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On A-statistical convergence and A-statistical Cauchy via ideal

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In [Analysis 1985, **5** (4), 301–313], J.A. Fridy proved an equivalence relation between statistical convergence and statistical Cauchy sequence. In this paper, we define A^{I^*}-statistical convergence and find under certain conditions, that it is equivalent to A^I-statistical convergence defined in [Appl. Math. Lett. 2012, **25** (4), 733–738]. Moreover, we define A^I- and A^{I^*}-statistical Cauchy sequences and find some equivalent relation with A^I- and A^{I^*}-statistical convergence.

Key words and phrases: I-convergence, A^I-statistical convergence, A^{I^*}-statistical convergence, A^I-statistical Cauchy convergence, A^{I^*}-statistical Cauchy convergence.

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1 Introduction

The natural density of $V \subseteq \mathbb{N}$ (the set of natural numbers) is defined by

$$\delta(V) = \lim_n \frac{1}{n} \Big| \{v \leq n : v \in V\} \Big|,$$

if the limit exists, where $|\cdot|$ denotes the cardinality of the enclosed set. The statistically convergence [14] of a sequence $\mu = (\mu_k)$ to the number L is obtained if $\forall\, \epsilon > 0$, $\delta(V(\epsilon)) = 0$, where $V(\epsilon) = \{k \in \mathbb{N} : |\mu_k - L| \geq \epsilon\}$, i.e. $st\text{-}\lim \mu = L$. For an infinite matrix $A = (a_{nk})$, a sequence $\mu = (\mu_k)$ is A-summable to L if $\lim_n A_n(\mu) = L$, where $A_n(\mu) = \sum_{k=1}^{\infty} a_{nk}\mu_k$ and the series converges for each n. A matrix A is regular if A transforms every convergent sequence into a convergent sequence leaving the limit invariant, i.e. $A\mu \in c$ for every $\mu \in c$ and $\lim_n A_n(\mu) = \lim_k \mu_k$. Let Ω denote the class of all non-negative regular matrices. If C_1 is replaced by $A \in \Omega$, then x is A-statistically convergent to L [3, 23], i.e. if $\delta_A(V(\epsilon)) = \lim_n \sum_{k \in V(\epsilon)} a_{nk} = 0$ for every $\epsilon > 0$, and we write $st_A\text{-}\lim_k \mu_k = L$. J.A. Fridy [16] defined statistically Cauchy (i.e. a sequence $\mu = (\mu_k)$ is statistically Cauchy if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $\delta(\{k \in \mathbb{N} : |\mu_k - \mu_{N_\epsilon}| \geq \epsilon\}) = 0$ and showed that it is equivalent to statistical convergence. In [23], the same results were shown for A-statistically Cauchy sequences. Several generalizations and variants can be found in [4, 9, 17, 18, 22, 30, 31, 36]. The concept of I-convergence [24] is one of such generalizations. Let $\varnothing \neq I$ $(F$, resp.$) \subseteq P(\mathbb{N})$, then I $(F$, resp.) is an ideal (filter, resp.) in $\mathbb{N}$ if for any $B, C \in I$ (F, resp.), we have (i) $B \cup C \in I$ ($B \cap C \in F$, resp.); (ii) $B \in I$ whenever $B \subseteq C$ and $C \in I$ ($C \in F$ whenever $B \subseteq C$ and $B \in F$, resp.); (iii) $\varnothing \in I$ ($\varnothing \notin F$, resp.).

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I is non-trivial if $\mathbb{N} \notin I$, and is admissible if I contains all finite subset of $\mathbb{N}$. Let Im denote the set of all non-trivial admissible ideals in $\mathbb{N}$. The filter associated with I is denoted by $F(I) = \{D = \mathbb{N} \setminus B : B \in I\}$. A sequence $\mu = (\mu_k)$ is I-convergent to $L \in \mathbb{R}$ if for every $\epsilon > 0$, the set $V(\epsilon) \in I$, and we write $I\text{-}\lim \mu_k = L$. The notion of I^*-convergence was introduced in [24] and it was shown under certain conditions the equivalence of I- and I^*-convergence. A real sequence $\mu = (\mu_k)$ is I^*-convergent to $L \in \mathbb{R}$ if there is a set $B \in I$ such that for $D = \mathbb{N} \setminus B = \{d_i\}_{i=1}^{\infty}$, we have $\lim_i \mu_{d_i} = L$; and we write $I^*\text{-}\lim \mu_k = L$. The notion of I-Cauchy sequence was studied by many authors see [7, 26, 35], which is a generalization of statistical Cauchy. A real sequence μ is I-Cauchy if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $\{k : |\mu_k - \mu_{N_\epsilon}| \geq \epsilon\} \in I$. A real sequence $\mu = (\mu_k)$ is I^*-Cauchy if there exists a set $D = \{d_i\}_{i=1}^{\infty} \in F(I)$ such that the subsequence (μ_{d_i}) is an ordinary Cauchy sequence in $\mathbb{R}$. One more generalization of A-statistical convergence is A^I-statistical convergence introduced by E. Savas at el. [39], see also [2, 7, 8, 13, 19, 21, 25, 27–29, 32–34, 38].

Definition 1. *Let $I \in$ Im and $A \in \Omega$. A real sequence $\mu = (\mu_k)$ is said to be A^I-statistically convergent to $L \in \mathbb{R}$ if for every $\epsilon > 0$ and $v > 0$, the set $\left\{n : \sum_{k \in V(\epsilon)} a_{nk} \geq v\right\}$ belongs to I, where $V(\epsilon)$ is same as above, and we write $I\text{-}st_A \lim x_k = L$.*

Remark 1. *(a) If $I = I_{fin} = \{V \subseteq \mathbb{N} : V \text{ is finite}\}$, then A^I-statistical convergence becomes A-statistical convergence due to [3].*

(b) If $A = C_1$, then A^I-statistically convergent becomes I-statistical convergence due to [37] and we write $I\text{-}st \lim \mu_k = L$.

For related notions, see [5, 6, 18, 20, 25].

Definition 2. *Let $I \in$ Im, $A \in \Omega$ and $\mu = (\mu_k)$ be a real sequence. Then for some $v > 0$*

$$I\text{-}st_A \limsup \mu = \begin{cases} \sup G_\mu, & \text{if } G_\mu \neq \varnothing, \\ -\infty, & \text{if } G_\mu = \varnothing, \end{cases}$$

and

$$I\text{-}st_A \liminf \mu = \begin{cases} \inf H_\mu, & \text{if } H_\mu \neq \varnothing, \\ \infty, & \text{if } H_\mu = \varnothing, \end{cases}$$

where

$$G_\mu = \left\{ g \in \mathbb{R} : \left\{ n \in \mathbb{N} : \sum_{\{k : \mu_k > g\}} a_{nk} > v \right\} \notin \mathfrak{I} \right\},$$

and

$$H_\mu = \left\{ h \in \mathbb{R} : \left\{ n \in \mathbb{N} : \sum_{\{k : \mu_k < h\}} a_{nk} > v \right\} \notin \mathfrak{I} \right\}.$$

Definition 3. *Let $I \in$ Im and $A \in \Omega$. Then $\mu = (\mu_k)$ is called A^I-statistically bounded if for any $v > 0$,*

$$\left\{ n \in \mathbb{N} : \sum_{\{k : |\mu_k| > t\}} a_{nk} > v \right\} \in \mathfrak{I}, \quad t \in \mathbb{R}.$$

Remark 2. *(a) A^I-statistical boundedness $\Rightarrow$ $I\text{-}st_A \limsup \mu$ and $I\text{-}st_A \liminf \mu$ are finite.*

(b) If μ is A^I-statistically convergent then μ is A^I-statistically bounded.

Remark 3. *Throughout the paper, $I \in$ Im and $A \in \Omega$.*

2 A-statistical convergence and A-statistical Cauchy via ideal

We introduce the notion of A^{I^*}-statistical convergence, A^I-statistical Cauchy and A^{I^*}-statistical Cauchy and obtain some results. We study under what conditions A^I-statistical convergence (or Cauchy) and A^{I^*}-statistical convergence (or Cauchy) are equivalent.

Definition 4. *A sequence $\mu = (\mu_k)$ is A^{I^*}-statistically convergent to the number L if there exists $D = \{d_i\}_{i=1}^{\infty} \in F(I)$ such that $\delta_A(D) = 1$ and μ is A^D-statistically convergent to L, i.e. for every $\epsilon > 0$, $\lim_i \sum_{k\in V(\epsilon)} a_{d_i k} = 0$, and we write $I^*\text{-}st_A \lim \mu_k = L$.*

Remark 4. *If $A = C_1$, then A^{I^*}-statistical convergence becomes I^*-statistical converges due to [11].*

Now to show the equivalence between A^I-statistical convergence and A^{I^*}-statistical convergence, we need to define (APO) condition which is similar to the condition used in [4,15,24].

Definition 5. *Let $I \in \mathrm{Im}$ and $A \in \Omega$, then I is said to satisfy (APO) condition if for every sequence (B_n) of (pairwise disjoint) sets from I there exist sets $C_n \in I$, $n \in \mathbb{N}$, such that the symmetric difference $B_n \Delta C_n$ is finite for every n, $\bigcup\limits_n C_n \in I$, $\delta_A(\bigcup\limits_n C_n) = 0$.*

The following proposition is an analogous to [1, Proposition 1].

Proposition 1. *Let $I \in \mathrm{Im}$ and $A \in \Omega$, then I satisfies (APO) if and only if for every sequence (B_n) of (pairwise disjoint) sets from I there exists $B \in I$, such that $B_n \setminus B$ is finite for every n and $\delta_A(B) = 0$.*

Theorem 1. *(a) $I^*\text{-}st_A \lim \mu_k = L \Rightarrow I\text{-}st_A \lim \mu_k = L$.*
(b) $I\text{-}st_A \lim \mu_k = L \Rightarrow I^*\text{-}st_A \lim \mu_k = L$, provided I satisfies (APO).

Proof. (a) Let $I^*\text{-}st_A \lim \mu_k = L$. Then there exists $B \in I$ such that $D = \{d_i\} = \mathbb{N} \setminus B \in F(\mathcal{I})$, $\delta_A(D) = 1$, and $\forall\, \epsilon > 0$, $\lim_i \sum_{k\in V(\epsilon)} a_{d_i k} = 0$, where $V(\epsilon) = \{k \le n : |\mu_k - L| \ge \epsilon\}$. Therefore for each $\nu > 0$, there exists N such that $\frac{1}{N} < \nu$, so

$$E = \left\{ n : \sum_{k\in V(\epsilon)} a_{nk} \ge \nu \right\} \subseteq B \cup \{d_1, d_2, \ldots, d_N\}.$$

Since $B \in I$ and $\{d_1, d_2, \ldots, d_N\} \in I$, we have $E \in I$. Hence $I\text{-}st_A \lim \mu_k = L$.

(b) Let $I\text{-}st_A \lim \mu_k = L$. Then $\forall\, \epsilon > 0$ and for each $\nu > 0$, we have

$$\left\{ n : \sum_{k\in V(\epsilon)} a_{nk} \ge \nu \right\} \in I,$$

where $V(\epsilon)$ is same as mentioned above. Therefore, define the sequence (B_i) of sets as

$$B_i = \left\{ n : \sum_{k\in V(\epsilon)} a_{nk} \ge \frac{1}{i} \right\}, \quad i \in \mathbb{N}.$$

Since I satisfies the condition (APO) and each $B_i \in I$, by Proposition 1, there exists a set $B \in I$ such that $\delta_A(B) = 0$ and $B_i \setminus B$ is finite for each i. Let $E = \mathbb{N} \setminus B$. Then $\delta_A(E) = 1$. Now for any $\eta > 0$, there is $N \in \mathbb{N}$ such that $\frac{1}{N} < \eta$. Therefore

$$B_N = \left\{ n : \sum_{k\in V(\epsilon)} a_{nk} \ge \frac{1}{N} \right\} \in I.$$

Now let us define the set D as

$$D = \left\{ n : \sum_{k \in V(\epsilon)} a_{nk} < \frac{1}{N} \right\} \setminus B.$$

Since B_N, $B \in I$, we have $D \in F(I)$ and $\delta_A(D) = 1$. Hence we have

$$\sum_{k \in V(\epsilon)} a_{nk} < \eta, \quad \forall n > N,\ n \in D,$$

i.e.

$$\lim_n \sum_{k \in V(\epsilon)} a_{nk} = 0, \quad n \in D.$$

Hence $st_{A^D}\text{-}\lim_k \mu_k = L$, $\delta_A(D) = 1$, i.e. $\mathcal{I}^*\text{-}st_A \lim \mu_k = L$. □

Remark 5. *The converse of Theorem 1 (a) need not be true.*

Example 1. *Let* $B_m = \left\{2^{m-1}(2n-1) : n \in \mathbb{N}\right\}$ *and*

$$I = \left\{ B \subset \mathbb{N} : B \text{ intersects only finite numbers of } {B_m}'s \right\},$$

then $I \in \mathrm{Im}$. *Define* $\mu = (\mu_k)$ *as*

$$\mu_k = \frac{1}{m}, \quad k^2 \in B_m,$$

and $A = (a_{nk}) = C_1$. *Now for any* $\epsilon > 0$, *let* $V(\epsilon) = \{k : |\mu_k - 0| \geq \epsilon\}$, *therefore for any* $\nu > 0$, *we have*

$$\left\{ n : \sum_{k \in V(\epsilon)} a_{nk} \geq \nu \right\} \subseteq \left\{ n \in \mathbb{N} : \frac{\sqrt{n}}{n} \geq \nu \right\} \in I.$$

Hence μ *is* A^I*-statistically convergent to zero. Now we need to show that* μ *is not* A^{I^*}*-statistically convergent to zero. Suppose if it is possible that* μ *is* A^{I^*}*-statistically convergent to zero, then there exists a set* $D = \mathbb{N} \setminus C = \{d_i\} \in F(I)$, *where* $C \in I$, $\delta_A(D) = 1$ *and* $\lim_i \sum_{k \in V(\epsilon)} a_{d_i k} = 0$. *Since* $C \in I$, *then there exists* $t \in \mathbb{N}$ *such that* t *is odd and* $C \subseteq B_1 \cup B_2 \cup \ldots . \cup B_t$. *So* $B_{t+1} \subseteq D$. *Therefore* $\mu_{d_i} = \frac{1}{t+1}$ *for infinitely many* i*'s. Now let us choose* $\eta > 0$ *such that* $\eta < \frac{1}{t+1}$. *Hence,*

$$\delta_A \left\{ d_i \in B_{t+1} : \sum_{k \in V(\epsilon)} a_{d_i k} \geq \eta \right\} = \frac{1}{2^{t+1}} \neq 0,$$

i.e. $I^*\text{-}s_A \lim_k \mu_k \neq 0$, *a contradiction. Hence,* μ *is not* A^{I^*}*-statistically convergent to zero.*

Definition 6. *A sequence $\mu = (\mu_k)$ is A^I-statistical Cauchy if for any $\epsilon > 0$ and for each $\nu > 0$ there is $N_\epsilon \in \mathbb{N}$ such that*

$$\left\{n : \sum_{k \in K(\epsilon)} a_{nk} \geq \nu\right\} \in \mathcal{I},$$

where $K(\epsilon) = \{k \leq n : |\mu_k - \mu_{N_\epsilon}| \geq \epsilon\}$.

Definition 7. *A sequence $\mu = (\mu_k)$ is A^{I^*}-statistical Cauchy if there is a set $D = \{d_i\}_{i=1}^{\infty} \in F(I)$ such that $\delta_A(D) = 1$ and (μ_k) is A^D-statistical Cauchy in $\mathbb{R}$, i.e. for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $\lim_i \sum_{k \in K(\epsilon)} a_{d_i k} = 0$.*

Remark 6. *If $A = C_1$, then Definition 6 and Definition 7 reduce to I-statistical Cauchy and I^*-statistical Cauchy due to [13].*

Theorem 2. *A^{I^*}-statistical convergence $\Leftrightarrow$ A^{I^*}-statistical Cauchy.*

Proof. Let μ be A^{I^*}-statistically convergent to L. Then μ is A^D-statistically convergent to L and hence by [23, Theorem 2.2] we can deduce that by replacing the regularity of A by A^D that μ is A^D-statistically convergent if and only if μ is A^D-statistical Cauchy. □

Lemma 1. *A^I-statistical Cauchy implies A^I-statistically bounded.*

Proof. If μ is A^I-statistically Cauchy, then for any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$T(\nu) = \left\{n : \sum_{k \in F(\frac{\epsilon}{2})} a_{nk} \geq \nu\right\} \in I \ \text{ for every } \nu > 0,$$

where $F(\frac{\epsilon}{2}) = \left\{k \leq n : |\mu_k - \mu_{N_\epsilon}| \geq \frac{\epsilon}{2}\right\}$. Therefore

$$M(\nu) = \left\{n : \sum_{k \in G(\frac{\epsilon}{2})} a_{nk} \geq \nu\right\} \in F(I),$$

where $G(\frac{\epsilon}{2}) = \left\{k \leq n : |\mu_k - \mu_{N_\epsilon}| < \frac{\epsilon}{2}\right\}$. Let us define the set $E(\epsilon)$ as

$$E(\epsilon) = \{k : |\mu_k| < \epsilon + |\mu_m|\},$$

where $m \in \mathbb{N}$ satisfies $|\mu_m - \mu_{N_\epsilon}| < \frac{\epsilon}{2}$, such m exists because I is an admissible ideal, otherwise $T\left(\frac{1}{4}\right) = \mathbb{N} \notin I$. Now for any $a \in G\left(\frac{\epsilon}{2}\right)$, we have

$$|\mu_a - \mu_m| \leq |\mu_a - \mu_{N_\epsilon}| + |\mu_{N\epsilon} - \mu_m| < \epsilon.$$

Therefore

$$|\mu_a| \leq |\mu_a - \mu_m| + |\mu_m| < \epsilon + |\mu_m|,$$

hence $a \in E(\epsilon)$. So we have $G(\frac{\epsilon}{2}) \subseteq E(\epsilon)$, therefore for every $\nu > 0$,

$$M(\nu) \subseteq \left\{n : \sum_{k \in E(\epsilon)} a_{nk} \geq \nu\right\},$$

since $M(\nu) \in F(I)$, we have

$$\left\{n : \sum_{k \in E(\epsilon)} a_{nk} \geq \nu\right\} \in F(I).$$

Hence $\left\{n : \sum_{\{k : |\mu_k| \geq \epsilon + |\mu_m|\}} a_{nk} \geq \nu\right\} \in I$, i.e. μ is A^I-statistically bounded. □

Theorem 3. *A^I-statistical convergence $\Leftrightarrow$ A^I-statistical Cauchy.*

Proof. Let $I\text{-}st_A \lim_k \mu_k = L$. Then for any $\epsilon > 0$ and $\forall\, \nu > 0$, we have

$$B(\nu) = \left\{ n : \sum_{k \in K(\frac{\epsilon}{2})} a_{nk} \geq \nu \right\} \in I,$$

where $K(\frac{\epsilon}{2}) = \left\{ k \leq n : |\mu_k - L| \geq \frac{\epsilon}{2} \right\}$. Let us define $G(\epsilon)$ as

$$G(\epsilon) = \{ k \leq n : |\mu_k - \mu_N| \geq \epsilon \},$$

where $N \notin K\left(\frac{\epsilon}{2}\right)$, such N exists because I is an admissible ideal, otherwise $B\left(\frac{1}{3}\right) = \mathbb{N} \notin I$. Now for any $a \in G(\epsilon)$, we have

$$\epsilon \leq |\mu_a - \mu_N| \leq |\mu_a - L| + |\mu_N - L|.$$

Since $N \notin K\left(\frac{\epsilon}{2}\right)$, we have

$$|\mu_N - L| < \frac{\epsilon}{2},$$

therefore

$$|\mu_a - L| > \frac{\epsilon}{2}.$$

Hence $a \in K\left(\frac{\epsilon}{2}\right)$, and so we have $G(\epsilon) \subseteq K\left(\frac{\epsilon}{2}\right)$. Therefore for any $\nu > 0$, we have

$$\left\{ n : \sum_{k \in G(\epsilon)} a_{nk} \geq \nu \right\} \subseteq B(\nu) \in I.$$

Hence μ is A^I-statistically Cauchy.

Conversely, let μ be A^I-statistical Cauchy. Then by Lemma 1, we have μ is A^I-statistically bounded. Therefore $I\text{-}st_A \liminf \mu$ and $I\text{-}st_A\text{-} \limsup \mu$ are finite. Using [20, Theorem 3], we have $u = I\text{-}st_A \liminf \mu \leq I\text{-}st_A \limsup \mu = w$. Since μ is A^I-statistical Cauchy sequence, then for any $\epsilon > 0$, there exists $N_{\frac{\epsilon}{2}} \in \mathbb{N}$ such that for every $\nu > 0$ we have

$$\left\{ n : \sum_{\left\{k : \left|\mu_k - \mu_{N_{\frac{\epsilon}{2}}}\right| \geq \frac{\epsilon}{2}\right\}} a_{nk} \geq \nu \right\} \in I.$$

Therefore

$$\left\{ n : \sum_{\left\{k : \mu_k > \mu_{N_{\frac{\epsilon}{2}}} + \frac{\epsilon}{2}\right\}} a_{nk} \geq \nu \right\} \in I,$$

hence by the definition of $I\text{-}st_A \limsup \mu$ and [20, Theorem 1], we have

$$w < \mu_{N_{\frac{\epsilon}{2}}} + \frac{\epsilon}{2}. \tag{1}$$

Also, we have

$$\left\{ n : \sum_{\left\{k : \mu_k < \mu_{N_{\frac{\epsilon}{2}}} - \frac{\epsilon}{2}\right\}} a_{nk} \geq \nu \right\} \in I,$$

hence by the definition of $I\text{-}st_A \liminf \mu$ and [20, Theorem 2], we have

$$\mu_{N_{\frac{\epsilon}{2}}} < u + \frac{\epsilon}{2}. \tag{2}$$

Using equations (1) and (2), we have

$$w < u + \epsilon.$$

Hence, for any $\lambda > 0$, we always have $w < u + \lambda$, therefore $w \leq u$. Hence $u = I\text{-}st_A \liminf \mu = I\text{-}st_A \limsup \mu = w$. Now by [20, Theorem 4], we have μ is A^I-statistically convergent. □

Theorem 4. *(a) If $\mu = (\mu_k)$ is A^{I^*}-statistical Cauchy then μ is A^I-statistical Cauchy.*
(b) A^I-statistical Cauchy $\Rightarrow$ A^{I^*}-statistical Cauchy, if I satisfies (APO).

Proof. (a) It follows from Theorem 2, Theorem 1 (a) and Theorem 3.
(b) The proof follows from Theorem 3, Theorem 1 (b) and Theorem 2. □

Remark 7. *The converse of Theorem 4 (a) is not true in general.*

Example 2. *From Example 1, since μ is A^I-statistically convergent to zero but not A^{I^*}-statistically convergent then from Theorem 2 and Theorem 3 we get the result.*

Theorem 5. *A sequence $\mu = (\mu_k)$ is A^I-statistically convergent to a number $L \Leftrightarrow$ there exists a subset $M \subseteq \mathbb{N}$ such that $\{n : \sum_{k\in M} a_{nk} \geq \nu\} \in F(I)$ and $\lim_{j\in M} \mu_{\mathrm{j}} = L$.*

Proof. If μ is A^I-statistically convergent to L, then $\forall\, r \in \mathbb{N}$ and for every $\nu > 0$, we have

$$\left\{n : \sum_{k\in V(r)} a_{nk} \geq \nu\right\} \in I, \quad \text{where} \quad V(r) = \left\{k \leq n : |\mu_k - L| \geq \frac{1}{r}\right\}.$$

Therefore $\left\{n : \sum_{k\in M(r)} a_{nk} \geq \nu\right\} \in F(I)$, where $M(r) = \left\{k \leq n : |\mu_k - L| < \frac{1}{r}\right\}$. Now we need to show that there exists $M(i)$ such that the subsequence (μ_j), $j \in M(i)$, is convergent to L. Suppose that for any r, the subsequence (μ_j), $j \in M\,(r)$, is not convergent to L. So for each r, there exists $\epsilon_r > 0$ such that

$$\left|\mu_j - L\right| \geq \varepsilon_r$$

for infinitely many terms in $M\,(r)$. So, we have

$$G\,(\epsilon_r) = \left\{j \in M\,(r) : \left|\mu_j - L\right| \geq \epsilon_r\right\} \neq \varnothing.$$

For every ϵ_r there exists $t_r \in \mathbb{N}$ such that

$$\frac{1}{t_r} < \epsilon_r < \frac{1}{r}, \quad \frac{1}{t_1} > \frac{1}{t_2} > \cdots > \frac{1}{t_r} > \cdots.$$

Now we construct a sequence of closed intervals $\{I_r\}_{r\in\mathbb{N}}$, where $I_r = \left[\frac{1}{t_r}, \frac{1}{r}\right]$. Since $\{I_r\}$ satisfies nested Intervals Theorem, there is a number $\alpha \in I_r$ for every r. Hence α must satisfy

$$0 < \alpha < \frac{1}{r}, \quad \forall r,$$

which contradicts Archimedean Property. Hence there exists $M(i)$ such that the subsequence (μ_j), $j \in M(i)$, is convergent to L.

Conversely, suppose that there exists a set $M \subseteq \mathbb{N}$, such that

$$D = \left\{ n : \sum_{k \in M} a_{nk} \geq \nu \right\} \in F(I)$$

and $\lim_j \mu_j = L,\ j \in M$. Therefore for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left|\mu_j - L\right| < \epsilon, \quad \forall j \geq N,\ j \in M.$$

Let $\{j_1, j_2, \ldots, j_{N-1}\} = F \in M$. Then we have

$$T(\epsilon) = \{k \leq n : |\mu_k - L| < \epsilon\} \supseteq M \setminus F,$$

therefore

$$\left\{ n : \sum_{k \in T(\epsilon)} a_{nk} \geq \nu \right\} \supseteq \left\{ n : \sum_{k \in M \setminus F} a_{nk} \geq \nu \right\}.$$

Since $D \in F(I)$ and $D \setminus \{\text{finite set}\} \in F(I)$, we have $\{n : \sum_{k \in M \setminus F} a_{nk} \geq \nu\} \in F(I)$. Therefore

$$\left\{ n : \sum_{k \in T(\epsilon)} a_{nk} \geq \nu \right\} \in F(I),$$

and so $\left\{ n : \sum_{k \in V(\epsilon)} a_{nk} \geq \nu \right\} \in I$, where $V(\epsilon) = \{k \leq n : |\mu_k - L| \geq \epsilon\}$. Hence μ is A^I-statistically convergent to L. □

From Theorem 3 and Theorem 5 we have equivalent statements.

Theorem 6. *(a) μ is A^I-statistically convergent to L;*
(b) μ is A^I-statistical Cauchy;
(c) μ is such a sequence that there exists a subset $M \subseteq \mathbb{N}$ such that $\lim_{j \in M} \mu_j = L$ and

$$\left\{ n : \sum_{k \in M} a_{nk} \geq \nu \right\} \in F(I).$$

Recall that a real sequence $\mu = (\mu_k)$ is said to be A^{I^*}-summable to L if there is a set $B \in I$, such that $D = \mathbb{N} \setminus B = \{d_i\} \in F(I)$ and $\lim_i \sum_k a_{d_i k} \mu_k = L$ and we write $A^{I^*}\text{-}\lim \mu_k = L$ [10]. Also we say that μ is statistically A^{I^*}-summable to L if there is a set $D = \{d_i\} \in F(I)$ and $\delta(D) = 1$, such that $st\text{-}\lim_i \sum_k a_{d_i k} \mu_k = L$, and $(A^{I^*})_{st}\text{-}\lim \mu = L$ [11].

Theorem 7. *If μ is bounded then A^{I^*}-statistical convergence implies μ is A^{I^*}-summable and hence statistically A^{I^*}-summable.*

Proof. Let μ be bounded and A^{I^*}-statistical convergent to L. Then μ is A^D-statistically convergent to L, where $D \in F(I)$. Now use [12, Theorem 2.1] by replacing the regularity of A by A^D, we have μ is A^{I^*}-summable to L and μ is statistically A^{I^*}-summable to L, i.e. $(A^{I^*})_{st}\text{-}\lim \mu = L$. □

Remark 8. *The converse of Theorem 7 need not be true.*

Example 3. *Let I be the class defined in Example 1 and $A = C_1$. Let $\mu = (\mu_k)$ be defined by*

$$\mu_k = \begin{cases} 1, & \text{if } k \in B_1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\sum\limits_k a_{nk}\mu_k = \frac{1}{2}$.

Let us define $B = \{b_i \in B_2 : b_i \text{ is square}\}$. Then $B \in I$, so $D = \mathbb{N} \setminus B = \{d_i\} \in F(I)$ and $\delta(D) = 1$. Now $\lim_i \sum_k a_{d_i k}\mu_k = \frac{1}{2}$, so μ is A^{I^*}-summable to $\frac{1}{2}$ and $st\text{-}\lim_i \sum_k a_{d_i k}\mu_k = \frac{1}{2}$, i.e. μ is also statistically A^{I^*}-summable to $\frac{1}{2}$. Now we show that μ is not A^I-statistically convergent to any number and hence μ is not A^{I^*}-statistically convergent. Since for $\epsilon = \frac{1}{3}$ and for any $L \in \mathbb{R}$, the set $K(\frac{1}{3}) = \left\{k : |\mu_k - L| \geq \frac{1}{3}\right\}$ contains either B_1 (the set of odd) or the set of even or both. So $\sum_{k \in K(\frac{1}{3})} a_{nk} = \frac{1}{2}$ or 1. Therefore for $\nu = \frac{1}{3}$, we have

$$\left\{n : \sum_{k \in K(\frac{1}{3})} a_{nk} \geq \frac{1}{3}\right\} = \mathbb{N} \notin I,$$

since $I \in \mathrm{Im}$, we have μ is not A^I-statistically convergent and hence μ is not A^{I^*}-statistically convergent.

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Еделі О.Х., Мурсалін М. *Про A-статистичну збіжність та A-статистичну збіжність за Коші через ідеал* // Карпатські матем. публ. — 2022. — Т.14, №2. — С. 442–452.

У статті [Analysis 1985, **5** (4), 301–313], J.A. Fridy довів еквівалентне відношення між статистичною збіжністю та статистичною послідовністю Коші. У цій статті ми визначаємо A^{I^*}-статистичну збіжність та доводимо, що за певних умов вона еквівалентна до A^I-статистичної збіжності, що визначена у [Appl. Math. Lett. 2012, **25** (4), 733–738]. Більше того, ми визначаємо A^I- та A^{I^*}-статистичну послідовність Коші та знаходимо певне еквівалентне співвідношення з A^I- та A^{I^*}-статистичною збіжністю.

Ключові слова і фрази: I-збіжність, A^I-статистична збіжність, A^{I^*}-статистична збіжність, A^I-статистична збіжність за Коші, A^{I^*}-статистична збіжність за Коші.