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On generalized fractional integral operator involving Fox's H-function and its applications to unified subclass of prestarlike functions with negative coefficients

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The aim of present paper is to find out different interesting properties and characterization of unified class $P_\gamma(A,B,\alpha,\sigma)$ of prestarlike functions with negative coefficients in the unit disc U. Furthermore, distortion theorems involving a generalized fractional integral operator involving well-known Fox's H-function for functions in this class are proved.

Key words and phrases: univalent function, distortion theorem, prestarlike function.

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Introduction

Let A be the class of analytic functions in the open unit disc $U = \{z : |z| < 1\}$ and S denote the subclass of A consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let throughout the paper γ be a real number satisfying $0 \le \gamma < 1$.

Let $S^*(\gamma)$ denote the subclass of S consisting of functions which satisfy

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma, \quad z \in U.$$

A function $f(z) \in S^*(\gamma)$ is said to be starlike of order γ in U.

Also, $K(\gamma)$ denote the subclass of S consisting of functions which satisfy

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \gamma, \quad z \in U.$$

A function $f(z) \in K(\gamma)$ is said to be convex of order γ in U.

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Let T be the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0. \tag{2}$$

The class T is introduced and studied by H. Silverman [20]. A function $f(z) \in T$ is said to be starlike of order γ, denoted by $T^*(\gamma)$, and is said to be convex of order γ, denoted by $K^*(\gamma)$.

The function

$$S_\gamma(z) = z(1-z)^{-2(1-\gamma)}$$

is the familiar extremal function for the class $S^*(\gamma)$. Setting

$$C(\gamma, n) = \frac{\prod_{i=2}^{n}(i-2\gamma)}{(n-1)!}, \quad n \in \mathbb{N} \setminus \{1\}, \tag{3}$$

where $\mathbb{N} = \{1, 2, 3, \ldots\}$, we obtain

$$S_\gamma(z) = z + \sum_{n=2}^{\infty} C(\gamma, n) z^n.$$

Clearly, $S_\gamma(z) \in S^*(\gamma)$ and $C(\gamma, n)$ is a decreasing function in γ and it satisfies

$$\lim_{n\to\infty} C(\gamma, n) = \begin{cases} \infty, & \gamma < \frac{1}{2}, \\ 1, & \gamma = \frac{1}{2}, \\ 0, & \gamma > \frac{1}{2}. \end{cases}$$

The convolution or Hadamard product of $f(z)$ and $g(z)$ is denoted by $(f * g)(z)$. Where $f(z)$ is given by (2) and $g(z)$ given by

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n, \quad b_n \geq 0,$$

then

$$(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n.$$

Let $R_\gamma(A, B, \alpha)$ be the class of prestarlike functions which was introduced by M.K. Aouf et al. [1] and $R_\gamma(A, B, \alpha)$ be the subclass of A consisting of functions $f(z)$ which satisfy the condition

$$\left| \frac{\frac{zh'(z)}{h(z)} - 1}{B\frac{zh'(z)}{h(z)} - [B + (A-B)(1-\alpha)]} \right| < 1,$$

where $h(z) = (f * S_\gamma)(z)$, A and B fixed, $-1 \leq A < B \leq 1$, $0 < B \leq 1$, $0 \leq \alpha < 1$, $0 \leq \gamma < 1$.

The class $C_\gamma(A, B, \alpha)$ also studied by M.K. Aouf et al. [1] and it is the subclass of A consisting of functions $f(z)$ such that $zf'(z) \in R_\gamma(A, B, \alpha)$.

Also we note that the subclasses

$$R_\gamma[A,B,\alpha] = R_\gamma(A,B,\alpha) \cap T, \quad C_\gamma[A,B,\alpha] = C_\gamma(A,B,\alpha) \cap T$$

were introduced and studied by M.K. Aouf et al. [1]. In addition, we make a note that such kind of classes were widely studied by T. Sheil-Small et al. [18], S. Owa and B.A. Uralegaddi [11], S.K. Lee and S.B. Joshi [6], H.M. Srivastava and M.K. Aouf [22]. For other related interesting topics one may also refer to M.K. Aouf and T. Seoudy [2,16], G.A. Anastassiou [4].

The following lemmas will be required in our present investigation.

Lemma 1 ([1]). *A function f defined by* (2) *is in the class $R_\gamma[A,B,\alpha]$ if and only if*

$$\sum_{n=2}^{\infty} C(\gamma,n)\big\{(n-1)(1+B)+(B-A)(1-\alpha)\big\}a_n \leq (B-A)(1-\alpha). \tag{4}$$

The result (4) *is sharp.*

Lemma 2 ([1]). *A function f defined by* (2) *is in the class $C_\gamma[A,B,\alpha]$ if and only if*

$$\sum_{n=2}^{\infty} C(\gamma,n)n\big\{(n-1)(1+B)+(B-A)(1-\alpha)\big\}a_n \leq (B-A)(1-\alpha). \tag{5}$$

The result (5) *is sharp.*

By considering Lemma 1 and Lemma 2, it would seem to be normal to introduce and study a remarkable unification of the classes $R_\gamma[A,B,\alpha]$ and $C_\gamma[A,B,\alpha]$ by introducing a new subclass $P_\gamma(A,B,\alpha,\sigma)$. In fact, we state that a function $f(z)$ defined by (2) belongs to the class $P_\gamma(A,B,\alpha,\sigma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{\{(n-1)(1+B)+(B-A)(1-\alpha)\}\,(1-\sigma+\sigma n)}{(B-A)(1-\alpha)}\, C(\gamma,n)a_n \leq 1, \tag{6}$$

where $-1 \leq A < B \leq 1$, $0 < B \leq 1$, $0 \leq \alpha < 1$, $0 \leq \gamma < 1$, $0 \leq \sigma \leq 1$.

Obviously, we have

$$P_\gamma(A,B,\alpha,\sigma) = (1-\sigma)R_\gamma[A,B,\alpha] + \sigma C_\gamma[A,B,\alpha], \tag{7}$$

where $0 \leq \sigma \leq 1$. Then

$$P_\gamma(A,B,\alpha,0) = R_\gamma[A,B,\alpha], \quad P_\gamma(A,B,\alpha,1) = C_\gamma[A,B,\alpha]. \tag{8}$$

Also we note that by specializing the parameters A, B, α and γ, we get the following subclasses studied by various authors:

(i) $P_\gamma(-1,1,\alpha,0) = R_\gamma[\alpha]$ (H. Silverman and E. Silvia [21]);

(ii) $P_\gamma(-1,1,\alpha,1) = C_\gamma[\alpha]$ (S. Owa and B.A. Uralegaddi [11]).

Prestarlike functions have numerous significant geometric properties. For an enlightenment of the fundamentals of the theory of prestarlike functions please refer to T. Sheil-Small [19] and S. Ruscheweyh [15]. Additionally, the unified subclasses of prestarlike functions was investigated by R.K. Raina and H.M. Srivastava [14] and subsequently by S.B. Joshi et al. [7].

The main goal of the present paper is to find out different interesting properties and characterization of the general class $P_\gamma(A,B,\alpha,\sigma)$. Furthermore, distortion theorem involving a generalized fractional integral operator for functions in this class are proved.

1 Main Results

Theorem 1. *Let the function* $f(z)$ *be defined by* (2). *If* $f(z) \in P_{\gamma}(A,B,\alpha,\sigma)$ *then*

$$a_n \leq \frac{(B-A)(1-\alpha)}{C(\gamma,n)\left\{(n-1)(1+B)+(B-A)(1-\alpha)\right\}(1-\sigma+\sigma n)}, \quad n \in \mathbb{N} \setminus \{1\}.$$

Equality holds true for the function $f(z)$ *given by*

$$f(z) = z - \frac{(B-A)(1-\alpha)}{C(\gamma,n)\left\{(n-1)(1+B)+(B-A)(1-\alpha)\right\}(1-\sigma+\sigma n)} z^n, \quad n \in \mathbb{N} \setminus \{1\}.$$

Proof. The proof of Theorem 1 is straightforward and hence details are omitted. □

Next, we will obtain distortion theorem for the class $P_{\gamma}(A,B,\alpha,\sigma)$.

Theorem 2. *Let the function* $f(z)$ *be defined by* (2). *If* $f(z) \in P_{\gamma}(A,B,\alpha,\sigma)$ *then*

$$\begin{aligned} |z| - \frac{(B-A)(1-\alpha)}{2\left\{(1+B)+(B-A)(1-\alpha)\right\}(1-\gamma)(1+\sigma)} |z|^2 &\leq |f(z)| \\ &\leq |z| + \frac{(B-A)(1-\alpha)}{2\left\{(1+B)+(B-A)(1-\alpha)\right\}(1-\gamma)(1+\sigma)} |z|^2 \end{aligned} \tag{9}$$

and

$$\begin{aligned} 1 - \frac{(B-A)(1-\alpha)}{\left\{(1+B)+(B-A)(1-\alpha)\right\}(1-\gamma)(1+\sigma)} |z| &\leq |f'(z)| \\ &\leq 1 + \frac{(B-A)(1-\alpha)}{\left\{(1+B)+(B-A)(1-\alpha)\right\}(1-\gamma)(1+\sigma)} |z|. \end{aligned} \tag{10}$$

Proof. Let

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n.$$

Given that $f(z) \in P_{\gamma}(A,B,\alpha,\sigma)$ and obviously $C(\gamma,n)$ defined by (3) is decreasing for $0 \leq \gamma \leq \frac{1}{2}$, then we find from (6) that

$$\sum_{n=2}^{\infty} a_n \leq \frac{(B-A)(1-\alpha)}{2\left\{(1+B)+(B-A)(1-\alpha)\right\}(1-\gamma)(1+\sigma)}, \quad n \in \mathbb{N} \setminus \{1\}. \tag{11}$$

Then by means of (2) and (11) for $z \in U$ we have

$$|f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} |a_n| \leq |z| + \frac{(B-A)(1-\alpha)}{2\left\{(1+B)+(B-A)(1-\alpha)\right\}(1-\gamma)(1+\sigma)} |z|^2$$

and

$$|f(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} |a_n| \geq |z| - \frac{(B-A)(1-\alpha)}{2\left\{(1+B)+(B-A)(1-\alpha)\right\}(1-\gamma)(1+\sigma)} |z|^2,$$

which gives the inequality (9) of Theorem 2.

Also for $z \in U$ we find that

$$|f'(z)| \le 1 + |z| \sum_{n=2}^{\infty} n\,|a_n| \le 1 + \frac{(B-A)(1-\alpha)}{\{(1+B)+(B-A)(1-\alpha)\}\,(1-\gamma)(1+\sigma)}\,|z|$$

and

$$|f'(z)| \ge 1 - |z| \sum_{n=2}^{\infty} n\,|a_n| \ge 1 - \frac{(B-A)(1-\alpha)}{\{(1+B)+(B-A)(1-\alpha)\}\,(1-\gamma)(1+\sigma)}\,|z|,$$

which gives the inequality (10) of Theorem 2. Hence the proof is completed.

Finally, it is easily seen that the inequalities (9) and (10) is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{(B-A)(1-\alpha)}{2\,\{(1+B)+(B-A)(1-\alpha)\}\,(1-\gamma)(1+\sigma)}\,z^2. \tag{12}$$

□

2 Closure theorems

Theorem 3. *The class $P_\gamma(A, B, \alpha, \sigma)$ is closed under convex linear combinations.*

Proof. Let $f_1(z),\ f_2(z) \in P_\gamma(A, B, \alpha, \sigma)$ and

$$f_1(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad f_2(z) = z - \sum_{n=2}^{\infty} b_n z^n.$$

It is sufficient to prove that the function H defined by

$$H(z) = \lambda f_1(z) + (1-\lambda) f_2(z), \quad 0 \le \lambda \le 1,$$

is also in the class $P_\gamma(A, B, \alpha, \sigma)$. Since

$$H(z) = z - \sum_{n=2}^{\infty} [\lambda a_n + (1-\lambda) b_n] z^n,$$

we get

$$\sum_{n=2}^{\infty} \frac{\{(n-1)(1+B)+(B-A)(1-\alpha)\}\,(1-\sigma+\sigma n)}{(B-A)(1-\alpha)}\, C(\gamma, n)[\lambda a_n + (1-\lambda) b_n] \le 1.$$

Hence $H \in P_\gamma(A, B, \alpha, \sigma)$. This completes the proof. □

Theorem 4. *If $f_1(z) = z$ and*

$$f_n(z) = z - \frac{(B-A)(1-\alpha)}{C(\gamma, n)\,\{(n-1)(1+B)+(B-A)(1-\alpha)\}\,(1-\sigma+\sigma n)} z^n, \quad n \ge 2,$$

then $f(z) \in P_\gamma(A, B, \alpha, \sigma)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z),$$

where $\lambda_n \ge 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$.

Proof. Let

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) = z - \sum_{n=2}^{\infty} \frac{(B-A)(1-\alpha)}{C(\gamma,n)\left\{(n-1)(1+B)+(B-A)(1-\alpha)\right\}(1-\sigma+\sigma n)} \lambda_n z^n$$
$$= z - \sum_{n=2}^{\infty} a_n z^n,$$

where

$$a_n = \frac{(B-A)(1-\alpha)}{C(\gamma,n)\left\{(n-1)(1+B)+(B-A)(1-\alpha)\right\}(1-\sigma+\sigma n)} \lambda_n \geq 0, \quad n \geq 2.$$

Clearly,

$$\sum_{n=2}^{\infty} \left[\frac{(B-A)(1-\alpha)}{C(\gamma,n)\left\{(n-1)(1+B)+(B-A)(1-\alpha)\right\}(1-\sigma+\sigma n)} \times \frac{C(\gamma,n)\left\{(n-1)(1+B)+(B-A)(1-\alpha)\right\}(1-\sigma+\sigma n)}{(B-A)(1-\alpha)} \right] \lambda_n$$
$$= \sum_{n=2}^{\infty} \lambda_n = \sum_{n=1}^{\infty} \lambda_n - \lambda_1 = 1 - \lambda_1 \leq 1\,.$$

Hence $f(z) \in P_\gamma(A, B, \alpha, \sigma)$.

Conversely, assume that $f(z) \in P_\gamma(A, B, \alpha, \sigma)$ and

$$a_n = \frac{(B-A)(1-\alpha)}{C(\gamma,n)\left\{(n-1)(1+B)+(B-A)(1-\alpha)\right\}(1-\sigma+\sigma n)} \lambda_n \geq 0, \quad n \geq 2.$$

Set

$$\lambda_n = \frac{C(\gamma,n)\left\{(n-1)(1+B)+(B-A)(1-\alpha)\right\}(1-\sigma+\sigma n)}{(B-A)(1-\alpha)}, \quad n \geq 2,$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n.$$

Hence we can see that

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z).$$

Thus we complete the proof of Theorem 4. □

3 Generalized fractional integral operator

Fractional calculus has gained great importance during the last three decades due to its applications in various fields of science and engineering such as probability, fluid flow, electrical networks, and rheology. In recent years, the theory of fractional calculus operator has been effectively applied to analytic functions. Furthermore generalized operator of fractional integrals (or derivatives) having kernels of different types of special functions (like Meijer's G-function or Fox's H-function) have generated strong interest in this area. For additional excellent information one may refer to V.S. Kiryakova and H.M. Srivastava [10], H.M. Srivastava and S. Owa [23], K. Uma et al. [24], V.S. Kiryakova [8,9], R.K. Raina and M. Saigo [13], H. Amsalu and D.L. Suthar [3], S.K. Sharma and A.S. Shekhawat [17] and R.K. Raina and M. Bolia [12].

L. Galue et al. [5] defined a multiple Erdelyi-Kober operator of Riemann-Liouville type involving the well-known Fox's H-function as below.

Definition 1. *Let $m \in \mathbb{N}$, $\beta_k, \lambda_k \in \mathbb{R}$ and $\gamma_k, \delta_k \in \mathbb{C}$, $\forall\ k = 1, 2, \ldots, m$. Then the integral operator*

$$\begin{aligned} I^{(\gamma_m),(\delta_m)}_{(\beta_m),(\lambda_m);m} f(z) &= I^{(\gamma_1,\ldots,\gamma_m),(\delta_1,\ldots,\delta_m)}_{(\beta_1,\ldots,\beta_m),(\lambda_1,\ldots,\lambda_m);m} f(z) \\ &= \begin{cases} \dfrac{1}{z}\displaystyle\int_0^z H^{m,0}_{m,m}\left[\dfrac{t}{z}\left|\begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m \\ \left(\gamma_k+1-\frac{1}{\lambda_k},\frac{1}{\lambda_k}\right)_1^m\end{matrix}\right.\right] f(t)\,dt, \text{ if } \displaystyle\sum_{k=1}^m \mathrm{Re}(\delta_k) > 0, \\ f(z), \text{ if } \delta_k = 0 \text{ and } \beta_k = \lambda_k, k = 1, \ldots, m, \end{cases} \end{aligned} \tag{13}$$

is said to be a multiple Erdelyi-Kober operator of Riemann-Liouville type of multiorder $\delta = (\delta_1, \ldots, \delta_m)$.

V.S. Kiryakova [8] denote by Δ a complex domain starlike with respect to the origin $z = 0$. Also $A(\Delta)$ denotes the space of functions analytic in Δ. If $A_\rho(\Delta)$ denotes the class of functions

$$A_\rho(\Delta) = \{f(z) = z^\rho \bar{f}(z) : \bar{f}(z) \in A(\Delta)\}, \quad \rho \geq 0,$$

then clearly $A_\rho(\Delta) \subseteq A_v(\Delta) \subseteq A(\Delta)$ for $\rho \geq v \geq 0$.

It is interesting to observe that the multiple Erdelyi-Kober operator (13) of Riemann-Liouville type includes a range of important and considerable fractional integral operators as special cases. For more details, one may refer to R.K. Raina and M. Saigo [13].

Throughout this paper $(\lambda)_k = \dfrac{\Gamma(\lambda+k)}{\Gamma(\lambda)}$.

The following property of the operator (13) was derived by L. Galue et al. [5] and is needed in present investigation.

Lemma 3 ([8]). *Let $\gamma_k > -\frac{p}{\lambda_k} - 1$, $\delta_k \geq 0$, $\forall\ k = 1, \ldots, m$, $\displaystyle\sum_{k=1}^m \frac{1}{\lambda_k} \geq \sum_{k=1}^m \frac{1}{\beta_k}$. Then the operator $I^{(\gamma_m),(\delta_m)}_{(\beta_m),(\lambda_m);m}$ maps the class $\Delta_p(G)$ into itself preserving the power functions $f(z) = z^p$ (up to a constant multiplier), namely*

$$I^{(\gamma_m),(\delta_m)}_{(\beta_m),(\lambda_m);m}\{z^p\} = \prod_{k=1}^m \left\{\frac{\Gamma\left(\frac{p}{\lambda_k}+\gamma_k+1\right)}{\Gamma\left(\frac{p}{\beta_k}+\gamma_k+\delta_k+1\right)}\right\} z^p. \tag{14}$$

Theorem 5. *Let $m \in \mathbb{N}$, $h_k, g_k \in \mathbb{R}_+$ and $\gamma_k, \delta_k \in \mathbb{R}$ such that $1+\gamma_k+\delta_k > 0$, $k = 1, \ldots, m$. Let*

$$\prod_{k=1}^m \left\{\frac{(1+\gamma_k+2g_k)_{g_k}}{(1+\gamma_k+\delta_k+2h_k)_{h_k}}\right\} \leq 1 \tag{15}$$

and $f(z)$ defined by (2) be in the class $P_\gamma(A,B,\alpha,\sigma)$ with $-1 \leq A < B \leq 1$, $0 < B \leq 1$, $0 \leq \alpha < 1$, $0 \leq \gamma \leq \frac{1}{2}$, $0 \leq \sigma \leq 1$. Then

$$\begin{aligned} \left|I^{(\gamma_m),(\delta_m)}_{(h_m^{-1}),(g_m^{-1});m} f(z)\right| \geq &\left\{\prod_{k=1}^m \left(\frac{\Gamma(1+\gamma_k+g_k)}{\Gamma(1+\gamma_k+\delta_k+h_k)}\right)\right. \\ &\left.\times\left[|z| - \frac{\Psi^*(B-A)(1-\alpha)}{2\{(1+B)+(B-A)(1-\alpha)\}(1-\gamma)(1+\sigma)}|z|^2\right]\right\} \end{aligned} \tag{16}$$

and

$$\left| I^{(\gamma_m),(\delta_m)}_{(h_m^{-1}),(g_m^{-1});m} f(z) \right| \leq \left\{ \prod_{k=1}^{m} \left(\frac{\Gamma(1+\gamma_k+g_k)}{\Gamma(1+\gamma_k+\delta_k+h_k)} \right) \times \left[|z| + \frac{\Psi^*(B-A)(1-\alpha)}{2\{(1+B)+(B-A)(1-\alpha)\}(1-\gamma)(1+\sigma)} |z|^2 \right] \right\} \tag{17}$$

for $z \in U$. *The inequalities in* (16) *and* (17) *are attained by the function* $f(z)$ *given by* (12), *where*

$$\Psi^* = \prod_{k=1}^{m} \left\{ \frac{(1+\gamma_k+g_k)_{g_k}}{(1+\gamma_k+\delta_k+h_k)_{h_k}} \right\}.$$

Proof. By making use of Lemma 3 and from $f(z)$ given by (2), we get

$$I^{(\gamma_m),(\delta_m)}_{(h_m^{-1}),(g_m^{-1});m} f(z) = \prod_{k=1}^{m} \left\{ \frac{\Gamma(1+\gamma_k+g_k)}{\Gamma(1+\gamma_k+\delta_k+h_k)} \right\} z - \sum_{n=2}^{\infty} \prod_{k=1}^{m} \left\{ \frac{\Gamma(1+\gamma_k+n\,g_k)}{\Gamma(1+\gamma_k+\delta_k+n\,h_k)} \right\} a_n z^n.$$

Taking

$$V(z) = \prod_{k=1}^{m} \left\{ \frac{\Gamma(1+\gamma_k+\delta_k+g_k)}{\Gamma(1+\gamma_k+h_k)} \right\} I^{(\gamma_m),(\delta_m)}_{(h_m^{-1}),(g_m^{-1});m} f(z) = z - \sum_{n=2}^{\infty} \psi(n) a_n z^n,$$

where

$$\psi(n) = \prod_{k=1}^{m} \left\{ \frac{(1+\gamma_k+g_k)_{g_k(n-1)}}{(1+\gamma_k+\delta_k+h_k)_{h_k(n-1)}} \right\}, \quad n \in \mathbb{N} \setminus \{1\}.$$

By the hypothesis of Theorem 5 (along with the conditions (15)), we can observe that $\psi(n)$ is non-increasing for integers n, $n \geq 2$, and we get

$$0 < \psi(n) \leq \psi(2) = \prod_{k=1}^{m} \left\{ \frac{(1+\gamma_k+g_k)_{g_k}}{(1+\gamma_k+\delta_k+h_k)_{h_k}} \right\} = \Psi^*, \quad n \in \mathbb{N} \setminus \{1\}. \tag{18}$$

Now by using equations (6) and (18), we get

$$|V(z)| \geq |z| - \psi(2)\, |z|^2 \sum_{n=2}^{\infty} a_n \geq \left\{ \prod_{k=1}^{m} \left(\frac{\Gamma(1+\gamma_k+g_k)}{\Gamma(1+\gamma_k+\delta_k+h_k)} \right) \times \left[|z| - \frac{\Psi^*(B-A)(1-\alpha)}{2\{(1+B)+(B-A)(1-\alpha)\}(1-\gamma)(1+\sigma)} |z|^2 \right] \right\}$$

and

$$|V(z)| \leq |z| + \phi(2)\, |z|^2 \sum_{n=2}^{\infty} a_n \leq \left\{ \prod_{k=1}^{m} \left(\frac{\Gamma(1+\gamma_k+g_k)}{\Gamma(1+\gamma_k+\delta_k+h_k)} \right) \times \left[|z| + \frac{\Psi^*(B-A)(1-\alpha)}{2\{(1+B)+(B-A)(1-\alpha)\}(1-\gamma)(1+\sigma)} |z|^2 \right] \right\}.$$

Finally, it is easily verified that the following inequalities are attained by the function $f(z)$ given by (12), namely

$$\left| I^{(\gamma_m),(\delta_m)}_{(h_m^{-1}),(g_m^{-1});m} f(z) \right| \geq \left\{ \prod_{k=1}^{m} \left(\frac{\Gamma(1+\gamma_k+g_k)}{\Gamma(1+\gamma_k+\delta_k+h_k)} \right) \times \left[|z| - \frac{\Psi^*(B-A)(1-\alpha)}{2\{(1+B)+(B-A)(1-\alpha)\}(1-\gamma)(1+\sigma)} |z|^2 \right] \right\}$$

and

$$\left| I^{(\gamma_m),(\delta_m)}_{(h_m^{-1}),(g_m^{-1});m} f(z) \right| \leq \left\{ \prod_{k=1}^{m} \left(\frac{\Gamma(1+\gamma_k+g_k)}{\Gamma(1+\gamma_k+\delta_k+h_k)} \right) \times \left[|z| + \frac{\Psi^*(B-A)(1-\alpha)}{2\{(1+B)+(B-A)(1-\alpha)\}(1-\gamma)(1+\sigma)} |z|^2 \right] \right\},$$

which are required in (16) and (17). This completes the proof. □

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Сойбаш Д., Джоші С.Б., Павар Х.Х. *Про узагальнений дробовий інтегральний оператор, що включає H-функцію Фокса та його застосування до уніфікованого підкласу передзірчастих функцій з від'ємними коефіцієнтами* // Карпатські матем. публ. — 2022. — Т.14, №1. — С. 289–298.

Метою цієї статті є знаходження різних цікавих властивостей та зарактеризації уніфікованого класу $P_\gamma(A, B, \alpha, \sigma)$ передзірчастих функцій з від'ємними коефіцієнтами в одиничному диску U. Більш того, ми доводимо теореми спотворення для функцій з цього класу, які включають узагальнений дробовий інтегральний оператор, що включає відому H-функцію Фокса.

Ключові слова і фрази: однолиста функція, теорема спотворення, передзірчаста функція.