On the Kantor product, II

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We describe the Kantor square (and Kantor product) of multiplications, extending the classification proposed in [J. Algebra Appl. 2017, **16** (9), 1750167]. Besides, we explicitly describe the Kantor square of some low dimensional algebras and give constructive methods for obtaining new transposed Poisson algebras and Poisson-Novikov algebras; and for classifying Poisson structures and commutative post-Lie structures on a given algebra.

Key words and phrases: Kantor product, Kantor square, non-associative algebra.

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Introduction

The idea of obtaining new objects from old ones by using derivative operations has long been known in algebra [1]. In its most general form, the idea was realized by A. Malcev [19]. Let M_n be an associative algebra of matrices of order n over a field $\mathbb{F}$. Assume that some finite collection $\Lambda = (a_{ij}, b_{ij}, c_{ij})$ of matrices in M_n is given. Denote by $M_n^{(\Lambda)}$ an algebra defined on a space of matrices in M_n with respect to new multiplication $x \cdot_\Lambda y = \sum_{i,j} a_{ij} x b_{ij} y c_{ij}$. It was proved that every n-dimensional algebra over $\mathbb{F}$ is isomorphic to a subalgebra of $M_n^{(\Lambda)}$ [19]. Other interesting ways to derive the initial multiplication are isotopes, homotopes and mutations [2,7,9,20]. The concept of an isotope was introduced by A.A. Albert [1]. Let algebras A and A_0 have a common linear space on which right multiplication operators R_x and $R_x^{(0)}$ are defined (for A and A_0, resp.). We say that A_0 and A are isotopic if there exist invertible linear operators ϕ, ψ, ξ such that $R_x^{(0)} = \phi R_{x\psi} \xi$. We call A_0 an isotope of A. Let A be an arbitrary associative algebra, and let p, q be two fixed elements of A. Then a new algebra is derived from A by using the same vector space structure of A but defining a new multiplication $x * y = xpy - yqx$. The resulting algebra is called the (p, q)-mutation of the algebra A.

The definition of the Kantor product of multiplications comes from the study of certain class of algebras. In 1972, I.L. Kantor introduced the class of conservative algebras [12], that contains many important classes of algebras (see [15]), for example, associative, Lie, Jordan and Leibniz algebras. To define what will be called Kantor product, we need to introduce the algebras $U(n)$ (see, for more details, [13,15]). Consider the space $U(n)$ of all bilinear multiplications on the n-dimensional vector space V_n. Now, fix a vector $u \in V_n$. For $A, B \in U(n)$ (two

multiplications) and $x, y \in V_n$, we set

$$x * y = [\![A, B]\!](x, y) := A(u, B(x, y)) - B(A(u, x), y) - B(x, A(u, y)).$$

This new multiplication is called the (left) Kantor product of the multiplications A and B (it is possible to define the right Kantor product). The Kantor product of a multiplication "$\cdot$" by itself will be the Kantor square of "$\cdot$":

$$x * y := u \cdot (x \cdot y) - (u \cdot x) \cdot y - x \cdot (u \cdot y).$$

It is easy to see that the Kantor square of a multiplication is a particular case of the Malcev construction in non-associative sense. On the other side,

1) in commutative associative case it coincides with a mutation,

2) in left commutative and left symmetric cases it coincides with an isotope.

As in [14], we will assume that the Kantor product is always the left Kantor product. In [14], it was studied the Kantor product and Kantor square of many well known algebras, for example, associative, (anti)-commutative, Lie, Leibniz, Novikov, dialgebras, Poisson, and obtained some results about derivations, automorphisms, ideals and nilpotent algebras.

In this paper, we will continue studying the Kantor product of multiplications for some other algebras. Besides, we compute the Kantor square of some finite dimensional algebras and we relate some cases with the ones studied in [14]. In the last section, we give a constructive method for classifying Poisson structures on a given algebra. Throughout this manuscript, algebras with product "$\cdot$" will have always the product written as $x \cdot y := xy$.

If we are considering an algebra A with product "$\bullet$", we will use the following notations:

$$As(x, y, z)_\bullet = (x \bullet y) \bullet z - x \bullet (y \bullet z),$$
$$J(x, y, z)_\bullet = (x \bullet y) \bullet z + (z \bullet x) \bullet y + (y \bullet z) \bullet x,$$
$$x \circ y = x \bullet y + y \bullet x.$$

Also, for an algebra **A** we will write $w \approx v$ if $w - v \in \text{Ann } \mathbf{A}$, where Ann **A** is the annihilator of **A**. To avoid some discussion during this manuscript, we consider that all algebras are over a field $\mathbb{F}$ of characteristic zero. We highlight that many of this results hold in positive characteristic.

1   Kantor square

In this section we will consider variety of algebras with a single multiplication and we will study the Kantor square in this cases.

1.1   Middle-commutative algebras

The variety of middle-commutative algebras (or reverse algebras) is defined by the identity $(xy)z = z(yx)$. This nomenclature appeared in [17] (see also [21]), in counterpart to the definition of left and right commutative algebras. Note that the variety of middle-commutative algebras contains the variety of (anti)-commutative algebras and it is contained in the variety of flexible algebras.

Proposition 1. *Let $(A, \cdot)$ be a middle-commutative algebra. Then $(A, *)$ is a middle-commutative algebra.*

Proof. We have

$$(x * y) * z = u((u(xy))z) - u(((ux)y)z) - u((x(uy))z) \tag{L1}$$
$$- (u(u(xy)))z + (u((ux)y))z + (u(x(uy)))z \tag{L2}$$
$$- (u(xy))(uz) + ((ux)y)(uz) + (x(uy))(uz), \tag{L3}$$
$$z * (y * x) = u(z(u(yx))) - u(z((uy)x)) - u(z(y(ux))) \tag{l1}$$
$$- (uz)(u(yx)) + (uz)((uy)x) + (uz)(y(ux)) \tag{l2}$$
$$- z(u(u(yx))) + z(u((uy)x)) + z(u(y(ux))). \tag{l3}$$

Computing $(x * y) * z - z * (y * x)$, we can see the following cancelations: (L1) with (l1), (L2) with (l3) and (L3) with (l2). Therefore, $(A, *)$ is a middle-commutative algebra. $\square$

1.2 Pseudo-flexible algebras

The variety of pseudo-flexible algebras is defined by

$$x(xy) = (yx)x$$

(see [17]). Note that the variety of pseudo-flexible algebras contains the variety of middle-commutative algebras.

Proposition 2. *Let $(A, \cdot)$ be a pseudo-flexible algebra. Then $(A, *)$ is a pseudo-flexible algebra.*

Proof. The proof follows the ideas in Proposition 1. $\square$

1.3 Weakly associative algebras

The variety of weakly associative algebras is defined by

$$As(x, y, z) + As(y, z, x) = As(y, x, z)$$

(see, for example, [22]). It contains commutative algebras, associative algebras, Lie algebras and symmetric Leibniz algebras as subvarieties. A direct computation shows that the following assertion is valid.

Proposition 3. *Let $(A, \cdot)$ be a weakly associative algebra. Then $(A, *)$ is a commutative algebra if and only if $(xu)y = y(ux)$ for all $x, y \in A$. In particular, $(A, *)$ is a commutative algebra if $(A, \cdot)$ is middle-commutative. Besides, $(A, *)$ is an anticommutative algebra if and only if $As(x, y, u) + As(y, x, u) = (xu)y + (yu)x$ for all $x, y \in A$.*

Besides, note that if $(A, \cdot)$ is a weakly associative algebra, it holds that

$$x * y = u(xy) - (ux)y - x(uy) = (xy)u - x(yu) - (xu)y,$$

that is, the weakly associative algebras are the algebras that the left Kantor square is equal to the right Kantor square. A consequence of this is the following theorem, that will give us the analogous results of some algebras considered in [14, Lemmas 5, 6, 8 and 9].

Proposition 4. *Let $(A, \cdot)$ be a weakly associative algebra.*

(a) If $(A, \cdot)$ is a right commutative algebra, then $(A, *)$ is a right-commutative algebra.

(b) If $(A, \cdot)$ is a right Leibniz algebra, then $* = 0$.

(c) If $(A, \cdot)$ is a right Zinbiel algebra, then $(A, *)$ is a right Zinbiel algebra.

(d) If $(A, \cdot)$ is a right Novikov algebra, then $(A, *)$ is a right Novikov algebra.

A direct consequence of [22], Proposition 1, is the following result.

Proposition 5. *Let $(A, \cdot)$ be an algebra such that $(A, *)$ is a weakly associative algebra. Then, $L_{xu} - R_{ux}$ is a derivation of $(A, \cdot)$ for all $u, x \in A$.*

1.4   Anti-associative algebra

The variety of anti-associative algebras (see, for example, [23]) is defined by

$$(xy)z = -x(yz).$$

Note that an anti-associative algebra is a nilpotent algebra of nilpotency index 4. A direct computation (or using [14], Lemma 25) shows that the next proposition is valid.

Proposition 6. *Let $(A, \cdot)$ be an anti-associative algebra. Then $(A, *)$ is a nilpotent algebra of nilpotency index at most 3, and therefore, is anti-associative.*

1.5   Quasi-commutative associative algebras

The variety of quasi-commutative associative algebras is defined by the relations

$$(xy)z = z(yx) \quad \text{and} \quad As(x, y, z) = 0.$$

Based on the work of M. Rais Khan (see [21]) these algebras were considered in [17] as a generalization of associative-commutative algebras.

Proposition 7. *Let $(A, \cdot)$ be a quasi-commutative associative algebra. Then $(A, *)$ is an associative-commutative algebra.*

Proof. First, by [14, Lemma 1], we know that $As(x, y, z)_* = 0$. Besides, note that by associativity and middle-commutativity we obtain

$$x * y = -x(uy) = -(yu)x = -y(ux) = y * x.$$

$\square$

1.6 Quasi-commutative alternative algebras

The variety of quasi-commutative alternative algebras is defined by the identities

$$(xy)z = z(yx), \quad (x,y,z) = -(y,x,z) = (y,z,x).$$

The variety of quasi-commutative Jordan algebras is defined by the identities

$$(xy)z = z(yx), \quad x^2(yx) = (x^2y)x.$$

This class of algebras appeared, for example, in [21]. The variety of quasi-commutative Jordan algebras contains (properly) the variety of Jordan algebras and it is contained properly in the variety of noncommutative Jordan algebras (see [21]).

Proposition 8. *Let* $(A, \cdot)$ *be a quasi-commutative alternative algebra. Then* $(A, *)$ *is a quasi-commutative Jordan algebra.*

Proof. First, by Proposition 1, we have $(x * y) * z = z * (y * x)$. It was already computed the Jordan property for alternative algebras in [14, Theorem 10]. Since middle-commutativity is stronger than the flexible property, we obtain

$$(x * x) * (y * x) - ((x * x) * y) * x = 2(((xuxu)y)(ux) - (xuxu)(y(ux))).$$

By middle-commutativity, $xuxu = uxux$. Since each 2-generated alternative algebra is associative, we have

$$(x * x) * (y * x) - ((x * x) * y) * x = 2As((xu)^2, y, ux) = 2As((ux)^2, y, ux) = 0.$$

$\square$

1.7 Mock-Lie algebras

The variety of mock-Lie algebras (or Jacobi-Jordan algebras; see [25]) is defined by the relations $xy = yx$ and $J(x, y, z) = 0$. Note that a mock-Lie algebra is a Jordan algebra.

Proposition 9. *Let* $(A, \cdot)$ *be a mock-Lie algebra. Then:*

  (a) $(A, *)$ *is a commutative algebra such that holds* Ann*-equality (2);*

  (b) $(A, *)$ *is a mock-Lie algebra if, and only if,* $(((xy)u)z) + (((zx)u)z) + (((yz)u)x) \approx 0;$

  (c) $(A, *)$ *is a Jordan algebra if, and only if,* $(((x^2u)y)u)x \approx (x^2u)((xy)u).$

Proof. By direct computation we obtain $x * y = 2(xy)u$, and thus $(x * y) * z = 4(((xy)u)z)u$. Therefore

$$J(x, y, z)_* = 4[(((xy)u)z) + (((zx)u)y) + (((yz)u)x)]u \tag{1}$$

and we obtain *(b)*. Since $J(x, y, z) = 0$, we can apply it in each term of equation (1) to obtain $J(x, y, z)_* = 4[(xy)(uz) + (zx)(uz) + (yz)(ux)]u$. Therefore,

$$(((xy)u)z) + (((zx)u)y) + (((yz)u)x) \approx (xy)(uz) + (zx)(uy) + (yz)(ux). \tag{2}$$

A direct computation shows item *(c)*.

$\square$

1.8   Almost-Lie algebras-1

The variety of almost-Lie algebras is defined (see [17]) by the identities

$$(xy)z = z(yx), \quad J(x,y,z) = 0.$$

Since (anti)-commutativity implies in the first property, the variety of almost-Lie algebras contains the variety of Lie and Mock-Lie algebras.

Proposition 10. *Let* $(A,\cdot)$ *be an almost-Lie algebra. Then* $(A,*)$ *is a commutative algebra. Besides,* $(A,*)$ *is a Mock-Lie algebra (see Section 1.7) if and only if*

$$(u(x \circ y))z + (u(x \circ z))y + (u(y \circ z))x \approx 0.$$

Proof. We have that

$$x * y = u(xy) - (ux)y - x(uy) = u(xy) + (xy)u = u(xy + yx) = y * x.$$

To complete the proof, note that the last equality implies that

$$J_*(x,y,z) = 2u[(u(xy + yx))z + (u(xz + zx))y + (u(yz + zy))x].$$

$\square$

1.9   Almost-Lie algebras-2

The variety of almost-Lie algebras is defined (see [16]) by the identities

$$xy = -yx, \quad J(x,y,z)t = 0.$$

The variety of almost-Lie algebras is formed by anticommutative central extensions of Lie algebras. The most interesting subvariety of the variety of almost-Lie algebras is the variety of anticommutative $\mathfrak{CD}$-algebras, which is defined by a common property of Lie and Jordan algebras: *every commutator for two right multiplications gives a derivation* [16]. For an anticommutative algebra $\mathfrak{L}$ we denote the set of all anticommutative k-dimensional central extensions as $\mathfrak{cent}(\mathfrak{L})$.

Proposition 11. *Let* $(A,\cdot)$ *be an almost-Lie algebra. Then* $(A,*)$ *is a 2-step nilpotent anticommutative algebra. In particular, for each Lie algebra* $\mathfrak{L}$*, we have*

$$(\mathfrak{cent}_k(\mathfrak{L}),*) \subseteq \mathfrak{cent}_k(\mathfrak{L},*).$$

Proof. We have that

$$(x * y) * z = (u(xy) - (ux)y - x(uy)) * z = -J(x,y,u) * z =$$
$$- (u(J(x,y,u)z) - (uJ(x,y,u))z - J(x,y,u)(uz)) = 0.$$

$\square$

1.10   Two-sided Leibniz algebras

The variety of two-sided Leibniz algebras [6] is defined by the identities

$$(xy)z = z(yx), \quad J(x,y,z) = 0, \quad (xy + yx)z = 0.$$

Since a two-sided Leibniz algebra is an almost-Lie algebra, we have the following result.

Proposition 12. *Let* $(A,\cdot)$ *be a two-sided Leibniz algebra. Then* $* = 0$.

1.11   CL- and CB-algebras

Remember that the centralizer of an element x in an algebra A is the set

$$C_A(x) = \{y \in A \; : \; xy = yx = 0\}.$$

An algebra A is a CL-algebra if every centralizer in A is an ideal of A (see, for example, [23]). Note that an associative-commutative algebra is a CL-algebra.

Proposition 13.   *(a) Let $(A, \cdot)$ be an associative-commutative algebra. Then, $(A, *)$ is an associative-commutative algebra and therefore a CL-algebra.*

(b) Let $(A, \cdot)$ be a CL-algebra. If $y \in C_A(x)$ then $yu \in C_A(x)$. Moreover, if $y \in C_A(ux)$ or $x \in C_A(x)$ then $y \in (C_A(x), *)$.

Now, let us consider CB-algebras.

Definition 1. *Let A be an algebra. We say that elements $x, y \in A$ (or the pair (x, y)) have commutative bonding (CB) if $xy = 0$ implies that $(xz)y = 0$ for all $z \in A$.*

An algebra A is a CB-algebra if every pair of elements of A have commutative bonding (see, for example, [23]). Note that, for example, right-commutative algebras are CB-algebras. In [23], it was proven that if A is an anticommutative algebra, than A is a CB-algebra if and only if it is an anti-associative algebra. Besides, by [23, Theorem 3.9], if A is an anticommutative algebra, then A is a CL-algebra if and only if A is a CB-algebra.

Proposition 14. *Let $(A, \cdot)$ be an anticommutative CB-algebra (CL-algebra). Then $(A, *)$ is an anticommutative CB-algebra (CL-algebra).*

Besides, we have the following result.

Proposition 15.   *(a) $(A, *)$ is a CB-algebra for any algebra $(A, \cdot)$ such that $(A, *)$ is right-commutative. In particular, if $(A, \cdot)$ is a weak-associative and right-commutative algebra, then $(A, *)$ is a right-commutative and, therefore, $(A, *)$ is a CB-algebra.*

(b) Let $(A, \cdot)$ be an associative CB-algebra (CL-algebra). Then $(A, *)$ is an associative CB-algebra (CL-algebra).

Proof. Item *(a)* is a consequence of Proposition 4. To prove item *(b)*, first note that associative property implies that if $x * y = 0$ then $xuy = 0$ for all $u \in A$. Therefore, if $x * y = 0$, we have $(x * y) * z = xuzuy = 0$ for all $u, z \in A$. In the case of CL-algebras, we need to show that $C_A(x)_* = \{a \in A; a * x = x * a = 0\}$ is an ideal of $(A, *)$. But the proof follows the same arguments. $\square$

1.12   Left-symmetric algebras

The variety of left-symmetric algebras (or left pre-Lie algebras) is defined (see [17]) by $As(x, y, z) = As(y, x, z)$. It contains, for example, the variety of right Novikov algebras.

Proposition 16. *Let $(A, \cdot)$ be a left-symmetric algebra. Then $(A, *)$ is a left-symmetric algebra if and only if $As(y, xu, u) \approx As(x, yu, u)$.*

Proof. Note that

$$x * y = u(xy) - (ux)y - x(uy) = x(uy) - (xu)y - x(uy) = -(xu)y.$$

Then,

$$\begin{aligned} As(x,y,z)_* &= (x*y)*z - x*(y*z) \\ &= (-(xu)y)*z - x*(-(yu)z) = (((xu)y)u)z - (xu)((yu)z). \end{aligned}$$

Now, we have

$$\begin{aligned} As(x,y,z)_* - As(y,x,z)_* &= (((xu)y)u)z - (xu)((yu)z) - (((yu)x)u)z + (yu)((xu)z) \\ &= (((xu)y - (yu)x)u)z + ((yu)(xu) - (xu)(yu))z \\ &= (((xu)y)u - ((yu)x)u + (yu)(xu) - (xu)(yu))z \\ &= (As(xu,y,u) - As(yu,x,u))z = (As(y,xu,u) - As(x,yu,u))z. \end{aligned}$$

The result follows.   □

1.13   Poisson algebras

A Poisson algebra $(A, \cdot, \{,\})$ is an algebra with two multiplications such that $(A, \cdot)$ is an associative-commutative algebra, $(A, \{,\})$ is a Lie algebra and they satisfy the following compatibility condition

$$\{x, yz\} = \{x, y\}z + y\{x, z\} \quad \text{(Leibniz rule)}.$$

Let $(A, \cdot, \{,\})$ be a Poisson algebra. If we consider the Kantor product of this two multiplications, it was proven in [14] that $[\![\{,\}, \cdot]\!] = 0$ and $(A, [\![\cdot, \{,\}]\!])$ is a Lie algebra. On the other hand, it is known that if we define the new multiplication $\circ = \cdot + \{,\}$ then $(A, \circ)$ is a noncommutative Jordan algebra.

Proposition 17. *Let $(A, \circ)$ be as before. Then $(A, *)$ is a noncommutative Jordan algebra.*

Proof. First of all, let us compute $x * y$ for any $x, y \in A$. Using all the properties of $(A, \cdot, \{,\})$ (except commutativity of $\cdot$) we obtain

$$\begin{aligned} x * y &= u \circ (x \circ y) - (u \circ x) \circ y - x \circ (u \circ y) \\ &= u(xy + \{x, y\}) + \{u, xy + \{x, y\}\} - (ux + \{u, x\})y \\ &\quad - \{ux + \{u, x\}, y\} - x(uy + \{u, y\}) - \{x, uy + \{u, y\}\} \\ &= u\{x, y\} - \{ux, y\} - x(uy) - \{x, uy\} = \{y, u\}x + u\{y, x\} - \{x, u\}y - xuy. \end{aligned}$$

By the last equation, we see that $(A, *)$, in general, is neither an (anti)-commutative algebra nor middle-commutative.

Let us now verify that $*$ is flexible. Note that

$$x * y + y * x = -xuy - yux = -2xyu. \tag{3}$$

Then,

$$\begin{aligned} (x*y)*x - x*(y*x) &= -x*(x*y) - 2xu(x*y) - x*(-x*y - 2xuy) \\ &= -2xu(x*y) + 2x*(xuy) \\ &= -2xu(\{y, u\}x + u\{y, x\} - \{x, u\}y - xuy) \\ &\quad + 2(\{xuy, u\}x + u\{xuy, x\} - \{x, u\}xuy - xuxuy) \\ &= -2xu(\{y, u\}x + u\{y, x\}) + 2(\{xuy, u\}x + u\{xuy, x\}) \\ &= -2(xy\{u, xu\} + uy\{x, xu\}) = 0. \end{aligned}$$

Now, let us check that $*$ satisfy the Jordan identity. By (3) we have $x * x = -x^2 u$. Therefore

$$(x * x)*(y * x)-((x * x)* y)* x = -(x^2 u) * (y * x) + (x^2 u * y) * x$$
$$= -\{y * x, u\}x^2 u - u\{y * x, x^2 u\} + \{x^2 u, u\}y * x + x^2 u^2 (y * x)$$
$$+ (\{y, u\}x^2 u + u\{y, x^2 u\} - \{x^2 u, u\}y - x^2 u^2 y) * x$$
$$= -2x^2 u\{y * x, u\} - 2xu^2\{y * x, x\} + \{x^2 u, u\}y * x + x^2 u^2 (y * x)$$
$$+ (2\{y, u\}x^2 u + 2u^2 x\{y, x\} + 2xyu\{u, x\} - x^2 u^2 y) * x.$$

Expanding the last expression we obtain the following

$$\begin{aligned}
&- 2x^2 u\{\{x, u\}y + u\{x, y\} - \{y, u\}x - xuy, u\}\\
&- 2xu^2\{\{x, u\}y + u\{x, y\} - \{y, u\}x - xuy, x\}\\
&+ \{x^2 u, u\}(\{x, u\}y + u\{x, y\} - \{y, u\}x - xuy)\\
&+ x^2 u^2(\{x, u\}y + u\{x, y\} - \{y, u\}x - xuy)\\
&+ \{x, u\}(2\{y, u\}x^2 u + 2u^2 x\{y, x\} + 2xyu\{u, x\} - x^2 u^2 y)\\
&+ u\{x, 2\{y, u\}x^2 u + 2u^2 x\{y, x\} + 2xyu\{u, x\} - x^2 u^2 y\}\\
&- \{2\{y, u\}x^2 u + 2u^2 x\{y, x\} + 2xyu\{u, x\} - x^2 u^2 y, u\}x\\
&- xu(2\{y, u\}x^2 u + 2u^2 x\{y, x\} + 2xyu\{u, x\} - x^2 u^2 y).
\end{aligned}\tag{4}$$

Now, let us compute separately the terms of the form $\{,\}$ (and without $\{,\}$), $\{\}\{\}$ and $\{,\{\}\}$, since they have no relation in A. In each case, we will show that these respective sums must be zero.

• Terms of the form $\{,\}$ and without $\{,\}$.

Collecting the respective terms in the last expression we obtain

$$\begin{aligned}
&- 2x^2 u\{u, xuy\} - 2xu^2\{x, xuy\} + xuy\{u, x^2 u\}\\
&+ x^2 u^2(\{x, u\}y + u\{x, y\} - \{y, u\}x - xuy)\\
&+ \{x, u\}(-x^2 u^2 y) - u\{x, x^2 u^2 y\} - \{u, x^2 u^2 y\}x\\
&- xu(2\{y, u\}x^2 u + 2u^2 x\{y, x\} + 2xyu\{u, x\} - x^2 u^2 y)\\
&= -2x^2 u^2(y\{u, x\} + x\{u, y\}) - 2x^2 u^2(y\{x, u\} + u\{x, y\})\\
&+ 2x^2 u^2 y\{u, x\} + x^2 u^2 y\{x, u\} + x^2 u^3\{x, y\} - x^3 u^2\{y, u\}\\
&- x^3 u^3 y - x^2 u^2 y\{x, u\} - ux^2(2uy\{x, u\} + u^2\{x, y\})\\
&- u^2 x(2xy\{u, x\} + x^2\{u, y\}) - 2x^3 u^2\{y, u\}\\
&- 2u^3 x^2\{y, x\} - 2x^2 yu^2\{u, x\} + x^3 u^3 y = 0.
\end{aligned}$$

Now, we consider the remaining terms in (4), that are

$$\begin{aligned}
&2x^2 u(\{u, \{x, u\}y\} + \{u, u\{x, y\}\} - \{u, \{y, u\}x\})\\
&+ 2xu^2(\{x, \{x, u\}y\} + \{x, u\{x, y\}\} - \{x, \{y, u\}x\})\\
&- 2xu\{u, x\}(\{x, u\}y + u\{x, y\} - \{y, u\}x)\\
&+ \{x, u\}(2\{y, u\}x^2 u + 2u^2 x\{y, x\} + 2xyu\{u, x\})\\
&+ u(\{x, 2\{y, u\}x^2 u\} + \{x, 2u^2 x\{y, x\}\} + \{x, 2xyu\{u, x\}\})\\
&+ x(\{u, 2\{y, u\}x^2 u\} + \{u, 2u^2 x\{y, x\}\} + \{u, 2xyu\{u, x\}\}).
\end{aligned}$$

Since $(A, \{,\})$ is a Lie algebra and using Leibniz rule, we obtain

$2x^2 u(y\{u, \{x,u\}\} + \{x,u\}\{u,y\} + u\{u, \{x,y\}\} - x\{u, \{y,u\}\} - \{y,u\}\{u,x\})$
$+ 2xu^2(y\{x, \{x,u\}\} + \{x,u\}\{x,y\} + u\{x, \{x,y\}\} + \{x,y\}\{x,u\} - x\{x, \{y,u\}\})$
$- 2xu\{u,x\}(\{x,u\}y + u\{x,y\} - \{y,u\}x) + \{x,u\}(2\{y,u\}x^2 u + 2u^2 x\{y,x\} + 2xyu\{u,x\})$
$+ 2x^2 u^2\{x, \{y,u\}\} + 2u\{y,u\}\{x, x^2 u\} + 2u^3 x\{x, \{y,x\}\} + 2u\{y,x\}\{x, u^2 x\}$
$+ 2xyu^2\{x, \{u,x\}\} + 2u\{u,x\}\{x, xyu\}2x^3 u\{u, \{y,u\}\} + 2x\{y,u\}\{u, x^2 u\} + 2u^2 x^2\{u, \{y,x\}\}$
$+ 2x\{y,x\}\{u, u^2 x\} + 2x^2 yu\{u, \{u,x\}\} + 2x\{u,x\}\{u, xyu\}.$

- Terms of the form $\{\}\{\}$.

$\quad 2x^2 u(\{x,u\}\{u,y\} - \{y,u\}\{u,x\}) + 2xu^2(\{x,u\}\{x,y\} + \{x,y\}\{x,u\})$
$\quad - 2xu\{u,x\}(\{x,u\}y + u\{x,y\} - \{y,u\}x)$
$\quad + \{x,u\}(2\{y,u\}x^2 u + 2u^2 x\{y,x\} + 2xyu\{u,x\})$
$\quad + 2u\{y,u\}\{x, x^2 u\} + 2u\{y,x\}\{x, u^2 x\} + 2u\{u,x\}\{x, xyu\} + 2x\{y,u\}\{u, x^2 u\}$
$\quad + 2x\{y,x\}\{u, u^2 x\} + 2x\{u,x\}\{u, xyu\}$
$\quad = 4xu^2\{x,u\}\{x,y\} - 2xu\{u,x\}(\{x,u\}y + u\{x,y\} - \{y,u\}x)$
$\quad + \{x,u\}(2\{y,u\}x^2 u + 2u^2 x\{y,x\} + 2xyu\{u,x\}) + 2ux^2\{y,u\}\{x,u\}$
$\quad + 4u^2 x\{y,x\}\{x,u\} + 2ux\{u,x\}(y\{x,u\} + u\{x,y\}) + 4x^2 u\{y,u\}\{u,x\}$
$\quad + 2xu^2\{y,x\}\{u,x\} + 2xu\{u,x\}(x\{u,y\} + y\{u,x\}) = 0.$

- Terms of the form $\{, \{\}\}$.

$2x^2 u(y\{u, \{x,u\}\} + u\{u, \{x,y\}\} - x\{u, \{y,u\}\})$
$+ 2xu^2(y\{x, \{x,u\}\} + u\{x, \{x,y\}\} - x\{x, \{y,u\}\}) + 2x^2 u^2\{x, \{y,u\}\} + 2u^3 x\{x, \{y,x\}\}$
$+ 2xyu^2\{x, \{u,x\}\} + 2x^3 u\{u, \{y,u\}\} + 2u^2 x^2\{u, \{y,x\}\} + 2x^2 yu\{u, \{u,x\}\} = 0.$

Therefore, $(A, *)$ is a noncommutative Jordan algebra. $\qquad\qquad\square$

1.14   On solvable algebras

Let $(A, \cdot)$ be a solvable algebra of solvability index $s > 0$. For example, a metabelian algebra is a solvable algebra of solvability index 2. We have:

Proposition 18. $(A, *)$ *is a solvable algebra of solvability index at most s.*

Proof. Since $x * y = u(xy) - (ux)y - x(uy)$, the result follows direct from an induction argument. $\qquad\qquad\square$

1.15   On the nucleus of an algebra

The nucleus of an algebra $(A, \cdot)$, that we will denote by $N(A, \cdot)$, is the set of elements $n \in (A, \cdot)$ such that
$$As(n, a, b) = As(a, n, b) = As(a, b, n) = 0$$
for all $a, b \in (A, \cdot)$.

We refer the reader to [4].

Proposition 19. *Let $(A, \cdot)$ be an algebra. Then we have:*

(a) *if $n, u \in N(A, \cdot)$, then $n \in N(A, *)$,*

(b) *if $n, nu, un \in N(A, \cdot)$, then $n \in N(A, *)$.*

Proof. First note that item *(b)* will follows the same ideas as item *(a)*, but with less steps. For item *(a)*, since $u \in N(A, \cdot)$, it follows that $x * y = -(xu)y$ for all $x, y \in A$. Therefore,

$$(n * y) * z - n * (y * z) = (((nu)y)u)z - (nu)((yu)z) = ((nu)y)(uz) - (nu)(y(uz))$$
$$= (n(uy))(uz) - (nu)(y(uz)) = n((uy)(uz)) - n(u(y(uz))) = 0.$$

The other cases are treated in the same way. This ends the proof. □

1.16   On algebras with involution

Let $(A, \cdot, \star)$ an algebra with involution $\star$. Then, we have the following result.

Proposition 20. *If u is self-adjoint (that is, $u^\star = u$) and is in the center of $(A, \cdot)$, then $\star$ is an involution of $(A, *)$.*

2   Kantor square of low dimensional algebras

In this section, we will study the Kantor square of some low dimensional algebras, computing explicitly them.

Remark 1. *Since all algebras $(A, \cdot)$ considered in this section are commutative or anticommutative, we have that $(A, *)$ will be also a commutative or anticommutative algebra, respectively. When we describe the multiplication from the basis of an algebra, the zero products will be omitted. Besides, since all computations in this section are standard, all of them will be omitted. The notation of algebras will be the same as in the cited papers.*

2.1   3-dimensional Jordan algebras

The Kantor square of an associative-commutative algebra is an associative-commutative (in this case, $x * y = -uxy$). In the case of non-associative Jordan algebras we do not have the same behaviour. For this purpose, consider the 3-dimensional Jordan algebra $\mathbb{T}_{02}^{US}$ given in [10] (using the same notation) by

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_3^2 = e_1 + e_2, \quad e_1 e_3 = e_2 e_3 = \frac{1}{2} e_3.$$

Write $u = u_1 e_1 + u_2 e_2 + u_3 e_3$, where $\{e_1, e_2, e_3\}$ is a basis of $\mathbb{T}_{02}^{US}$. Then, $(\mathbb{T}_{02}^{US}, *)$ is given by

$$e_1 * e_1 = -u_1 e_1, \quad e_2 * e_2 = -u_2 e_2, \quad e_3 * e_3 = -u_2 e_1 - u_1 e_2 - u_3 e_3,$$

$$e_1 * e_3 = -\frac{u_1}{2} e_3 - u_3 e_1, \quad e_2 * e_3 = -\frac{u_2}{2} e_3 - u_3 e_2.$$

A standard computation shows that $(\mathbb{T}_{02}^{US}, *)$ is a Jordan algebra if and only if $u_1 = u_2 = 0$ or $u_3 = 0$.

In this case, we have:

1) if $u_1 = u_2 = 0$ and $u_3 \neq 0$, then $(\mathbb{T}_{02}^{US}, *) \cong (\mathbb{T}_{08}^{AU}, \cdot)$,

2) if $u_1 = 0, u_2 \neq 0$ (or $u_1 \neq 0, u_2 = 0$) and $u_3 = 0$, then $(\mathbb{T}_{02}^{US}, *) \cong (\mathbb{T}_{13}, \cdot)$,

3) if $u_1 \neq 0, u_2 \neq 0$ and $u_3 = 0$, then $(\mathbb{T}_{02}^{US}, *) \cong (\mathbb{T}_{02}^{US}, \cdot)$.

On the other hand, if $u_3 \neq 0$ and u_1, u_2 are not both zero, $(\mathbb{T}_{02}^{US}, *)$ will be a non-Jordan commutative algebra, which does not satisfy almost-Jordan identity

$$2((yx)x)x + yx^3 = 3(yx^2)x,$$

generalizing Jordan identity.

The Jordan algebras $\mathbb{T}_{13}$ and $\mathbb{T}_{14}$ from [10], for example, give us Jordan algebras independent from the choice of u. In fact, since

$$(\mathbb{T}_{13}, \cdot) : \quad e_1^2 = e_1, \ e_1e_2 = \tfrac{1}{2}e_2, \ e_2^2 = e_3,$$
$$(\mathbb{T}_{14}, \cdot) : \quad e_1^2 = e_1, \ e_1e_2 = \tfrac{1}{2}e_2,$$

a standard computation shows that

$$(\mathbb{T}_{13}, *) : \quad e_1 * e_1 = -u_1e_1, \ e_1 * e_2 = -u_1(\tfrac{1}{2}e_2), \ e_2 * e_2 = -u_1e_3,$$
$$(\mathbb{T}_{14}, *) : \quad e_1 * e_1 = -u_1e_1, \ e_1 * e_2 = -u_1(\tfrac{1}{2}e_2).$$

Therefore, if $u_1 \neq 0$, then $(\mathbb{T}_{13}, *) \cong (\mathbb{T}_{13}, \cdot)$ and $(\mathbb{T}_{14}, *) \cong (\mathbb{T}_{14}, \cdot)$.

2.2 3-dimensional anticommutative algebras

Let us remember that the Kantor square of a Lie algebra is zero. It is known, that each 3-dimensional binary-Lie (and Malcev) algebra is Lie. Hence, we are interested in considering only non-Lie algebras. By [11], we have the following 3-dimensional non-Lie anticommutative algebras over $\mathbb{C}$:

$$A_1^\alpha : e_1e_2 = e_3, \ e_1e_3 = e_1 + e_3, \ e_2e_3 = \alpha e_2,$$
$$A_2 : e_1e_2 = e_1, \ e_2e_3 = e_2,$$
$$A_3 : e_1e_2 = e_3, \ e_1e_3 = e_1, \ e_2e_3 = e_2.$$

Write $u = u_1e_1 + u_2e_2 + u_3e_3$. Therefore, we have (using notation from [11, Table $A.1$]) the following.

Proposition 21. *Let A be a 3-dimensional non-Lie anticommutative algebra, then $(A, *)$ is a metabelian Lie algebra. Namely,*

(a) *$(A_1^\alpha, *)$ is defined by*

$$e_1 * e_2 = u_3((1+\alpha)e_3 - \alpha e_2), \ e_1 * e_3 = -u_2((1+\alpha)e_3 - \alpha e_2), \ e_2 * e_3 = u_1((1+\alpha)e_3 - \alpha e_2).$$

(b) *$(A_2, *)$ is a Lie algebra defined by*

$$e_1 * e_2 = u_3e_1, \ e_1 * e_3 = -u_2e_1, \ e_2 * e_3 = u_1e_1.$$

(c) $(A_3, *)$ is a Lie algebra defined by

$$e_1 * e_2 = 2u_3 e_3, \ e_1 * e_3 = -2u_2 e_3, \ e_2 * e_3 = 2u_1 e_3.$$

(d) The sets of algebras $\{(A_1^0, *)\}_{u \in V_3}$, $\{(A_1^{-1}, *)\}_{u \in V_3}$, $\{(A_2, *)\}_{u \in V_3}$, and $\{(A_3, *)\}_{u \in V_3}$ are coincident. It includes algebras isomorphic to the nilpotent 3-dimensional Lie algebra and the metabelian non-nilpotent 3-dimensional Lie algebra with 1-dimensional square. The set $\{(A_1^{\alpha \neq 0, -1}, *)\}_{u \in V_3}$ includes only one non-isomorphic algebra, that is the metabelian non-nilpotent 3-dimensional Lie algebra with 1-dimensional square.

2.3   4-dimensional binary Lie algebras

The variety of binary Lie algebras (see [18]) is defined by the relations $xy = -yx$ and $J(xy, x, y) = 0$. Note that every Malcev algebra and anticommutative $\mathfrak{CD}$-algebra is a binary Lie algebra. From [18], we have the following non-Lie binary Lie algebras of dimension 4:

$$A^0 : e_1 e_2 = e_3, \ e_3 e_4 = e_3,$$

$$A_{\alpha \neq 2} : e_1 e_2 = e_3, \ e_1 e_4 = e_1, \ e_2 e_4 = e_2, \ e_3 e_4 = \alpha e_3,$$

where A_2 is a Lie algebra, A_{-1} is a Malcev (non-Lie) algebra, and A_0 is an anticommutative (non-Lie) $\mathfrak{CD}$-algebra. Write $u = u_1 e_1 + u_2 e_2 + u_3 e_3 + u_4 e_4$.

Proposition 22. *Let A be a 4-dimensional binary Lie (non-Lie) algebra, then $(A, *)$ is a nilpotent Lie algebra, which is isomophic to the algebra with multiplication table $e_1 e_2 = e_3$.*

Proof. It is easy to see that:

1) $(A^0, *)$ is a Lie algebra defined by

$$e_1 * e_2 = -u_4 e_3, \ e_1 * e_4 = u_2 e_3, \ e_2 * e_4 = -u_1 e_3,$$

2) $(A_\alpha, *)$ is a Lie algebra defined by

$$e_1 * e_2 = (2 - \alpha)u_4 e_3, \ e_1 * e_4 = -(2 - \alpha)u_2 e_3, \ e_2 * e_4 = (2 - \alpha)u_1 e_3,$$

for $\alpha \neq 0, 2$. We have $(A_\alpha, *) \cong (A^0, *)$.

Hence, $(A_\alpha, *)$ and $(A^0, *)$ are 4-dimensional 2-step nilpotent anticommutative algebras. It is know that there is only one algebra with this property and it is isomorphic to an algebra with multiplication table $e_1 e_2 = e_3$. $\square$

3   Kantor product

3.1   Generic Poisson structures

Let $(A, \cdot)$ be an algebra, then an anticommutative bilinear mapping $\{,\}$ is called a generic Poisson structure if it satisfies the following compatibility condition:

$$\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\} \quad \text{(Leibniz rule)}.$$

Note that Poisson, non-commutative Poisson, non-associative Poisson, Poisson-Malcev, Malcev-Poisson-Jordan, etc. algebras are particular cases of generic Poisson structures with its underlying algebra.

Proposition 23. *Let $\{,\}$ be a generic Poisson structure on $(A, \cdot)$. Then $[\![\{,\}, \cdot]\!] = 0$ and $(A, [\![\cdot, \{,\}]\!])$ is an anticommutative algebra.*

Proof. By the Leibniz rule $[\![\{,\}, \cdot]\!] = 0$ (see [14, Lemma 5]). On the other hand, $[\![\cdot, \{,\}]\!]$ is anticommutative, by a direct computation using Leibniz rule and anti-commutativity of $\{,\}$. $\square$

3.2　Transposed Poisson algebras

A transposed Poisson algebra $(A, \cdot, [,])$ (see [3]) is an algebra with two bilinear operations such that $(A, \cdot)$ is an associative-commutative algebra and $(A, [,])$ is a Lie algebra that satisfy the following compatibility condition

$$2z[x, y] = [zx, y] + [x, zy] \quad \text{(dual Leibniz rule)}. \tag{5}$$

Proposition 24. *Let* $(A, \cdot, [,])$ *be a transposed Poisson algebra. Then* $(A, [\![\cdot, [,]]\!])$ *is a Lie algebra and* $(A, [\![[,], \cdot]\!])$ *is a commutative algebra.*

Proof. First, writing $* := [\![\,\cdot, [,]]\!]$, we have $x * y = u[x, y] - [ux, y] - [x, uy] = -u[x, y]$. Therefore, $(A, [\![\cdot, [,]]\!])$ is an anticommutative algebra. Since $(x * y) * z = u[u[x, y], z]$, we have

$$J(x, y, z)_* = u[u[x, y], z] + u[u[z, x], y] + u[u[y, z], x] = 0,$$

by [3, Theorem 2.5. (7)].

Note that commutativity of $[\![[,], \cdot]\!]$ is direct from the definition of Kantor product. $\quad\square$

Let us now consider a special example of transposed Poisson algebras constructed in [8, Theorem 25]. The transposed Poisson algebra $(\mathcal{W}, \cdot, [,])$ is spanned by $\{L_i, I_j\}_{i,j \in \mathbb{Z}}$. These generators satisfy

$$[L_m, L_n] = (m - n)L_{m+n}, \quad [L_m, I_n] = (m - n - a)I_{m+n}, \quad L_m \cdot L_n = wL_{m+n}, \quad L_m \cdot I_n = wI_{m+n},$$

where w is an fixed element from the vector space generated by $\{L_i, I_j\}_{i,j \in \mathbb{Z}}$ and the multiplication given by juxtaposition satisfies $L_i L_j = L_{i+j}$ and $L_i I_j = I_{i+j}$.

Proposition 25. *Let* $\star = [\![[,], \cdot]\!]$ *and* $\{,\} = [\![\,\cdot, [,]]\!]$ *be new multiplications defined on multiplications of the transposed Poisson algebra* $(\mathcal{W}, \cdot, [,])$ *defined above. Then* $(\mathcal{W}, \star, \{,\})$ *is a transposed Poisson algebra.*

Proof. Let $u = \sum_k (u_k^1 L_k + u_k^2 I_k)$ be the element to define the Kantor product and $w = \sum_k (w_k^1 L_k + w_k^2 I_k)$ be the element to define the multiplication "$\cdot$". Then by a straightful calculation we have only the following nonzero multiplications:

$$L_i \star L_j = -\sum_{k,n} \Big((k + n)u_k^1 w_n^1 L_{i+j+k+n} + (k + n + a)(u_k^1 w_n^2 + u_k^2 w_n^1)I_{i+j+k+n} \Big),$$

$$L_i \star I_j = -\sum_{k,n} (k + n)u_k^1 w_n^1 I_{i+j+k+n},$$

$$\{L_i, L_j\} = (j - i)\sum_{k,n} \Big(u_k^1 w_n^1 L_{i+j+k+n} + (u_k^1 w_n^2 + u_k^2 w_n^1)I_{i+j+k+n} \Big),$$

$$\{L_i, I_j\} = (j - i)\sum_{k,n} u_k^1 w_n^1 I_{i+j+k+n}.$$

Now let us define $\Omega := i + j + m + k_1 + n_1 + k_2 + n_2$. Hence,

$$
\begin{aligned}
(L_i \star L_j) \star L_m &= \sum_{k_1, n_1, k_2, n_2} (k_1 + n_1)(k_2 + n_2)u_{k_1}^1 w_{n_1}^1 u_{k_2}^1 w_{n_2}^1 L_\Omega \\
&\quad + \sum_{k_1, n_1, k_2, n_2} (k_1 + n_1)(k_2 + n_2 + a)u_{k_1}^1 w_{n_1}^1 (u_{k_2}^1 w_{n_2}^2 + u_{k_2}^2 w_{n_2}^1)I_\Omega \\
&\quad + \sum_{k_1, n_1, k_2, n_2} (k_1 + n_1 + a)(k_2 + n_2)u_{k_2}^1 w_{n_2}^1 (u_{k_1}^1 w_{n_1}^2 + u_{k_1}^2 w_{n_1}^1)I_\Omega \\
&= L_i \star (L_j \star L_m)
\end{aligned}
$$

and

$$(L_i \star L_j) \star I_m = \sum_{k_1,n_1,k_2,n_2} (k_1 + n_1)(k_2 + n_2) u_{k_1}^1 w_{n_1}^1 u_{k_2}^1 w_{n_2}^1 I_\Omega = L_i \star (L_j \star I_m).$$

Then $\star$ is associative.

It is easy to see that

$$L_i \star \{L_j, L_m\} = (j - m)\Big(\sum_{k_1,n_1,k_2,n_2} (k_1 + n_1) u_{k_1}^1 w_{n_1}^1 u_{k_2}^1 w_{n_2}^1 L_\Omega$$
$$+ \sum_{k_1,n_1,k_2,n_2} (k_1 + k_2 + n_1 + n_2 + a) u_{k_1}^1 w_{n_1}^1 (u_{k_2}^1 w_{n_2}^2 + u_{k_2}^2 w_{n_2}^1) I_\Omega \Big).$$

On the other hand,

$$\{L_i \star L_j, L_m\} = \sum_{k_1,n_1,k_2,n_2} (i + j + k_1 + n_1 - m)(k_1 + n_1) u_{k_1}^1 w_{n_1}^1 u_{k_2}^1 w_{n_2}^1 L_\Omega$$
$$+ \sum_{k_1,n_1,k_2,n_2} (i + j + k_1 + n_1 - m)(k_1 + n_1)(u_{k_2}^1 w_{n_2}^2 + u_{k_2}^2 w_{n_2}^1) u_{k_1}^1 w_{n_1}^1 I_\Omega$$
$$+ \sum_{k_1,n_1,k_2,n_2} (i + j + k_1 + n_1 - m)(k_1 + n_1 + a)(u_{k_1}^1 w_{n_1}^2 + u_{k_1}^2 w_{n_1}^1) u_{k_2}^1 w_{n_2}^1 I_\Omega$$

and

$$\{L_j, L_i \star L_m\} = \sum_{k_1,n_1,k_2,n_2} (j - i - m - k_1 - n_1)(k_1 + n_1) u_{k_1}^1 w_{n_1}^1 u_{k_2}^1 w_{n_2}^1 L_\Omega$$
$$+ \sum_{k_1,n_1,k_2,n_2} (j - i - m - k_1 - n_1)(k_1 + n_1)(u_{k_2}^1 w_{n_2}^2 + u_{k_2}^2 w_{n_2}^1) u_{k_1}^1 w_{n_1}^1 I_\Omega$$
$$+ \sum_{k_1,n_1,k_2,n_2} (j - i - m - k_1 - n_1)(k_1 + n_1 + a)(u_{k_1}^1 w_{n_1}^2 + u_{k_1}^2 w_{n_1}^1) u_{k_2}^1 w_{n_2}^1 I_\Omega,$$

which gives the dual Leibniz rule (5) for $\{L_i, L_j, L_m\}$. By similar calculation, we have the dual Leibniz rule (5) for $\{L_i, L_j, I_m\}$, which conclude the proof of the statement. $\qquad\square$

Another example of a transposed Poisson algebra can be constructed in the following way (see [3, Proposition 2.2]). Let $(A, \cdot)$ be an associative-commutative algebra and D be a derivation of A. Define the multiplication

$$[x, y] := xD(y) - D(x)y$$

for all $x, y \in A$. Therefore, $(A, \cdot, [,])$ is a transposed Poisson algebra. Then, we have the following result.

Proposition 26. *Let* $\circ = [\![[,], \cdot]\!]$ *and* $\{,\} = [\![\cdot, [,]]\!]$ *be new multiplications defined on multiplications of the transposed Poisson algebra* $(A, \cdot, [,])$ *defined above. Then* $(A, \circ, \{,\})$ *is a transposed Poisson algebra.*

Proof. A direct computation shows that $x \circ y = xyD(u)$ (that is, associative) and $\{x, y\} = -u[x, y]$. By Theorem 24, we need only to check the dual Leibniz rule. For this purpose, we have

$$\{z \circ x, y\} + \{x, z \circ y\} = -u([zxD(u), y] + [x, zyD(u)])$$
$$= -u(zxD(u)D(y) - D(zxD(u))y + xD(zyD(u) - zyD(u)D(x)))$$
$$= -u(zxD(u)D(y) - zyD(u)D(x) - y(xD(zD(u))$$
$$+ D(x)zD(u))) - ux(yD(zD(u)) + D(y)zD(u))$$
$$= -2uzD(u)(xD(y) - yD(x)) = -2uzD(u)[x, y] = 2z \circ \{x, y\}.$$

$\qquad\square$

3.3   Pre-Lie Poisson algebras

A pre-Lie Poisson algebra $(A, \cdot, \circ)$ (see [3]) is an algebra with two bilinear operations such that $(A, \cdot)$ is an associative-commutative algebra, $(A, \circ)$ is a left pre-Lie algebra and the following conditions hold:

$$(xy) \circ z = x(y \circ z), \quad (x \circ y)z - (y \circ x)z = x \circ (yz) - y \circ (xz).$$

Here, by a pre-Lie Poisson algebra we mean a right pre-Lie Poisson algebra, using an analogous nomenclature as left and right Novikov-Poisson algebras (see [14]). For the definition of left pre-Lie Poisson algebra, see [14] for the analogous properties (see section of Novikov-Poisson algebras). Note that a Novikov-Poisson algebra is a pre-Lie Poisson algebra.

Proposition 27. *Let* $(A, \cdot, \circ)$ *be a right pre-Lie Poisson algebra. Then* $(A, \llbracket \circ, \cdot \rrbracket)$ *is a commutative algebra and* $(A, \llbracket \cdot, \circ \rrbracket)$ *is a left pre-Lie algebra.*

Proof. $(A, \llbracket \circ, \cdot \rrbracket)$ is a commutative algebra direct from definition of the product. Now, writing $* := \llbracket \cdot, \circ \rrbracket$, we have that $x * y = -x \circ (uy)$. Therefore

$$
\begin{aligned}
(x * y) * z - (y * x) * z &= (x \circ (uy)) \circ (uz) - (y \circ (ux)) \circ (uz) \\
&= ((x \circ y)u) \circ (uz) - ((y \circ x)u) \circ (uz) \\
&= x \circ (y(u \circ (uz))) - y \circ (x(u \circ (uz))) \\
&= x \circ ((uy) \circ (uz))) - y \circ ((ux) \circ (uz))) \\
&= x \circ (u(y \circ (uz))) - y \circ (u(x \circ (uz))) = x * (y * z) - y * (x * z),
\end{aligned}
$$

and the result follows.   □

By the same computations as in the case of left Novikov-Poisson algebras (in [14]), we obtain the following result.

Proposition 28. *Let* $(A, \cdot, \circ)$ *be a left pre-Lie Poisson algebra. Then* $(A, \llbracket \circ, \cdot \rrbracket)$ *is an associative-commutative algebra and* $(A, \llbracket \cdot, \circ \rrbracket)$ *is a right pre-Lie algebra.*

3.4   On Novikov-Poisson algebras

A (left) Novikov-Poisson algebra $(A, \cdot, \circ)$ (see [14]) is an algebra with two bilinear operations such that $(A, .)$ is an associative-commutative algebra, $(A, \circ)$ is a left Novikov algebra and the following conditions hold:

$$x \circ (yz) = (x \circ y)z \tag{6}$$

and

$$(xy) \circ z - x(y \circ z) = (xz) \circ y - x(z \circ y). \tag{7}$$

In [14], it was proved that $(A, \llbracket \cdot, \circ \rrbracket)$ is a left Novikov algebra and $(A, \llbracket \circ, \cdot \rrbracket)$ is an associative-commutative algebra. With this we can obtain the following assertion.

Corollary 1. *Let $(A, \cdot, \circ)$ be a left Novikov-Poisson algebra. Then $(A, [\![\circ, \cdot]\!], [\![\cdot, \circ]\!])$ is a left Novikov-Poisson algebra.*

Proof. We need only to proof the remaining properties (6) and (7). For this purpose, write $\star = [\![\circ, \cdot]\!]$ and $\bullet = [\![\cdot, \circ]\!]$. First, observe that $x \star y = -u \circ (xy)$ and $x \bullet y = -(ux) \circ y$. Thus

$$x \bullet (y \star z) = (ux) \circ (u \circ (yz)) = u \circ ((ux) \circ (yz)) = u \circ (((ux) \circ y)z) = (x \bullet y) \star z.$$

For the second property we have

$$(x \star y) \bullet z - x \star (y \bullet z) - ((x \star z) \bullet y - x \star (z \bullet y))$$
$$= (u(u \circ (xy))) \circ z - u \circ (x((uy) \circ z)) - (u(u \circ (xz))) \circ y + u \circ (x((uz) \circ y))$$
$$= (u(u \circ (xy))) \circ z - (u(u \circ (xz))) \circ y - u \circ (x((uy) \circ z - (uz) \circ y))$$
$$= ((u(u \circ x))y) \circ z - ((u(u \circ x))z) \circ y - u \circ (x(u(y \circ z) - u(z \circ y)))$$
$$= (u(u \circ x))(y \circ z) - (u(u \circ x))(z \circ y) - u \circ ((xu)(y \circ z - z \circ y)) = 0.$$

$\square$

Remark 2. *Since we do not use the left-commutativity of $\circ$ in the above computation, we obtain an analogous result in the case of left pre-Lie Poisson algebra (see Section 3.3).*

In [14, Theorem 29], it was proved that if $(A, \cdot)$ is a finite dimensional associative algebra, then $(A, \cdot)$ and $(A, *)$ are isomorphic if and only if A is a skew field. By the above corollary, we can formulate the following question: are $(A, \cdot, \circ)$ and $(A, [\![\circ, \cdot]\!], [\![\cdot, \circ]\!])$ isomorphic?

In general, this is not true. For this purpose, we will use the classification of 3-dimensional (right) Novikov-Poisson algebras given in [24, Theorem 3]. If we consider the left Novikov-Poisson algebras given in this theorem (after considering in the Novikov product the opposite product) given by the left Novikov algebras A_5, A_6 and A_7 (notation from [24]), all of them results in $[\![\circ, \cdot]\!] = 0$ and $[\![\cdot, \circ]\!] = 0$.

On the other hand, we will show that, in the next example, $(A, \cdot, \circ)$ and $(A, [\![\circ, \cdot]\!], [\![\cdot, \circ]\!])$ are isomorphic. Consider the following algebra (given by C_8 in [24, Theorem 3]).

Left Novikov product is given by:

$$e_3 \circ e_1 = e_1, \quad e_3 \circ e_2 = e_2, \quad e_3 \circ e_3 = e_3.$$

Associative-commutative product is given by:

$$e_1 e_3 = ae_1 + be_2, \quad e_2^2 = ce_2, \quad e_2 e_3 = ae_2, \quad e_3^2 = de_1 + fe_2 + ae_3,$$

where $fc = ab = bc = 0$.

In the base $\{e_1, e_2, e_3\}$, write $u = u_1 e_1 + u_2 e_2 + u_3 e_3$. We will suppose that $u_3 \neq 0$ and $a \neq 0$. Computing the Kantor product, we obtain:
- left Novikov product $[\![\cdot, \circ]\!]$

$$[\![\cdot, \circ]\!](e_3, e_1) = -u_3 a e_1, \quad [\![\cdot, \circ]\!](e_3, e_2) = -u_3 a e_2, \quad [\![\cdot, \circ]\!](e_3, e_3) = -u_3 a e_3;$$

- associative-commutative product $[\![\circ, \cdot]\!]$

$$[\![\circ, \cdot]\!](e_1, e_3) = -u_3(ae_1 + be_2), \quad [\![\circ, \cdot]\!](e_2, e_2) = -u_3 c e_2,$$

$$[\![\circ, \cdot]\!](e_2, e_3) = -u_3 a e_2, \quad [\![\circ, \cdot]\!](e_3, e_3) = -u_3(de_1 + fe_2 + ae_3).$$

4 A method for classifying Poisson structures and commutative post-Lie structures with a given algebra

4.1 A method for classifying Poisson structures with a given algebra

Let $(\mathfrak{A}, \cdot)$ be an algebra of dimension n. We say that the multiplication $[,]$, defined on the underlying space of the algebra $\mathfrak{A}$, gives a Poisson structure, if $(\mathfrak{A}, [,])$ is a Lie algebra and multiplications $\cdot$ and $[,]$ are satisfying the Leibniz rule

$$[x \cdot y, z] = [x, z] \cdot y + x \cdot [y, z].$$

Then for algebras $(\mathfrak{A}, \cdot)$ and $(\mathfrak{A}, [,])$ we can associate two elements $\mathfrak{a}$ and $\mathfrak{l}$ from our "big" algebra $U(n)$ (constructed by the Kantor way). It is easy to see that

$$[\![\mathfrak{l}, \mathfrak{a}]\!] = 0 \quad \text{and} \quad [\![\mathfrak{l}, \mathfrak{l}]\!] = 0.$$

Moreover, if the basis in $U(n)$ will be chosen by a similar way of [15, Section 3], $\mathfrak{l}$ is anti-symmetric on lower indices. Hence, we have a constructive method for description of all Poisson structures on a given algebra. We will illustrate it for a basic example below.

Example 1. *There are no nontrivial Poisson structures defined on a 2-dimensional Jordan algebra $\mathfrak{A}$ with multiplication table given by*

$$e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = \frac{1}{2}e_2, \quad e_2 \cdot e_1 = \frac{1}{2}e_2.$$

Proof. Let $\mathfrak{l}$ be the element from $U(2)$, which is associated to a Poisson structure defined on $\mathfrak{A}$. Thanks to [15], the multiplication of $U(2)$ is given in terms of "elementary" multiplications $\alpha_{i,j}^k$ $(i, j, k = 1, 2)$, such that $\alpha_{i,j}^k(v_t, v_l) = \delta_{it}\delta_{jl}v_k$ for all $t, l = 1, 2$, (where v_k are basis vectors of 2-dimensional space) and presented below.

$$
\begin{array}{llll}
[\![\alpha_{11}^1, \alpha_{11}^1]\!] = -\alpha_{11}^1 & [\![\alpha_{12}^1, \alpha_{11}^1]\!] = -\alpha_{12}^1 - \alpha_{21}^1 & [\![\alpha_{11}^2, \alpha_{11}^1]\!] = \alpha_{11}^2 & [\![\alpha_{12}^2, \alpha_{11}^1]\!] = 0 \\
[\![\alpha_{11}^1, \alpha_{12}^1]\!] = 0 & [\![\alpha_{12}^1, \alpha_{12}^1]\!] = -\alpha_{22}^1 & [\![\alpha_{11}^2, \alpha_{12}^1]\!] = -\alpha_{11}^1 + \alpha_{12}^2 & [\![\alpha_{12}^2, \alpha_{12}^1]\!] = -\alpha_{12}^1 \\
[\![\alpha_{11}^1, \alpha_{21}^1]\!] = 0 & [\![\alpha_{12}^1, \alpha_{21}^1]\!] = -\alpha_{22}^1 & [\![\alpha_{11}^2, \alpha_{21}^1]\!] = -\alpha_{11}^1 + \alpha_{21}^2 & [\![\alpha_{12}^2, \alpha_{21}^1]\!] = -\alpha_{21}^1 \\
[\![\alpha_{11}^1, \alpha_{22}^1]\!] = \alpha_{22}^1 & [\![\alpha_{12}^1, \alpha_{22}^1]\!] = 0 & [\![\alpha_{11}^2, \alpha_{22}^1]\!] = -\alpha_{12}^1 - \alpha_{21}^1 + \alpha_{22}^2 & [\![\alpha_{12}^2, \alpha_{22}^1]\!] = -2\alpha_{22}^1 \\
[\![\alpha_{11}^1, \alpha_{11}^2]\!] = -2\alpha_{11}^2 & [\![\alpha_{12}^1, \alpha_{11}^2]\!] = \alpha_{11}^1 - \alpha_{21}^2 - \alpha_{12}^2 & [\![\alpha_{11}^2, \alpha_{11}^2]\!] = 0 & [\![\alpha_{12}^2, \alpha_{11}^2]\!] = \alpha_{11}^2 \\
[\![\alpha_{11}^1, \alpha_{12}^2]\!] = -\alpha_{12}^2 & [\![\alpha_{12}^1, \alpha_{12}^2]\!] = \alpha_{12}^1 - \alpha_{22}^2 & [\![\alpha_{11}^2, \alpha_{12}^2]\!] = -\alpha_{11}^2 & [\![\alpha_{12}^2, \alpha_{12}^2]\!] = 0 \\
[\![\alpha_{11}^1, \alpha_{21}^2]\!] = -\alpha_{21}^2 & [\![\alpha_{12}^1, \alpha_{21}^2]\!] = \alpha_{21}^1 - \alpha_{22}^2 & [\![\alpha_{11}^2, \alpha_{21}^2]\!] = -\alpha_{11}^2 & [\![\alpha_{12}^2, \alpha_{21}^2]\!] = 0 \\
[\![\alpha_{11}^1, \alpha_{22}^2]\!] = 0 & [\![\alpha_{12}^1, \alpha_{22}^2]\!] = \alpha_{22}^1 & [\![\alpha_{11}^2, \alpha_{22}^2]\!] = -\alpha_{12}^2 - \alpha_{21}^2 & [\![\alpha_{12}^2, \alpha_{22}^2]\!] = -\alpha_{22}^2 \\
\end{array}
$$

Hence, $\mathfrak{a} = \alpha_{11}^1 + \frac{1}{2}(\alpha_{12}^2 + \alpha_{21}^2)$ and $\mathfrak{l} = \gamma_1(\alpha_{12}^1 - \alpha_{21}^1) + \gamma_2(\alpha_{12}^2 - \alpha_{21}^2)$. From $[\![\mathfrak{l}, \mathfrak{a}]\!] = 0$, we conclude that $\gamma_1 = \gamma_2 = 0$ and it follows that all Poisson structures on $\mathfrak{a}$ are trivial. ☐

4.2 A method for classifying commutative post-Lie structures on a given Lie algebra

A commutative post-Lie structure on a Lie algebra $(A, [,])$ is a bilinear product $x \cdot y$ that satisfy (see [5])

$$x \cdot y = y \cdot x, \quad [x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z) \quad \text{and} \quad x \cdot [y, z] = [x \cdot y, z] + [y, x \cdot z].$$

A direct consequence is the following proposition.

Proposition 29. *Let $(A, \cdot)$ be a commutative post-Lie structure on a Lie algebra $(A, [,])$. Then*

(a) $[\![\,\cdot\,, [,]\,]\!] = 0$,

(b) $(A, [\![\,[,], \cdot\,]\!])$ *is a commutative algebra.*

Let $(\mathfrak{A}, [,])$ be a Lie algebra of dimension n with a given commutative post-Lie structure $(\mathfrak{A}, \cdot)$. Then for algebras $(\mathfrak{A}, \cdot)$ and $(\mathfrak{A}, [,])$ we can associate two elements $\mathfrak{a}$ and $\mathfrak{l}$ from our "big" algebra $U(n)$ (constructed by the Kantor way). It is easy to see that

$$[\![\mathfrak{a}, \mathfrak{l}]\!] = 0.$$

Moreover, if the basis in $U(n)$ will be chosen by a similar way of [15, Section 3], $\mathfrak{a}$ is symmetric on lower indices. If there are some non-zero elements $\mathfrak{a}$ satisfying the relation indicated above, we choose only such commutative multiplications which satisfies the second post-Lie identity

$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z).$$

Hence, we have a constructive method for description of all commutative post-Lie structures on a given Lie algebra. The present method can be inverted to find commutative post-Lie algebra structures for a given commutative algebra. We will illustrate it for the following basic example.

Example 2. *Let $\mathfrak{S}_2$ be the solvable 2-dimensional Lie algebra with the multiplication table given by $[e_1, e_2] = e_2$, $[e_2, e_1] = -e_2$. Then a nonzero commutative post-Lie structure on $\mathfrak{S}_2$ is given (after a changing of the bases in $\mathfrak{S}_2$) by one of the following commutative multiplications:*

$$
\begin{array}{ll}
\text{(I)} & e_1 \cdot e_1 = e_2, \\
\text{(II)} & e_1 \cdot e_2 = e_2, \\
\text{(III)} & e_1 \cdot e_1 = e_2,\ e_1 \cdot e_2 = e_2.
\end{array}
$$

Proof. Let $\mathfrak{l}$ be the element from $U(2)$, which is associated to a commutative post-Lie structure defined on $\mathfrak{S}_2$. Thanks to [15], the multiplication of $U(2)$ is given in the Example 1. Hence,

$$\mathfrak{l} = \alpha_{12}^2 - \alpha_{21}^2 \quad \text{and} \quad \mathfrak{a} = \sum_{j=1}^{2} (\gamma_1^j \alpha_{11}^j + \gamma_2^j (\alpha_{12}^j + \alpha_{21}^j) + \gamma_3^j \alpha_{22}^j).$$

From $[\![\mathfrak{a}, \mathfrak{l}]\!] = 0$, we conclude that

$$\mathfrak{a} = \gamma_3^1 \alpha_{22}^1 + \gamma_1^2 \alpha_{11}^2 + \gamma_2^2 (\alpha_{21}^2 + \alpha_{12}^2) + \gamma_3^2 \alpha_{22}^2.$$

Hence,

$$e_1 \cdot e_1 = \gamma_1^2 e_2, \quad e_1 \cdot e_2 = \gamma_2^2 e_2, \quad e_2 \cdot e_2 = \gamma_3^1 e_1 + \gamma_3^2 e_2.$$

It follows that "$\cdot$", for each $z = z_1 e_1 + z_2 e_2$, satisfies only the following additional relation

$$e_2 \cdot z = e_1 \cdot (e_2 \cdot z) - e_2 \cdot (e_1 \cdot z).$$

Hence,

$$z_2 \gamma_3^1 = -z_1 \gamma_1^2 \gamma_3^1 - z_2 \gamma_2^2 \gamma_3^1, \quad z_1 \gamma_2^2 + z_2 \gamma_3^2 = z_1 (\gamma_2^2)^2 + z_2 \gamma_1^2 \gamma_3^1 - z_1 \gamma_1^2 \gamma_3^2,$$

which gives us that γ_1^2 is an arbitrary element, $\gamma_3^1 = \gamma_3^2 = 0$, and $\gamma_2^2 = 0$ or $\gamma_2^2 = 1$. Then we have only two types of commutative post-Lie structures defined on $\mathfrak{S}_2$:

$$(1) \quad e_1 \cdot e_1 = \gamma_1^2 e_2,$$
$$(2) \quad e_1 \cdot e_1 = \gamma_1^2 e_2, \quad e_1 \cdot e_2 = e_2.$$

Hence, if $\gamma_1^2 \neq 0$, then by a changing of bases $e_1^* = e_1$, $e_2^* = \gamma_1^2 e_2$, we have the statement of our example. $\qquad\square$

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Фелберг Жуніор Р., Кайгородов І. *Про добуток Кантора, II* // Карпатські матем. публ. — 2022. — Т.14, №2. — С. 543–563.

Ми описуємо квадрат Кантора (і добуток Кантора) множень, розширюючи класифікацію, запропоновану в [J. Algebra Appl. 2017, **16** (9), 1750167]. Крім того, ми явно описуємо квадрат Кантора деяких алгебр малої розмірності і наводимо конструктивні методи для отримання нових транспонованих алгебр Пуассона і алгебр Пуассона-Новікова; а також для класифікації пуассонівських структур і комутативних постлієвських структур на заданій алгебрі.

Ключові слова і фрази: добуток Кантора, квадрат Кантора, неасоціативна алгебра.