2, 3\}$ and $F(c) = \{1\}$. Then F is weakly $p\iota$-continuous but F is not $p\iota$-continuous.*

Definition 5. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be weakly $p\iota$-continuous if for each point $x \in X$ and each $\star$-open set V of Y containing $f(x)$, there exists a pre-$\mathscr{I}$-open set U of X containing x such that $f(U) \subset Cl^\star(V)$.*

Corollary 2. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) f is weakly $p\iota$-continuous;

(2) $f^{-1}(V) \subset Int(Cl^\star(f^{-1}(Cl^\star(V))))$ for every $\star$-open set V of Y;

(3) $Cl(Int^\star(f^{-1}(V))) \subset f^{-1}(Cl^\star(V))$ for every $\star$-open set V of Y;

(4) $Cl(Int^\star(f^{-1}(Int^\star(K)))) \subset f^{-1}(K)$ for every $\star$-closed set K of Y;

(5) $p\iota Cl(f^{-1}(Int^\star(K))) \subset f^{-1}(K)$ for every $\star$-closed set K of Y;

(6) $p\iota Cl(f^{-1}(Int^\star(Cl^\star(B)))) \subset f^{-1}(Cl^\star(B))$ for every subset B of Y;

(7) $f^{-1}(Int^\star(B)) \subset p\iota Int(f^{-1}(Cl^\star(Int^\star(B))))$ for every subset B of Y;

(8) $f^{-1}(V) \subset p\iota Int(f^{-1}(Cl^\star(V)))$ for every $\star$-open set V of Y;

(9) $p\iota Cl(f^{-1}(V)) \subset f^{-1}(Cl^\star(V))$ for every $\star$- open set V of Y.

Lemma 7 ([12]). *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) if A is $\star$-open in X, then $Cl^\star(A) = \star_\theta Cl(A)$;

(2) $\star_\theta Cl(A)$ is $\star$-closed in X.

Recall (see [13]) that a subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *R-$\mathscr{I}^\star$-open* (respectively *$\mathscr{I}^\star$-preopen*) if $A = \mathrm{Int}^\star(\mathrm{Cl}^\star(A))$ (respectively $A \subset \mathrm{Int}^\star(\mathrm{Cl}^\star(A))$). The complement of an *R-$\mathscr{I}^\star$*-open (respectively $\mathscr{I}^\star$-preopen) set is called *R-$\mathscr{I}^\star$-closed* (respectively *$\mathscr{I}^\star$-preclosed*).

Theorem 5. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) F is weakly pι-continuous;

(2) $p\iota Cl\big(F^-\big(Int^\star\big(\star_\theta\, Cl(B_1)\big)\big) \cup F^+(Int^\star(\star_\theta Cl(B_2)))\big) \subset F^-\big(\star_\theta\, Cl(B_1)\big) \cup F^+\big(\star_\theta\, Cl(B_2)\big)$ for every subsets B_1, B_2 of Y;

(3) $p\iota Cl\big(F^-\big(Int^\star\big(Cl^\star(B_1)\big)\big) \cup F^+\big(Int^\star\big(Cl^\star(B_2)\big)\big)\big) \subset F^-\big(\star_\theta\, Cl(B_1)\big) \cup F^+\big(\star_\theta\, Cl(B_2)\big)$ for every subsets B_1, B_2 of Y;

(4) $p\iota Cl\big(F^-\big(Int^\star\big(Cl^\star(V_1)\big)\big) \cup F^+\big(Int^\star\big(Cl^\star(V_2)\big)\big)\big) \subset F^-\big(Cl^\star(V_1)\big) \cup F^+\big(Cl^\star(V_2)\big)$ for every $\star$-open sets V_1, V_2 of Y;

(5) $p\iota Cl\big(F^-\big(Int^\star\big(Cl^\star(V_1)\big)\big) \cup F^+\big(Int^\star\big(Cl^\star(V_2)\big)\big)\big) \subset F^-\big(Cl^\star(V_1)\big) \cup F^+\big(Cl^\star(V_2)\big)$ for every $\mathscr{J}^\star$-preopen sets V_1, V_2 of Y;

(6) $p\iota Cl\big(F^-\big(Int^\star(K_1)\big) \cup F^+\big(Int^\star(K_2)\big)\big) \subset F^-(K_1) \cup F^+(K_2)$ for every R-$\mathscr{J}^\star$-closed sets K_1, K_2 of Y.

Proof. (1) ⇒ (2) Let B_1, B_2 be any subsets of X. By Lemma 7, we have $\star_\theta \mathrm{Cl}\,(B_1)$ and $\star_\theta \mathrm{Cl}\,(B_2)$ are $\star$-closed in Y. By Theorem 4, we get

$$p\iota\mathrm{Cl}\big(F^-\big(\mathrm{Int}^\star\big(\star_\theta\,\mathrm{Cl}(B_1)\big)\big) \cup F^+\big(\mathrm{Int}^\star\big(\star_\theta\,\mathrm{Cl}(B_2)\big)\big)\big) \subset F^-\big(\star_\theta\,\mathrm{Cl}(B_1)\big) \cup F^+\big(\star_\theta\,\mathrm{Cl}(B_2)\big).$$

(2) ⇒ (3) This is obvious since $\mathrm{Cl}^\star(B) \subset \star_\theta\mathrm{Cl}(B)$ for every subset B of Y.

(3) ⇒ (4) This is obvious since $\mathrm{Cl}^\star(V) = \star_\theta\mathrm{Cl}(V)$ for every $\star$-open set V of Y.

(4) ⇒ (5) Let V_1, V_2 be any $\mathscr{J}^\star$-preopen sets of Y. Then we have $V_i \subset \mathrm{Int}^\star\big(\mathrm{Cl}^\star\,(V_i)\big)$ and $\mathrm{Cl}^\star(V_i) = \mathrm{Cl}^\star\big(\mathrm{Int}^\star\big(\mathrm{Cl}^\star(V_i)\big)\big)$ for $i = 1, 2$. Now, put $U_i = \mathrm{Int}^\star\big(\mathrm{Cl}^\star(V_i)\big)$, then U_i is $\star$-open in Y and $\mathrm{Cl}^\star\,(U_i) = \mathrm{Cl}^\star\,(V_i)$. By *(4)*, we obtain

$$p\iota\mathrm{Cl}\big(F^-\big(\mathrm{Int}^\star\big(\mathrm{Cl}^\star(V_1)\big)\big) \cup F^+\big(\mathrm{Int}^\star\big(\mathrm{Cl}^\star(V_2)\big)\big)\big) \subset F^-\big(\mathrm{Cl}^\star(V_1)\big) \cup F^+\big(\mathrm{Cl}^\star(V_2)\big).$$

(5) ⇒ (6) Let K_1, K_2 be any R-$\mathscr{J}^\star$-closed sets of Y. Then we have $\mathrm{Int}^\star\,(K_1)$ and $\mathrm{Int}^\star\,(K_2)$ are $\mathscr{J}^\star$-preopen in Y. By *(5)*, we get

$$\begin{aligned} p\iota\mathrm{Cl}\big(F^-\big(\mathrm{Int}^\star(K_1)\big) \cup F^+\big(\mathrm{Int}^\star(K_2)\big)\big) &\\ &= p\iota\mathrm{Cl}\big(F^-\big(\mathrm{Int}^\star\big(\mathrm{Cl}^\star\big(\mathrm{Int}^\star(K_1)\big)\big)\big) \cup F^+\big(\mathrm{Int}^\star\big(\mathrm{Cl}^\star\big(\mathrm{Int}^\star(K_2)\big)\big)\big)\big) \\ &\subset F^-\big(\mathrm{Cl}^\star\big(\mathrm{Int}^\star(K_1)\big)\big) \cup F^+\big(\mathrm{Cl}^\star\big(\mathrm{Int}^\star(K_2)\big)\big) = F^-(K_1) \cup F^+(K_2). \end{aligned}$$

(6) ⇒ (1) Let V_1, V_2 be any $\star$-open sets of Y. Then $\mathrm{Cl}^\star\,(V_1)$ and $\mathrm{Cl}^\star\,(V_2)$ are R-$\mathscr{J}^\star$-closed sets of Y. By *(6)*, we have

$$\begin{aligned} p\iota\mathrm{Cl}\big(F^-(V_1) \cup F^+(V_2)\big) &\subset p\iota\mathrm{Cl}\big(F^-\big(\mathrm{Int}^\star\big(\mathrm{Cl}^\star(V_1)\big)\big) \cup F^+\big(\mathrm{Int}^\star\big(\mathrm{Cl}^\star(V_2)\big)\big)\big) \\ &\subset F^-\big(\mathrm{Cl}^\star(V_1)\big) \cup F^+\big(\mathrm{Cl}^\star(V_2)\big). \end{aligned}$$

It follows from Theorem 4 that F is weakly $p\iota$-continuous. □

Corollary 3. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) f is weakly $p\iota$-continuous;

(2) $p\iota Cl\big(f^{-1}\big(Int^{\star}\big(\star_{\theta} Cl(B)\big)\big)\big) \subset f^{-1}\big(\star_{\theta} Cl(B)\big)$ for every subset B of Y;

(3) $p\iota Cl\big(f^{-1}\big(Int^{\star}\big(Cl^{\star}(B)\big)\big)\big) \subset f^{-1}\big(\star_{\theta} Cl(B)\big)$ for every subset B of Y;

(4) $p\iota Cl\big(f^{-1}\big(Int^{\star}\big(Cl^{\star}(V)\big)\big)\big) \subset f^{-1}\big(Cl^{\star}(V)\big)$ for every $\star$-open set V of Y;

(5) $p\iota Cl\big(f^{-1}\big(Int^{\star}\big(Cl^{\star}(V)\big)\big)\big) \subset f^{-1}\big(Cl^{\star}(V)\big)$ for every $\mathscr{J}^{\star}$-preopen set V of Y;

(6) $p\iota Cl\big(f^{-1}\big(Int^{\star}(K)\big)\big) \subset f^{-1}(K)$ for every R-$\mathscr{J}^{\star}$-closed set K of Y.

Recall that a subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *semi-$\mathscr{I}^{\star}$-open* (respectively *semi-$\mathscr{I}^{\star}$-preopen*) if $A \subset \mathrm{Cl}^{\star}\big(\mathrm{Int}^{\star}(A)\big)$ $\big($respectively if $A \subset \mathrm{Cl}^{\star}\big(\mathrm{Int}^{\star}\big(\mathrm{Cl}^{\star}(A)\big)\big)\big)$. The complement of a semi-$\mathscr{I}^{\star}$-open (respectively of a semi-$\mathscr{I}^{\star}$-preopen) set is called *semi-$\mathscr{I}^{\star}$-closed* (respectively *semi-$\mathscr{I}^{\star}$-preclosed*) (see [12]).

Theorem 6. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) F is weakly $p\iota$-continuous;

(2) $p\iota Cl\big(F^{-}\big(Int^{\star}\big(Cl^{\star}(V_1)\big)\big) \cup F^{+}\big(Int^{\star}\big(Cl^{\star}(V_2)\big)\big)\big) \subset F^{-}\big(Cl^{\star}(V_1)\big) \cup F^{+}\big(Cl^{\star}(V_2)\big)$ for every semi-$\mathscr{J}^{\star}$-preopen sets V_1, V_2 of Y;

(3) $p\iota Cl\big(F^{-}\big(Int^{\star}\big(Cl^{\star}(V_1)\big)\big) \cup F^{+}\big(Int^{\star}\big(Cl^{\star}(V_2)\big)\big)\big) \subset F^{-}\big(Cl^{\star}(V_1)\big) \cup F^{+}\big(Cl^{\star}(V_2)\big)$ for every semi-$\mathscr{J}^{\star}$-open sets V_1, V_2 of Y;

(4) $p\iota Cl\big(F^{-}\big(Int^{\star}\big(Cl^{\star}(V_1)\big)\big) \cup F^{+}\big(Int^{\star}\big(Cl^{\star}(V_2)\big)\big)\big) \subset F^{-}\big(Cl^{\star}(V_1)\big) \cup F^{+}\big(Cl^{\star}(V_2)\big)$ for every $\mathscr{J}^{\star}$-preopen sets V_1, V_2 of Y.

Proof. (1) $\Rightarrow$ (2) Let V_1, V_2 be any semi-$\mathscr{J}^{\star}$-preopen sets of Y. Then we have

$$V_i \subset \mathrm{Cl}^{\star}\big(\mathrm{Int}^{\star}\big(\mathrm{Cl}^{\star}(V_i)\big)\big) \qquad \text{and} \qquad \mathrm{Cl}^{\star}(V_i) = \mathrm{Cl}^{\star}\big(\mathrm{Int}^{\star}\big(\mathrm{Cl}^{\star}(V_i)\big)\big)$$

for $i = 1, 2$. Since $\mathrm{Cl}^{\star}(V_1)$ and $\mathrm{Cl}^{\star}(V_2)$ are R-$\mathscr{J}^{\star}$-closed sets, by Theorem 5, we get

$$p\iota\mathrm{Cl}\big(F^{-}\big(\mathrm{Int}^{\star}\big(\mathrm{Cl}^{\star}(V_1)\big)\big) \cup F^{+}\big(\mathrm{Int}^{\star}\big(\mathrm{Cl}^{\star}(V_2)\big)\big)\big) \subset F^{-}\big(\mathrm{Cl}^{\star}(V_1)\big) \cup \mathrm{Cl}^{\star}(V_2).$$

(2) $\Rightarrow$ (3) This is obvious since every semi-$\mathscr{J}^{\star}$-open set is semi-$\mathscr{J}^{\star}$-preopen.

(3) $\Rightarrow$ (4) Let V_1, V_2 be any $\mathscr{J}^{\star}$-preopen sets of Y. Then we have that $\mathrm{Cl}^{\star}(V_1)$ and $\mathrm{Cl}^{\star}(V_2)$ are R-$\mathscr{J}^{\star}$-closed sets of Y. Thus, $\mathrm{Cl}^{\star}(V_1)$ and $\mathrm{Cl}^{\star}(V_2)$ are semi-$\mathscr{J}^{\star}$-open sets of Y. By *(3)*, we have

$$p\iota\mathrm{Cl}\big(F^{-}\big(\mathrm{Int}^{\star}\big(\mathrm{Cl}^{\star}(V_1)\big)\big) \cup F^{+}\big(\mathrm{Int}^{\star}\big(\mathrm{Cl}^{\star}(V_2)\big)\big)\big) \subset F^{-}\big(\mathrm{Cl}^{\star}(V_1)\big) \cup F^{+}\big(\mathrm{Cl}^{\star}(V_2)\big).$$

(4) $\Rightarrow$ (1) Let V_1, V_2 be any $\star$-open sets of Y. Then we have that V_1, V_2 are $\mathscr{J}^{\star}$-preopen sets of Y and by *(4)*, we obtain

$$\begin{aligned} p\iota\mathrm{Cl}\big(F^{-}(V_1) \cup F^{+}(V_2)\big) &\subset p\iota\mathrm{Cl}\big(F^{-}\big(\mathrm{Int}^{\star}\big(\mathrm{Cl}^{\star}(V_1)\big)\big) \cup F^{+}\big(\mathrm{Int}^{\star}\big(\mathrm{Cl}^{\star}(V_2)\big)\big)\big) \\ &\subset F^{-}\big(\mathrm{Cl}^{\star}(V_1)\big) \cup F^{+}\big(\mathrm{Cl}^{\star}(V_2)\big). \end{aligned}$$

It follows from Theorem 4, that F is weakly $p\iota$-continuous. □

Corollary 4. *For a function* $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$*, the following properties are equivalent:*

(1) f *is weakly pι-continuous;*

(2) $p\iota Cl\big(f^{-1}\big(Int^{\star}\big(Cl^{\star}(V)\big)\big)\big) \subset f^{-1}\big(Cl^{\star}(V)\big)$ *for every semi-*$\mathscr{J}^{\star}$*-preopen set* V *of* Y*;*

(3) $p\iota Cl\big(f^{-1}\big(Int^{\star}\big(Cl^{\star}(V)\big)\big)\big) \subset f^{-1}\big(Cl^{\star}(V)\big)$ *for every semi-*$\mathscr{J}^{\star}$*-open set* V *of* Y*;*

(4) $p\iota Cl\big(f^{-1}\big(Int^{\star}\big(Cl^{\star}(V)\big)\big)\big) \subset f^{-1}\big(Cl^{\star}(V)\big)$ *for every* $\mathscr{J}^{\star}$*-preopen set* V *of* Y*.*

Theorem 7. *For a multifunction* $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$*, the following properties are equivalent:*

(1) F *is weakly pι-continuous;*

(2) $Cl\big(Int^{\star}\big(F^{-}(V_1) \cup F^{+}(V_2)\big)\big) \subset F^{-}\big(Cl^{\star}(V_1)\big) \cup F^{+}\big(Cl^{\star}(V_2)\big)$ *for every* $\mathscr{J}^{\star}$*-preopen sets* V_1, V_2 *of* Y*;*

(3) $p\iota Cl\big(F^{-}(V_1) \cup F^{+}(V_2)\big) \subset F^{-}\big(Cl^{\star}(V_1)\big) \cup F^{+}\big(Cl^{\star}(V_2)\big)$ *for every* $\mathscr{J}^{\star}$*-preopen sets* V_1, V_2 *of* Y*;*

(4) $F^{+}(V_1) \cap F^{-}(V_2) \subset p\iota\big(F^{+}\big(Cl^{\star}(V_1)\big) \cap F^{-}\big(Cl^{\star}(V_2)\big)\big)$ *for every* $\mathscr{J}^{\star}$*-preopen sets* V_1, V_2 *of* Y*.*

Proof. (1) ⇒ *(2)* Let V_1, V_2 be any $\mathscr{J}^{\star}$-preopen sets of Y. Since F is weakly $p\iota$-continuous, by Theorem 4 and Lemma 3, we obtain

$$\begin{aligned}\mathrm{Cl}\big(\mathrm{Int}^{\star}\big(F^{-}(V_1) \cup F^{+}(V_2)\big)\big) &\subset \mathrm{Cl}\big(\mathrm{Int}^{\star}\big(F^{-}\big(\mathrm{Int}^{\star}\big(\mathrm{Cl}^{\star}(V_1)\big)\big) \cup F^{+}\big(\mathrm{Int}^{\star}\big(\mathrm{Cl}^{\star}(V_2)\big)\big)\big)\big)\\ &\subset F^{-}\big(\mathrm{Cl}^{\star}(V_1)\big) \cup F^{+}\big(\mathrm{Cl}^{\star}(V_2)\big).\end{aligned}$$

(2) ⇒ *(3)* Let V_1, V_2 be any $\mathscr{J}^{\star}$-preopen sets of Y. By *(2)* and Lemma 3, we get

$$\begin{aligned}p\iota\mathrm{Cl}\big(F^{-}(V_1) \cup F^{+}(V_2)\big) &= \big(F^{-}(V_1) \cup F^{+}(V_2)\big) \cup \mathrm{Cl}\big(\mathrm{Int}^{\star}\big(F^{-}(V_1) \cup F^{+}(V_2)\big)\big)\\ &\subset F^{-}\big(\mathrm{Cl}^{\star}(V_1)\big) \cup F^{+}\big(\mathrm{Cl}^{\star}(V_2)\big).\end{aligned}$$

(3) ⇒ *(4)* Let V_1, V_2 be any $\mathscr{J}^{\star}$-preopen sets of Y. Then by *(3)*, we have

$$\begin{aligned}&X - p\iota\mathrm{Int}\big(F^{+}\big(\mathrm{Cl}^{\star}(V_1)\big) \cap F^{-}\big(\mathrm{Cl}^{\star}(V_2)\big)\big)\\ &\qquad = p\iota\mathrm{Cl}\big(X - \big(F^{+}\big(\mathrm{Cl}^{\star}(V_1)\big) \cap F^{-}\big(\mathrm{Cl}^{\star}(V_2)\big)\big)\big)\\ &\qquad = p\iota\mathrm{Cl}\big(\big(X - F^{+}\big(\mathrm{Cl}^{\star}(V_1)\big)\big) \cup \big(X - F^{-}\big(\mathrm{Cl}^{\star}(V_2)\big)\big)\big)\\ &\qquad = p\iota\mathrm{Cl}\big(F^{-}\big(Y - \mathrm{Cl}^{\star}(V_1)\big) \cup F^{+}\big(Y - \mathrm{Cl}^{\star}(V_2)\big)\big)\\ &\qquad \subset F^{-}\big(\mathrm{Cl}^{\star}\big(Y - \mathrm{Cl}^{\star}(V_1)\big)\big) \cup F^{+}\big(\mathrm{Cl}^{\star}\big(Y - \mathrm{Cl}^{\star}(V_2)\big)\big)\\ &\qquad = \big(X - F^{+}\big(\mathrm{Int}^{\star}\big(\mathrm{Cl}^{\star}(V_1)\big)\big)\big) \cup \big(X - F^{-}\big(\mathrm{Int}^{\star}\big(\mathrm{Cl}^{\star}(V_2)\big)\big)\big)\\ &\qquad = X - \big(F^{+}\big(\mathrm{Int}^{\star}\big(\mathrm{Cl}^{\star}(V_1)\big)\big) \cap F^{-}\big(\mathrm{Int}^{\star}\big(\mathrm{Cl}^{\star}(V_2)\big)\big)\big)\\ &\qquad \subset X - \big(F^{+}(V_1) \cap F^{-}(V_2)\big)\end{aligned}$$

and hence

$$F^{+}(V_1) \cap F^{-}(V_2) \subset p\iota\mathrm{Int}\big(F^{+}\big(\mathrm{Cl}^{\star}(V_1)\big) \cap F^{-}\big(\mathrm{Cl}^{\star}(V_2)\big)\big).$$

(4) ⇒ (1) Let V_1, V_2 be any $\star$-open sets of Y. Then we have that V_1, V_2 are $\mathscr{J}^\star$-preopen sets of Y and by *(4)*, we obtain $F^+(V_1) \cap F^-(V_2) \subset p\imath\text{Int}\big(F^+(\text{Cl}^\star(V_1)) \cap F^-(\text{Cl}^\star(V_2))\big)$. It follows from Theorem 4 that F is weakly $p\imath$-continuous. □

Corollary 5. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) f is weakly $p\imath$-continuous;

(2) $Cl\big(Int^\star(f^{-1}(V))\big) \subset f^{-1}\big(Cl^\star(V)\big)$ for every $\mathscr{J}^\star$-preopen set V of Y;

(3) $p\imath Cl\big(f^{-1}(V)\big) \subset f^{-1}\big(Cl^\star(V)\big)$ for every $\mathscr{J}^\star$-preopen set V of Y;

(4) $f^{-1}(V) \subset p\imath Int\big(f^{-1}\big(Cl^\star(V)\big)\big)$ for every $\mathscr{J}^\star$-preopen set V of Y.

4 Conclusion

In this paper, we have introduced two classes of multifunctions called $p\imath$-continuous multifunctions and weakly $p\imath$-continuous multifunctions. Also, we have discussed the relationships between $p\imath$-continuity and weak $p\imath$-continuity. Furthermore, several characterizations and fundamental properties concerning $p\imath$-continuous multifunctions and weakly $p\imath$-continuous multifunctions are established. In the upcoming work, we plan to apply the concepts initiated in this paper to study a new generalization of $p\imath$-continuous multifunctions, namely almost $p\imath$-continuous multifunctions. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is called almost $p\imath$-continuous if for each $x \in X$ and each $\star$-open sets V_1, V_2 of Y such that $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \varnothing$, there exists a pre-$\mathscr{I}$-open set U of X containing x such that $F(U) \subset \text{Int}\big(\text{Cl}^\star(V_1)\big)$ and $F(z) \cap \text{Int}\big(\text{Cl}^\star(V_2)\big) \neq \varnothing$ for every $z \in U$. The class of almost $p\imath$-continuous multifunctions included in the class of weakly $p\imath$-continuous multifunctions.

References

[1] Abd El-Monsef M.E., Lashien E.F., Nasef A.A. *On $\mathscr{I}$-open sets and $\mathscr{I}$-continuous functions*. Kyungpook Math. J. 1992, **32** (1), 21–30.

[2] Abd El-Monsef M.E., El-Deeb S.N., Mahmoud R.A. *β-open sets and β-continuous mappings*. Bull. Fac. Sci. Assiut Univ. 1983, **12**, 77–90.

[3] Açikgöz A., Noiri T., Yüksel Ş. *A decomposition of continuity in ideal topological spaces*. Acta Math. Hungar. 2004, **105**, 285–289. doi:10.1023/B:AMHU.0000049280.10577.4e

[4] Al-Omeri W., Noiri T. *On almost e-$\mathscr{I}$-continuous functions*. Demonstr. Math. 2021, **54** (1), 168–177. doi: 10.1515/dema-2021-0014

[5] Al-Omeri W. *On mixed b-fuzzy topological spaces*. Int. J. Fuzzy Log. Intell. Syst. 2020, **20** (3), 242–246. doi:10.5391/IJFIS.2020.20.3.242

[6] Al-Omeri W., Khalil O. H., Ghareeb A. *Degree of (L, M)-fuzzy semi-precontinuous and (L, M)-fuzzy semi-preirresolute functions*. Demonstr. Math. 2018, **51** (1), 182–197. doi:10.1515/dema-2018-0014

[7] Al-Omeri W., Noorani M.S.Md., Al-Omari A., Noiri T. *New types of multifunctions in ideal topological spaces via e-$\mathscr{I}$-open sets and δβ-$\mathscr{I}$-open sets*. Bol. Soc. Parana. Mat. (3) 2016, **34** (1), 213–223. doi: 10.5269/bspm.v34i1.26193

[8] Al-Omeri W., Noorani M.S.Md., Al-Omari A. *On e-$\mathscr{I}$-open sets, e-$\mathscr{I}$-continuous functions and decomposition of continuity*. J. Math. Appl. 2015, **38**, 15–31. doi:10.7862/rf.2015.2

[9] Al-Omeri W., Noorani M.S.Md., Al-Omari A. *New forms of contra-continuity in ideal topological spaces*. Missouri J. Math. Sci. 2014, **26** (1), 33–47. doi:10.35834/mjms/1404997107

[10] Berge C. Espaces topologiques. Fonctions multivoques. Dunod., Paris, 1959.

[11] Boonpok C. *On some types of continuity for multifunctions in ideal topological spaces*. Adv. Math. Sci. 2020, **9** (3), 859–886. doi:10.37418/amsj.9.3.13

[12] Boonpok C. *Weak quasi continuity for multifunctions in ideal topological spaces*. Adv. Math. Sci. 2020, **9** (1), 339–355. doi:10.37418/amsj.9.1.28

[13] Boonpok C. *On continuous multifunctions in ideal topological spaces*. Lobachevskii J. Math. 2019, **40** (1), 24–35. doi:10.1134/S1995080219010049

[14] Dontchev J. *Idealization of Ganster-Reilly decomposition theorems*. arXiv:math/9901017 [math.GN]. doi: 10.48550/arXiv.math/9901017

[15] Ekici E., Noiri T. *⋆-hyperconnected ideal topological spaces*. An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) 2012, **58**, 121–129. doi:10.2478/v10157-011-0045-9

[16] Hatir E., Keskin A., Noiri T. *A note on strong β-$\mathscr{I}$-open sets and strongly β-$\mathscr{I}$-continuous functions*. Acta Math. Hungar. 2005, **108** (1–2), 87–94. doi:10.1007/s10474-005-0210-2

[17] Hatir E., Noiri T. *On semi-$\mathscr{I}$-open sets and semi-$\mathscr{I}$-continuous functions*. Ata Math. Hungar. 2005, **107** (4), 345–353. doi:10.1007/s10474-005-0202-2

[18] Hatir E., Noiri T. *On decompositions of continuity via idealization*. Acta Math. Hungar. 2002, **96** (4), 341–349. doi:10.1023/A:1019760901169

[19] Husain T. *Almost continuous mappings*. Comment. Math. 1966, **10** (1), 1–7.

[20] Janković D., Hamlett T.R. *New topologies from old via ideals*. Amer. Math. Monthly 1990, **97**, 295–310.

[21] Kuratowski K. Topology. Vol. I. Academic Press, INC. New York, 1966.

[22] Levine N. *Semi-open sets and semi-continuity in topological spaces*. Amer. Math. Monthly 1963, **70**, 36–41. doi:10.1080/00029890.1963.11990039

[23] Mashhour A.S., Abd El-Monsef M.E., El-Deeb S.N. *On precontinuous and weak precontinuous mappings*. Proc. Math. Phys. Soc. Egypt 1982, **53**, 47–53.

[24] Njåstad O. *On some classes of nearly open sets*. Pacific J. Math. 1965, **15** (3), 961–970.

[25] Popa V., Noiri T. *Properties of weakly precontinuous multifunctions*. Istanb. Univ. Sci. Fac. J. Math. Phys. Astronom. 2001, **57-58**, 41–52.

[26] Popa V., Noiri T. *A note on precontinuity and quasicontinuity for multifunctions*. Demonstr. Math. 1997, **30**, 271–278.

[27] Popa V., Noiri T. *Characterizations of α-continuous multifunctions*. Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 1993, **23** (1), 29–38.

[28] Przemski M. *Some generalizations of continuity and quasicontinuity for multivalued maps*. Demonstr. Math. 1993, **26** (2), 381–400. doi:10.1515/dema-1993-0212

[29] Vaidyanathaswamy R. *The localisation theory in set topology*. Proc. Indian Acad. Sci. Math. Sci. 1945, **20**, 51–61. doi:10.1007/BF03048958

[30] Yüksel Ş., Simsekler T.H., Guzel Z., Noiri T. *Strongly θ-pre-$\mathscr{I}$-continuous functions*. Sci. Stud. Res. Ser. Math. Inform. 2010, **20**, 111–126.

[31] Yüksel Ş., Açikgöz A., Gursel E. *On a new type of continuous functions in ideal topological spaces*. J. Indian Math. Soc. (N.S.) 2007, **28**, 427–438.

[32] Yüksel Ş., Açikgöz A., Gursel E. *On β-$\mathscr{I}$-regular sets*. Dal'nevost. Mat. Zh. 2007, **25**, 353–366.

[33] Yüksel Ş., Açikgöz A., Noiri T. *On δ-$\mathscr{I}$-continuous functions*. Turkish J. Math. 2005, **29** (1), 39–51.

Received 04.06.2021
Revised 20.05.2024

Бунпок Ч. *pι-Неперервність та слабка pι-неперервність* // Карпатські матем. публ. — 2025. — Т.17, №1. — С. 171–186.

У цій роботі ми вводимо поняття *pι*-неперервних мультфункцій та слабко *pι*-неперервних мультфункцій, використовуючи поняття перед-$\mathscr{I}$-відкритих множин в ідеальних топологічних просторах. Також досліджено деякі характеризації *pι*-неперервних мультфункцій та слабко *pι*-неперервних мультфункцій. Крім того, обговорюються взаємозв'язки між *pι*-неперервними та слабко *pι*-неперервними мультфункціями.

Ключові слова і фрази: перед-$\mathscr{I}$-відкрита множина, *pι*-неперервна мультфункція, слабко *pι*-неперервна мультфункція.