ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2021, **13** (3), 676–686
doi:10.15330/cmp.13.3.676-686

https://journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2021, Т.13, №3, С.676–686

Recovery of continuous functions of two variables from their Fourier coefficients known with error

Pozharskyi A.A., Pozharska K.V.✉

In this paper, we continue to study the classical problem of optimal recovery for the classes of continuous functions. The investigated classes $W^{\psi}_{2,p}$, $1 \le p < \infty$, consist of functions that are given in terms of generalized smoothness ψ. Namely, we consider the two-dimensional case which complements the recent results from [Res. Math. 2020, **28** (2), 24–34] for the classes W^{ψ}_{p} of univariate functions.

As to available information, we are given the noisy Fourier coefficients $y^{\delta}_{i,j} = y_{i,j} + \delta \xi_{i,j}$, $\delta \in (0,1)$, $i, j = 1, 2, \ldots,$ of functions with respect to certain orthonormal system $\{\varphi_{i,j}\}_{i,j=1}^{\infty}$, where the noise level is small in the sense of the norm of the space l_p, $1 \le p < \infty$, of double sequences $\xi = (\xi_{i,j})_{i,j=1}^{\infty}$ of real numbers. As a recovery method, we use the so-called Λ-method of summation given by certain two-dimensional triangular numerical matrix $\Lambda = \{\lambda^{n}_{i,j}\}_{i,j=1}^{n}$, where n is a natural number associated with the sequence ψ that define smoothness of the investigated functions. The recovery error is estimated in the norm of the space $C([0,1]^2)$ of continuous on $[0,1]^2$ functions.

We showed, that for $1 \le p < \infty$, under the respective assumptions on the smoothness parameter ψ and the elements of the matrix Λ, it holds

$$\Delta(W^{\psi}_{2,p}, \Lambda, l_p) = \sup_{y \in W^{\psi}_{2,p}} \sup_{\|\xi\|_{l_p} \le 1} \Big\| y - \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda^{n}_{i,j} (y_{i,j} + \delta \xi_{i,j}) \varphi_{i,j} \Big\|_{C([0,1]^2)} \ll \frac{n^{\beta+1-1/p}}{\psi(n)}.$$

Key words and phrases: Fourier series, method of regularization, Λ-method of summation.

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereschenkivska str., 01024, Kyiv, Ukraine
✉ Corresponding author
E-mail: pozharskyio@gmail.com (Pozharskyi A.A.), kate.shvai@gmail.com (Pozharska K.V.)

Notation

Let $L_p([0,1]^2)$, $1 \le p < \infty$, be the space of real-valued summable with pth power on the square $[0,1]^2$ functions of two variables $f\colon [0,1]^2 \to \mathbb{R}$ equipped with the norm

$$\|f\|_{L_2([0,1]^2)} := \left(\int_0^1 \int_0^1 |f(t,\tau)|^p \, \mathrm{d}t \mathrm{d}\tau \right)^{1/p}, \quad 1 \le p < \infty;$$

$C([0,1]^2)$ be the space of continuous on $[0,1]^2$ functions $f\colon [0,1]^2 \to \mathbb{R}$ with the norm

$$\|f\|_{C([0,1]^2)} := \max_{t,\tau \in [0,1]^2} |f(t,\tau)|;$$

УДК 517.5
2020 *Mathematics Subject Classification:* 41A10, 46E35, 47A52.

The work was partially financially supported by the budget program "Support of the development of priority branches of scientific research in 2020" (KPKVK 6541230).

l_p, $1 \le p < \infty$, be the set of double sequences $\xi = (\xi_{i,j})_{i,j=1}^{\infty}$ of real numbers, such that

$$\|\xi\|_{l_p} := \left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\xi_{i,j}|^p \right)^{1/p} < \infty;$$

l_∞ be the space of bounded double sequences $\xi = (\xi_{i,j})_{i,j=1}^{\infty}$ of real numbers with the norm

$$\|\xi\|_{l_\infty} := \sup_{i,j \in \mathbb{N}} |\xi_{i,j}|.$$

For a function $y \in C([0,1]^2)$ let us consider its Fourier series

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} y_{i,j} \varphi_{i,j}(t,\tau),$$

where

$$y_{i,j} := \langle y, \varphi_{i,j} \rangle_{L_2([0,1]^2)} = \int_0^1 \int_0^1 y(t,\tau) \varphi_{i,j}(t,\tau)\, dt d\tau$$

are the Fourier coefficients of the function y with respect to certain system $\{\varphi_{i,j}\}_{i,j=1}^{\infty}$ of continuous on $[0,1]^2$ functions, such that $\varphi_{i,j}(t,\tau) = \varphi_i(t)\varphi_j(\tau)$, where $\Phi := \{\varphi_k\}_{k=1}^{\infty}$ is a complete orthonormal system in the space $L_2([0,1])$ of square summable on $[0,1]$ functions.

Assume also, that functions from the system Φ satisfy the condition

$$\|\varphi_k\|_{C([0,1])} \le C_1 k^{\beta}, \quad k = 1, 2, \ldots, \tag{1}$$

where $C([0,1])$ is the space of continuous on $[0,1]$ functions with usual norm, $C_1 > 0$, $\beta \ge 0$ are some constants. The set of such systems we denote by K^{β}.

In what follows, the notation $A \asymp B$ for a positive number sequence $A = (A_n)_{n=1}^{\infty}$ and function $B = B(\delta)$, $\delta \in (0,1)$, that may depend on some set of parameters, means that for all admissible values of this parameters under certain connection between $n \in \mathbb{N}$ and $\delta \in (0,1)$ the relations $c_1 B \le A \le c_2 B$ are true with certain positive quantities c_1 and c_2 that do not depend on $n \in \mathbb{N}$ and $\delta \in (0,1)$. We also use symbols $\ll$ and $\gg$, i.e. $A \ll B$ $(A \gg B)$, if $A \le cB$ $(B \le cA)$ for some $c > 0$ that does not depend on $n \in \mathbb{N}$ and $\delta \in (0,1)$. In the case of two positive number sequences $A = (A_n)_{n=1}^{\infty}$ and $D = (D_n)_{n=1}^{\infty}$, under the indicated above conditions we write $A \asymp D$, $A \ll D$ and $A \gg D$ (the constants c_i in corresponding inequalities do not depend on the parameter $n \in \mathbb{N}$).

Note also, that quantities C_i, may depend on some parameters. This dependence is usually not important in the investigated context.

1 Problem statement and history overview

Further, for $y \in C([0,1]^2)$, let us know only approximate values $y_{i,j}^{\delta}$ of their Fourier coefficients $y_{i,j}$, such that

$$y_{i,j}^{\delta} = y_{i,j} + \delta \xi_{i,j}, \quad i,j = 1, 2, \ldots,$$

where $\delta \in (0,1)$ and $\xi = (\xi_{i,j})_{i,j=1}^{\infty}$ is a noise, for which

$$\|\xi\|_{l_p} \le 1. \tag{2}$$

Put $y^{\delta,p} := (y^{\delta}_{i,j})_{i,j=1}^{\infty}$ and for a function class $F \subset C([0,1]^2)$ denote by $Y^{\delta,p}(F)$ the set of all accordingly given approximative Fourier coefficients $y^{\delta,p}$ of functions $y \in F$.

A recovery problem for functions $y \in F$ from their coefficients $y^{\delta,p}$ consists in determination or choosing a mapping $A\colon Y^{\delta,p}(F) \to C([0,1]^2)$ (method of the recovery) such that the quantity

$$\varepsilon(\delta) = \Delta(F, A, l_p, C([0,1]^2)) := \sup_{y\in F} \sup_{\|\xi\|_{l_p}\le 1} \|y - Ay^{\delta,p}\|_{C([0,1]^2)}$$

tends to zero as $\delta \to 0$.

Quite complete information on a general problem statement for the optimal recovery in normed spaces, as well as corresponding results for the classes of smooth and analytic functions defined on various compact manifolds can be found in [8]. In the paper by A.M. Tikhonov [17] a method of series summation was suggested based on the idea of regularization. Note, that earlier [16] the scientist has formulated general ideas on regularization.

As to recovery problems for functions in the case where one knows exact values of the respective Fourier coefficients instead of noisy one, they are well studied for different classes and error norms. In the context of linear methods of summation of the Fourier series defined by triangular matrices, we mention the well-known names of A.N. Kolmogorov, S.M. Nikol'skii, S.B. Stechkin, N.P. Korneichuk, V.K. Dzyadyk, A.I. Stepanets and others. Among recent results in this direction, we refer to the following papers: [9], where order estimates of the uniform approximations by Zygmund sums on the classes of convolutions of periodic functions were obtained, [1] for direct and inverse theorems of approximation by the methods of Zygmund, Abel-Poisson, Taylor-Abel-Poisson in the Orlicz-type spaces, and [11] for the investigation of saturation of several linear methods of summation of the Fourier series.

Another important direction of research deals with investigation of summation methods defined by a set of functions of a natural argument, e.g., the approximation of functions from different classes by the classical, biharmonic or three-harmonic Poisson integrals, Weierstrass integrals, etc. (see the papers [2–4, 6, 7], were one can find further references).

As to "ill-posed" problems (for the perturbed Fourier coefficients of respective functions), for a more detailed information concerning the univariate case we refer to the paper [5].

The two-dimensional case is much less investigated. In particular, initially here appears the problem of choosing an algorithm for the construction of an effective apparatus of approximation. A core point here is determining a structure of the finite sets of points (i,j) in $\mathbb{Z}^2$ which form the base for building subsets in the set $\{\varphi_{i,j}\}_{i,j=1}^{\infty}$, that generate finite dimensional approximative subsets. Usually, such sets are squares, circles, so-called "hyperbolic crosses", etc. So, in solving the recovery problem on the classes of functions of Sobolev type of smoothness and functions with dominating mixed partial derivatives in the paper [10] rectangles have been used, in [12] hyperbolic crosses, and in [13] triangles and non-uniform hyperbolic crosses.

In the present paper we use a method of series summation of the following form

$$T_n^{\Lambda}(y^{\delta})(t,\tau) = \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda^n_{i,j} y^{\delta}_{i,j} \varphi_{i,j}(t,\tau) \tag{3}$$

as a recovery method, where n is a natural number associated with δ, $\Lambda = \{\lambda^n_{i,j}\}_{i,j=1}^{n}$ is certain two-dimensional triangular numerical matrix, the elements of which satisfy the condition

$$|1 - \lambda^n_{i,j}| \le C_2 \Big(\frac{ij}{n^2}\Big)^{\theta}, \quad i,j = \overline{1,n}, \quad n \in \mathbb{N}, \tag{4}$$

for some number $\theta > 0$ and positive constant C_2. One says, that the method $T_n^{\Lambda}(y^{\delta})$ is a summation method on the squares

$$\square_n := \{(i,j) \in \mathbb{Z}^2 \colon\ 1 \le i \le n,\ 1 \le j \le n\},$$

and, if the condition (4) holds, it is of the order θ.

The aim of the present paper is to estimate the quantity

$$\Delta(W^{\psi}_{2,p}, \Lambda, l_p) := \sup_{y \in W^{\psi}_{2,p}} \sup_{\|\xi\|_{l_p} \le 1} \Big\| y - \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda^n_{i,j} (y_{i,j} + \delta \xi_{i,j}) \varphi_{i,j} \Big\|_{C([0,1]^2)},$$

$1 \le p < \infty$, $\delta \in (0,1)$, when:

1) $\Phi \in K^{\beta}$;

2) for elements of the matrix Λ the condition (4) is satisfied;

3) $W^{\psi}_{2,p} = \Big\{ y \in L_2([0,1]^2) \colon\ \|y\|^p_{W^{\psi}_{2,p}} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (\psi(i)\psi(j))^p |y_{i,j}|^p \le 1 \Big\}$, where the sequence $\{\psi(k), k \in \mathbb{N}\}$ belongs to the further defined set Ψ_{γ_1,γ_2}, $0 < \gamma_1 < \gamma_2$.

Therefore, in this problem statement, firstly, the case is covered when the noise ξ is stronger than in the space l_2 (in view of $\|\xi\|_{l_2} \le \|\xi\|_{l_p}$ if $1 \le p \le 2$), but still not stochastic. Secondly, a quite wide spectrum of the classes $W^{\psi}_{2,p}$, $1 \le p < \infty$, is considered of functions to recover. As to the classes $W^{\psi}_{2,p}$, we additionally note that for fixed system Φ and function ψ the following embedding $W^{\psi}_{2,p} \subseteq W^{\psi}_{2,q}$, $1 \le p \le q < \infty$, holds.

We should also mention, that in the paper [14] the classes (and spaces) S^p were introduced with non-symmetric metric that in partial case coincide with corresponding classes $W^{\psi}_{1,p}$ of one-variable functions

$$W^{\psi}_{1,p} = \Big\{ y \in L_2([0,1]) \colon\ \|y\|^p_{W^{\psi}_{1,p}} = \sum_{k=1}^{\infty} \psi^p(k) |y_k|^p \le 1 \Big\},$$

where $y_k = \langle y, \varphi_k \rangle_{L_2([0,1])} = \int_0^1 y(t) \varphi_k(t)\, dt$, $k = 1, 2, \dots$, are the Fourier coefficients of y with respect to certain complete orthonormal in the space $L_2([0,1])$ with the inner product $\langle \cdot, \cdot \rangle_{L_2([0,1])}$ system $\{\varphi_k\}_{k=1}^{\infty}$ of continuous on $[0,1]$ functions. There (see also [15]) the estimates are obtained for the best approximations and Kolmogorov widths of q-ellipsoids in this spaces.

2 Estimates of the recovery error

By Ψ_{γ_1,γ_2}, $0 < \gamma_1 < \gamma_2$, we denote the set of continuous, positive and strictly increasing on $[1,\infty)$ real-valued functions ψ that satisfy the conditions:

1) $\psi(1) = 1$;

2) for some γ, $\gamma_1 < \gamma < \gamma_2$, the function $\phi_-(\tau) := \tau^{\gamma}/\psi(\tau)$ does not increase for $\tau \ge 1$, and $\phi_-(\tau) \to 0$ as $\tau \to \infty$;

3) the function $\phi_+(\tau) := \tau^{\gamma_2}/\psi(\tau)$ does not decrease for $\tau \ge 1$.

In particular, power functions $\psi(\tau) = \tau^{\alpha_1}$ with $\gamma_1 < \alpha_1 \le \gamma_2$, and also functions of the form $\psi(\tau) = \tau^{\alpha_1}(\log^+(\tau))^{\alpha_2}$, $\gamma_1 < \alpha_1 < \gamma_2$, $\alpha_2 \in \mathbb{R}$ or $\psi(\tau) = \tau^{\gamma_2}(\log^+(\tau))^{\alpha}$, $\alpha < 0$, belong to the set Ψ_{γ_1,γ_2}, where $\log^+(\tau) = \max\{1, \log(\tau)\}$, and the symbol log denotes a logarithm with arbitrary base $a > 0$, $a \ne 1$.

Theorem. *Let $1 \le p \le 2$, $\psi \in \Psi_{\beta+\frac{1}{2},\theta}$, or $2 < p < \infty$, $\psi \in \Psi_{\beta+1,\theta}$ and $T_n^{\Lambda}(y^{\delta})$ is a method of recovery defined by the formula* (3). *Then for $n^{\beta+1-1/p}\psi(n) \asymp 1/\delta$ we have the estimate*

$$\Delta(W^{\psi}_{2,p}, \Lambda, l_p) \ll \frac{n^{\beta+1-1/p}}{\psi(n)}.$$

Proof. First note that under the conditions of the theorem the set $W^{\psi}_{2,p}$ consists of continuous on $[0,1]^2$ functions, and for any $y \in W^{\psi}_{2,p}$ its Fourier series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} y_{i,j}\varphi_{i,j}(t,\tau)$ with respect to the system $\{\varphi_{i,j}\}_{i,j=1}^{\infty}$ converges uniformly on $[0,1]^2$ to y, because

$$\Big\| \sum_{(i,j)\in\mathbb{Z}^2\setminus\square_n} y_{i,j}\varphi_{i,j}\Big\|_{C([0,1]^2)} \to 0 \quad \text{as} \quad n \to \infty$$

(see further estimates of the terms I_2, I_3 and I_4 of the relation (5)).

Therefore, for any $y \in W^{\psi}_{2,p}$ we can write

$$\begin{aligned} \|y - T_n^{\Lambda}(y^{\delta})\|_{C([0,1]^2)} \le & \Big\| \sum_{i=1}^{n}\sum_{j=1}^{n}(1-\lambda^n_{i,j})y_{i,j}\varphi_{i,j}\Big\|_{C([0,1]^2)} + \Big\| \sum_{i=1}^{n}\sum_{j=n+1}^{\infty} y_{i,j}\varphi_{i,j}\Big\|_{C([0,1]^2)} \\ & + \Big\| \sum_{i=n+1}^{\infty}\sum_{j=1}^{n} y_{i,j}\varphi_{i,j}\Big\|_{C([0,1]^2)} + \Big\| \sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty} y_{i,j}\varphi_{i,j}\Big\|_{C([0,1]^2)} \\ & + \delta\Big\| \sum_{i=1}^{n}\sum_{j=1}^{n}\lambda^n_{i,j}\xi_{i,j}\varphi_{i,j}\Big\|_{C([0,1]^2)} =: I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{5}$$

Let us estimate first the term I_1 from (5). Taking into account the condition (4) and norm properties, we obtain

$$\begin{aligned} I_1 = \Big\| \sum_{i=1}^{n}\sum_{j=1}^{n}(1-\lambda^n_{i,j})y_{i,j}\varphi_{i,j}\Big\|_{C([0,1]^2)} & \le C_2 \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{ij}{n^2}\right)^{\theta}|y_{i,j}|\|\varphi_{i,j}\|_{C([0,1]^2)} \\ & = C_2 \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{ij}{n^2}\right)^{\theta}\psi(i)\psi(j)|y_{i,j}|\frac{\|\varphi_{i,j}\|_{C([0,1]^2)}}{\psi(i)\psi(j)}. \end{aligned} \tag{6}$$

Now note the following. In further speculations, in particular, when using the Hölder's inequality for number sequences, an important thing in the form of writing is that $p \ne 1$. Nevertheless, this speculations as well as their result remain true also for $p = 1$ after their corresponding correction with respect to the definition of norm in the space l_{∞}.

Therefore, applying the Hölder's inequality to the right-hand side of (6) for $1/p + 1/p' = 1$, $p \ne 1$, we get

$$\begin{aligned} I_1 & \le C_2 \Big(\sum_{i=1}^{n}\sum_{j=1}^{n}(\psi(i)\psi(j))^p|y_{i,j}|^p\Big)^{1/p}\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\left(\frac{ij}{n^2}\right)^{\theta}\frac{\|\varphi_{i,j}\|_{C([0,1]^2)}}{\psi(i)\psi(j)}\Big)^{p'}\Big)^{1/p'} \\ & \le C_2\|y\|_{W^{\psi}_{2,p}}\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\frac{(ij)^{\theta}}{\psi(i)\psi(j)}\frac{\|\varphi_{i,j}\|_{C([0,1]^2)}}{n^{2\theta}}\Big)^{p'}\Big)^{1/p'}. \end{aligned} \tag{7}$$

Now, taking into account, that $\|y\|_{W^{\psi}_{2,p}} \leq 1$, the function ψ satisfies condition 3) in the definition of the set Ψ_{γ_1,γ_2} with $\gamma_2 = \theta$, and in view of (1) from (7) we have

$$\begin{aligned} I_1 &\leq C_2 \frac{n^{2\theta}}{(\psi(n))^2 n^{2\theta}} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \|\varphi_{i,j}\|^{p'}_{C([0,1]^2)} \right)^{1/p'} \leq \frac{C_2(C_1)^2}{(\psi(n))^2} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} (ij)^{\beta p'} \right)^{1/p'} \\ &= \frac{C_2(C_1)^2}{(\psi(n))^2} \left(\sum_{i=1}^{n} i^{\beta p'} \right)^{1/p'} \left(\sum_{j=1}^{n} j^{\beta p'} \right)^{1/p'} \leq \frac{C_2(C_1)^2}{(\psi(n))^2} (n^{\beta p'} n)^{\frac{2}{p'}} = \frac{C_2(C_1)^2}{(\psi(n))^2} n^{2(\beta+1-1/p)}. \end{aligned} \tag{8}$$

The same estimate holds also for $p = 1$.

Further we move to estimation of the term I_2 from the right-hand side of (5). First, applying the Hölder's inequality (for $p \neq 1$) and using (1), we can write

$$\begin{aligned} I_2 &= \left\| \sum_{i=1}^{n} \sum_{j=n+1}^{\infty} y_{i,j} \varphi_{i,j} \right\|_{C([0,1]^2)} = \left\| \sum_{i=1}^{n} \sum_{j=n+1}^{\infty} \psi(i)\psi(j) y_{i,j} \frac{\varphi_{i,j}}{\psi(i)\psi(j)} \right\|_{C([0,1]^2)} \\ &\leq \left(\sum_{i=1}^{n} \sum_{j=n+1}^{\infty} (\psi(i)\psi(j))^p |y_{i,j}|^p \right)^{1/p} \left(\sum_{i=1}^{n} \sum_{j=n+1}^{\infty} \left(\frac{\|\varphi_{i,j}\|_{C([0,1]^2)}}{\psi(i)\psi(j)} \right)^{p'} \right)^{1/p'} \\ &\leq (C_1)^2 \|y\|_{W^{\psi}_{2,p}} \left(\sum_{i=1}^{n} \sum_{j=n+1}^{\infty} \left(\frac{(ij)^{\beta}}{\psi(i)\psi(j)} \right)^{p'} \right)^{1/p'}. \end{aligned} \tag{9}$$

In what follows, we split estimation of the right-hand side of (9) into two cases: when $1 < p \leq 2$ and $2 < p < \infty$.

Let $1 < p \leq 2$, then from (9) taking into account that $\|y\|_{W^{\psi}_{2,p}} \leq 1$, we obtain

$$I_2 \leq (C_1)^2 \left(\sum_{i=1}^{n} \left(\frac{i^{\beta+1/2+\varepsilon}}{\psi(i)} \frac{1}{i^{1/2+\varepsilon}} \right)^{p'} \right)^{1/p'} \left(\sum_{j=n+1}^{\infty} \left(\frac{j^{\beta+1/2+\varepsilon}}{\psi(j)} \frac{1}{j^{1/2+\varepsilon}} \right)^{p'} \right)^{1/p'} \tag{10}$$

(here $\varepsilon > 0$ is arbitrary).

By the conditions of the theorem, the function ψ belongs to $\Psi_{\beta+1/2,\theta}$, therefore for sufficiently small $\varepsilon > 0$ (10) yields

$$I_2 \leq \frac{(C_1)^2}{\psi(1)} \left(\sum_{i=1}^{n} i^{-(1/2+\varepsilon)p'} \right)^{1/p'} \frac{(n+1)^{\beta+1/2+\varepsilon}}{\psi(n+1)} \left(\sum_{j=n+1}^{\infty} j^{-(1/2+\varepsilon)p'} \right)^{1/p'}. \tag{11}$$

Obviously, for any $\alpha > 1$ the following relations hold

$$\begin{aligned} \sum_{k=1}^{n} k^{-\alpha} = 1 + \sum_{k=2}^{n} k^{-\alpha} \leq 1 + \int_1^n x^{-\alpha} dx &= 1 + \frac{1}{1-\alpha}(n^{-\alpha+1} - 1) \\ &= 1 + \frac{1}{\alpha - 1} - \frac{1}{(\alpha-1)n^{\alpha-1}} \leq \frac{\alpha}{\alpha-1} \end{aligned} \tag{12}$$

and

$$\sum_{k=n+1}^{\infty} k^{-\alpha} \leq \int_n^{\infty} x^{-\alpha} dx = \frac{1}{1-\alpha}(0 - n^{-\alpha+1}) = \frac{n^{-\alpha+1}}{\alpha - 1}. \tag{13}$$

In view of $1 < p \le 2$ it holds $(1/2+\varepsilon)\, p' > 1$, besides $\psi(1) = 1$ (see condition 1) from the definition of the set Ψ_{γ_1,γ_2}), and hence using (12) and (13), from (11) we derive

$$\begin{aligned} I_2 &\le (C_1)^2 \left(\frac{(1/2+\varepsilon)p'}{((1/2+\varepsilon)p'-1)^2} \right)^{1/p'} \frac{(n+1)^{\beta+1/2+\varepsilon} n^{1/p'-(1/2+\varepsilon)}}{\psi(n+1)} \\ &\le (C_1)^2 \left(\frac{(1/2+\varepsilon)p'}{((1/2+\varepsilon)p'-1)^2} \right)^{1/p'} \frac{n^{\beta+1-1/p}}{\psi(n)}. \end{aligned} \tag{14}$$

It is easy to show, that for $p = 1$ it holds

$$I_2 \le (C_1)^2 \frac{n^\beta}{\psi(n)}. \tag{15}$$

Assume now, that $2 < p < \infty$. Here, as in the previous case, we take into account the condition $\psi \in \Psi_{\beta+1,\theta}$, and for sufficiently small $\varepsilon > 0$ from (9) we get

$$\begin{aligned} I_2 &\le (C_1)^2 \left(\sum_{i=1}^{n} \left(\frac{i^{\beta+1+\varepsilon}}{\psi(i)} \frac{1}{i^{1+\varepsilon}} \right)^{p'} \right)^{1/p'} \left(\sum_{j=n+1}^{\infty} \left(\frac{j^{\beta+1+\varepsilon}}{\psi(j)} \frac{1}{j^{1+\varepsilon}} \right)^{p'} \right)^{1/p'} \\ &\le \frac{(C_1)^2}{\psi(1)} \left(\sum_{i=1}^{n} i^{-(1+\varepsilon)p'} \right)^{1/p'} \frac{n^{\beta+1+\varepsilon}}{\psi(n)} \left(\sum_{j=n+1}^{\infty} j^{-(1+\varepsilon)p'} \right)^{1/p'}. \end{aligned} \tag{16}$$

Since $1 < p' < 2$, then $(1+\varepsilon)p' > 1$, and therefore (16) yields

$$I_2 \le (C_1)^2 \left(\frac{(1+\varepsilon)p'}{((1+\varepsilon)p'-1)^2} \right)^{1/p'} \frac{n^{\beta+1-1/p}}{\psi(n)}. \tag{17}$$

From the relations (14), (15) and (17) we conclude, that

$$I_2 \le C_3(p) \frac{n^{\beta+1-1/p}}{\psi(n)}, \quad 1 \le p < \infty, \tag{18}$$

where

$$C_3(p) = \begin{cases} (C_1)^2, & p = 1, \\ (C_1)^2 \big((1/2+\varepsilon)p' / ((1/2+\varepsilon)p'-1)^2\big)^{1/p'}, & 1 < p \le 2, \\ (C_1)^2 \big((1+\varepsilon)p' / ((1+\varepsilon)p'-1)^2\big)^{1/p'}, & 2 < p < \infty. \end{cases}$$

Making similar speculations, we obtain the same estimate also for the term I_3 from the right-hand side of (5):

$$I_3 = \Big\| \sum_{i=n+1}^{\infty} \sum_{j=1}^{n} y_{i,j} \varphi_{i,j} \Big\|_{C([0,1]^2)} \le C_3(p) \frac{n^{\beta+1-1/p}}{\psi(n)}, \quad 1 \le p < \infty. \tag{19}$$

Estimate now the fourth term from the right-hand side of (5). If $1 < p \le 2$, arguing similarly as in proving the estimate (10), for arbitrary $\varepsilon > 0$ we write

$$I_4 = \Big\| \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} y_{i,j} \varphi_{i,j} \Big\|_{C([0,1]^2)} \le (C_1)^2 \left(\sum_{i=n+1}^{\infty} \left(\frac{i^{\beta+1/2+\varepsilon}}{\psi(i)} \frac{1}{i^{1/2+\varepsilon}} \right)^{p'} \right)^{2/p'}. \tag{20}$$

Then, in view of (13) and the condition $\psi \in \Psi_{\beta+1,\theta}$, from (20) we get for sufficiently small $\varepsilon > 0$, that

$$I_4 \le (C_1)^2 \left[\left[\frac{(n+1)^{\beta+1/2+\varepsilon}}{\psi(n+1)} \right]^{p'} \frac{n^{1-(1/2+\varepsilon)p'}}{(1/2+\varepsilon)p'-1} \right]^{2/p'} \le \frac{(C_1)^2}{((1/2+\varepsilon)p'-1)^{2/p'}} \frac{n^{2(\beta+1-1/p)}}{(\psi(n))^2}. \quad (21)$$

One can easily verify that for $p = 1$ it holds

$$I_4 \le (C_1)^2 \frac{n^{2\beta}}{(\psi(n))^2}. \quad (22)$$

In the case $2 < p < \infty$ we obtain the estimate

$$I_4 \le \frac{(C_1)^2}{((1+\varepsilon)p'-1)^{2/p'}} \frac{n^{2(\beta+1-1/p)}}{(\psi(n))^2}. \quad (23)$$

Respectively, combining (21), (22) and (23) for $1 \le p < \infty$ we obtain

$$I_4 \le C_4(p) \frac{n^{2(\beta+1-1/p)}}{(\psi(n))^2}, \quad (24)$$

where

$$C_4(p) = \begin{cases} (C_1)^2, & p = 1, \\ (C_1)^2/((1/2+\varepsilon)p'-1)^{2/p'}, & 1 < p \le 2, \\ (C_1)^2/((1+\varepsilon)p'-1)^{2/p'}, & 2 < p < \infty. \end{cases}$$

It remains to estimate the term I_5 from the right-hand side of (5). Note, that the condition (4) yields uniform boundness of the elements $\lambda_{i,j}^n$ of the matrix Λ, moreover $|\lambda_{i,j}^n| \le C_2 + 1$, $i, j = \overline{1, n}$, $n \in \mathbb{N}$. So, we can write

$$I_5 = \delta \left\| \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{i,j}^n \xi_{i,j} \varphi_{i,j} \right\|_{C([0,1]^2)} \le (C_2+1)\delta \sum_{i=1}^{n} \sum_{j=1}^{n} |\xi_{i,j}| \|\varphi_{i,j}\|_{C([0,1]^2)}. \quad (25)$$

Using the Hölder's inequality to the right-hand side of (25) for $1/p + 1/p' = 1$, $p \ne 1$, and then taking into account the conditions (2), (1), we get

$$I_5 \le (C_2+1)\delta \left[\sum_{i=1}^{n} \sum_{j=1}^{n} |\xi_{i,j}|^p \right]^{1/p} \left[\sum_{i=1}^{n} \sum_{j=1}^{n} \|\varphi_{i,j}\|_{C([0,1]^2)}^{p'} \right]^{1/p'} \le \frac{C_2+1}{(C_1)^{-2}} \delta \|\xi\|_{l_p} \left[\sum_{i=1}^{n} \sum_{j=1}^{n} (ij)^{\beta p'} \right]^{1/p'}.$$

Hence, making similar speculations to that in proving the estimate (8), we obtain

$$I_5 \le (C_2+1)(C_1)^2 \delta n^{2(\beta+1-1/p)}. \quad (26)$$

The same estimate holds also for $p = 1$.

Further, combining the initial inequality (5) with the estimates (8), (18), (19), (24) and (26), we derive

$$\begin{aligned} \|y - T_n^\Lambda(y^\delta)\|_{C([0,1]^2)} &\le ((C_1)^2 C_2 + C_4(p)) \left(\frac{n^{\beta+1-1/p}}{\psi(n)} \right)^2 + 2C_3(p) \frac{n^{\beta+1-1/p}}{\psi(n)} \\ &\quad + (C_2+1)(C_1)^2 \delta n^{2(\beta+1-1/p)} \\ &= n^{2(\beta+1-1/p)} \left(\frac{(C_1)^2 C_2 + C_4(p)}{(\psi(n))^2} + \frac{2C_3(p)}{n^{\beta+1-1/p}\psi(n)} + (C_2+1)(C_1)^2\delta \right) \quad (27) \\ &= n^{2(\beta+1-1/p)} \left(\frac{((C_1)^2 C_2 + C_4(p)) \frac{n^{\beta+1-1/p}}{\psi(n)} + 2C_3(p)}{n^{\beta+1-1/p}\psi(n)} + (C_2+1)(C_1)^2\delta \right). \end{aligned}$$

Assume first, that $1 \leq p \leq 2$, $\psi \in \Psi_{\beta+1/2,\theta}$. Then for any $n \in \mathbb{N}$ and a sufficiently small $\varepsilon > 0$ we have

$$1 = \frac{\psi(1)}{1^{\beta+1/2+\varepsilon}} \leq \frac{\psi(n)}{n^{\beta+1/2+\varepsilon}},$$

therefore

$$\frac{n^{\beta+1/2+\varepsilon}}{\psi(n)} \leq 1.$$

Then, from (27), in view of $1/p - 1/2 + \varepsilon > 0$, it follows that

$$\begin{aligned} &\|y - T_n^{\Lambda}(y^{\delta})\|_{C([0,1]^2)} \\ &\leq n^{2(\beta+1-1/p)} \left(\frac{\frac{(C_1)^2 C_2 + C_4(p)}{n^{1/p-1/2+\varepsilon}} \frac{n^{\beta+1/2+\varepsilon}}{\psi(n)} + 2C_3(p)}{n^{\beta+1-1/p}\psi(n)} + (C_2+1)(C_1)^2\delta \right) \\ &\leq n^{2(\beta+1-1/p)} \Big(\frac{(C_1)^2 C_2 + C_4(p) + 2C_3(p)}{n^{\beta+1-1/p}\psi(n)} + (C_2+1)(C_1)^2\delta \Big). \end{aligned} \tag{28}$$

Arguing similarly in the case $2 < p < \infty$, $\psi \in \Psi_{\beta+1,\theta}$, we get

$$\frac{n^{\beta+1+\varepsilon}}{\psi(n)} \leq 1.$$

Respectively, from (27) we have the estimate

$$\begin{aligned} &\|y - T_n^{\Lambda}(y^{\delta})\|_{C([0,1]^2)} \\ &\leq n^{2(\beta+1-1/p)} \left(\frac{\frac{(C_1)^2 C_2 + C_4(p)}{n^{1/p+\varepsilon}} \frac{n^{\beta+1+\varepsilon}}{\psi(n)} + 2C_3(p)}{n^{\beta+1-1/p}\psi(n)} + (C_2+1)(C_1)^2\delta \right) \\ &\leq n^{2(\beta+1-1/p)} \Big(\frac{(C_1)^2 C_2 + C_4(p) + 2C_3(p)}{n^{\beta+1-1/p}\psi(n)} + (C_2+1)(C_1)^2\delta \Big). \end{aligned} \tag{29}$$

Hence, according to (28) and (29), for all $1 \leq p < \infty$ it holds

$$\|y - T_n^{\Lambda}(y^{\delta})\|_{C([0,1]^2)} \leq n^{2(\beta+1-1/p)} \Big(\frac{(C_1)^2 C_2 + C_4(p) + 2C_3(p)}{n^{\beta+1-1/p}\psi(n)} + (C_2+1)(C_1)^2\delta \Big). \tag{30}$$

Now, if $n \in \mathbb{N}$ and $\delta \in (0,1)$ are such that $n^{\beta+1-1/p}\psi(n) \asymp \frac{1}{\delta}$, then (30) yields

$$\|y - T_n^{\Lambda}(y^{\delta})\|_{C([0,1]^2)} \ll \frac{n^{\beta+1-1/p}}{\psi(n)}.$$

Theorem is proved. □

Remark 1. *In the case $\psi(k) = k^{\mu}$, $p = 2$, Theorem yields the estimate for $\Delta(W^{\mu}_{2,2}, \Lambda, l_2)$, where*

$$W^{\mu}_{2,2} = \Big\{ y \in L_2([0,1]^2) \colon \ \|y\|^2_{W^{\mu}_{2,2}} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} (ij)^{2\mu} |y_{i,j}|^2 \leq 1 \Big\},$$

with $\beta + 1/2 < \mu \leq \theta$, $n \asymp \delta^{-2/(2\mu+2\beta+1)}$, namely

$$\Delta(W^{\mu}_{2,2}, \Lambda, l_2) \ll n^{\beta+1/2-\mu},$$

or, in terms of the parameter δ

$$\Delta(W^{\mu}_{2,2}, \Lambda, l_2) \ll \delta^{(2\mu-2\beta-1)/(2\mu+2\beta+1)}.$$

This estimate was obtained earlier in the paper [10].

Remark 2. *The estimate of the quantity $\Delta(W^{\psi}_{1,p}, \Lambda, l_p)$, $1 \le p < \infty$, for the introduced above classes $W^{\psi}_{1,p}$ of univariate functions is obtained in the paper [5] (with appropriately modified restrictions (4) on the elements λ^n_k, $k = 1, \ldots, n$, of the matrix Λ).*

Here, for $1 \le p \le 2$, $\psi \in \Psi_{\beta+1/2,\theta}$, or $2 < p < \infty$, $\psi \in \Psi_{\beta+1,\theta}$ and $n \asymp \psi^{-1}([1/\delta])$ (notation ψ^{-1} stands for inverse of ψ, $[a]$ for integer part of $a \in \mathbb{R}$) it holds

$$\Delta(W^{\psi}_{1,p}, \Lambda, l_p) \ll \frac{n^{\beta+1-1/p}}{\psi(n)},$$

that is,

$$\Delta(W^{\psi}_{1,p}, \Lambda, l_p) \ll \delta(\psi^{-1}([1/\delta]))^{\beta+1-1/p}.$$

Note, that here, when defining the set Ψ_{γ_1,γ_2}, $0 < \gamma_1 < \gamma_2$, the assumption $\psi(1) = 1$ is not obligatory.

References

[1] Chaichenko S., Savchuk V., Shidlich A. *Approximation of functions by linear summation methods in the Orlicz-type spaces*. J. Math. Sci. 2020, **249** (5), 705–719. doi:10.1007/s10958-020-04967-y (translation of Ukrain. Mat. Visnyk 2020, **17** (2), 152–170. (in Ukrainian))

[2] Hrabova U.Z., Kal'chuk I.V. *Approximation of the classes $W^r_{\beta,\infty}$ by three-harmonic Poisson integrals*. Carpathian Math. Publ. 2019, **11** (2), 321–334. doi:10.15330/cmp.11.2.321-334

[3] Hrabova U.Z., Kal'chuk I.V., Stepaniuk T.A. *On the approximation of the classes $W^r_\beta H^\alpha$ by biharmonic Poisson integrals*. Ukrainian Math. J. 2018, **70** (5), 719–729. doi:10.1007/s11253-018-1528-6 (translation of Ukrain. Mat. Zh. 2018, **70** (5), 625–634. (in Ukrainian))

[4] Hrabova U.Z., Kal'chuk I.V., Stepaniuk T.A. *Approximative properties of the Weierstrass integrals on the classes $W^r_\beta H^\alpha$*. J. Math. Sci. 2018, **231** (1), 41–47. doi:10.1007/s10958-018-3804-2 (translation of Ukrain. Mat. Visnyk 2017, **14** (3), 361–369. (in Ukrainian))

[5] Pozharska K.V., Pozharskyi A.A. *Recovery of continuous functions from their Fourier coefficients known with error*. Res. Math. 2020, **28** (2), 24–34. doi:10.15421/242008

[6] Kal'chuk I.V., Kharkevych Yu.I., Pozharska K.V. *Asymptotics of approximation of functions by conjugate Poisson integrals*. Carpathian Math. Publ. 2020, **12** (1), 138–147. doi:10.15330/cmp.12.1.138-147

[7] Kharkevych Yu.I., Pozharska K.V. *Asymptotics of approximation of conjugate functions by Poisson integrals*. Acta Comment. Univ. Tartu. Math. 2018, **22** (2), 235–243. doi:10.12697/ACUTM.2018.22.19

[8] Magaril-Il'yaev G.G., Osipenko K.Y. *Optimal recovery of functions and their derivatives from Fourier coefficients prescribed with an error*. Sb. Math. 2002, **193** (3), 387–407. doi:10.1070/SM2002v193n03ABEH000637 (translation of Mat. Sbornik 2002, **193** (3), 79–100. doi:10.4213/sm637 (in Russian))

[9] Serdyuk A.S., Hrabova U.Z. *Order estimates of the uniform approximations by Zygmundsums on the classes of convolutions of periodic functions*. Carpathian Math. Publ. 2021, **13** (1), 68–80. doi:10.15330/cmp.13.1.68-80

[10] Sharipov K. *On the recovery of continuous functions of two variables from noisy Fourier coefficients*. J. Numer. & Appl. Math. 2012, **109** (3), 116–124.

[11] Shydlich A.L. *Saturation of the linear methods of summation of Fourier series in the spaces S^p_φ* Ukrainian Math. J. 2008, **60** (6), 950–966. doi:10.1007/s11253-008-0107-7 (translation of Ukrain. Mat. Zh. 2008, **60** (6), 815–828. (in Ukrainian))

[12] Solodky S.G., Sharipov K.K. *Summation of smooth functions of two variables with perturbed Fourier coefficients*. J. Inverse Ill-Posed Probl. 2015, **23** (3), 287–297. doi:10.1515/jiip-2013-0076

[13] Solodky S.G., Stasyuk S.A. *Estimates of efficiency for two methods of stable numerical summation of smooth functions*. J. Complexity 2020, **56**, 101422. doi:10.1016/j.jco.2019.101422

[14] Stepanets A.I., Rukasov V.I. *Spaces S^p with nonsymmetric metric*. Ukrainian Math. J. 2003, **55** (2), 322–338. doi:10.1023/A:1025472514408 (translation of Ukrain. Mat. Zh. 2003, **55** (2), 264–277. (in Russian))

[15] Stepanets A.I. *Best approximations of q-ellipsoids in spaces $S^{p,\mu}_{\varphi}$*. Ukrainian Math. J. 2004, **56** (10), 1646–1652. doi:10.1007/s11253-005-0140-8 (translation of Ukrain. Mat. Zh. 2004, **56** (10), 1378–1383. (in Russian))

[16] Tihonov A.N. *On the solution of ill-posed problems and the method of regularization*. Dokl. Akad. Nauk SSSR 1963, **151** (3), 501–504. (in Russian)

[17] Tihonov A.N. *On stable summability methods for Fourier series*. Dokl. Akad. Nauk SSSR 1964, **156** (2), 268–271. (in Russian)

Received 03.07.2021
Revised 05.08.2021

Пожарський О.А., Пожарська К.В. *Відновлення неперервних функцій двох змінних за їхніми коефіцієнтами Фур'є, що задані з похибкою* // Карпатські матем. публ. — 2021. — Т.13, №3. — С. 676–686.

У даній роботі ми продовжуємо вивчати класичну задачу оптимального відновлення на класах неперервних функцій. А саме, розглянуто класи $W^{\psi}_{2,p}$, $1 \leq p < \infty$, функцій, що задаються у термінах узагальненої гладкості ψ. Досліджено двовимірний випадок, який доповнює недавні результати роботи [Res. Math. 2020, **28** (2), 24–34] для класів W^{ψ}_{p} функцій однієї змінної.

Вважаємо, що для функцій відомі їхні коефіцієнти Фур'є $y^{\delta}_{i,j} = y_{i,j} + \delta\xi_{i,j}$, $\delta \in (0,1)$, $i,j = 1,2,\ldots$, відносно деякої ортонормованої системи $\{\varphi_{i,j}\}_{i,j=1}^{\infty}$, які збурені шумом. При цьому, рівень шуму вважаємо малим в сенсі норми простору l_p, $1 \leq p < \infty$, подвійних послідовностей $\xi = (\xi_{i,j})_{i,j=1}^{\infty}$ дійсних чисел.

У якості методу відновлення, взято так званий Λ-метод підсумовування, що задається деякою двовимірною числовою матрицею $\Lambda = \{\lambda^n_{i,j}\}_{i,j=1}^{n}$, де n — натуральне число, яке певним чином пов'язане із послідовністю ψ, що визначає гладкість досліджуваних функцій. Похибку наближення оцінено в нормі простору $C([0,1]^2)$ неперервних на $[0,1]^2$ функцій.

Показано, що при $1 \leq p < \infty$, за відповідних умов на гладкісний параметр ψ та елементи матриці Λ, справедлива оцінка

$$\Delta(W^{\psi}_{2,p}, \Lambda, l_p) = \sup_{y \in W^{\psi}_{2,p}} \sup_{\|\xi\|_{l_p} \leq 1} \Big\| y - \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda^n_{i,j} (y_{i,j} + \delta\xi_{i,j}) \varphi_{i,j} \Big\|_{C([0,1]^2)} \ll \frac{n^{\beta+1-1/p}}{\psi(n)}.$$

Ключові слова і фрази: ряд Фур'є, метод регуляризації, Λ-метод підсумовування.