ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2022, **14** (2), 345–353
doi:10.15330/cmp.14.2.345-353

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2022, Т.14, №2, С.345–353

On the derivations of cyclic Leibniz algebras

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Let L be an algebra over a field F. Then L is called a left Leibniz algebra, if its multiplication operation $[-,-]$ additionally satisfies the so-called left Leibniz identity: $[[a,b],c] = [a,[b,c]] - [b,[a,c]]$ for all elements $a,b,c \in L$. A linear transformation f of a Leibniz algebra L is called a derivation of an algebra L, if $f([a,b]) = [f(a),b] + [a, f(b)]$ for all elements $a,b \in L$. It is well known that the set of all derivations $\mathrm{Der}(L)$ of a Leibniz algebra L is a subalgebra of the Lie algebra $\mathrm{End}_F(L)$ of all linear transformations of an algebra L. The algebras of derivations of Leibniz algebras play an important role in the study of structure of Leibniz algebras. Their role is similar to that played by groups of automorphisms in the study of group structure.

In this paper, a complete description of the algebra of derivations of nilpotent cyclic Leibniz algebra is obtained. In particular, it was proved that this algebra is metabelian and supersoluble Lie algebra, and its dimension is equal to the dimension of an algebra L.

Key words and phrases: (cyclic) Leibniz algebra, Lie algebra, ideal, derivation.

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Introduction

Let L be an algebra over a field F with the binary operations $+$ and $[-,-]$. Then L is called a *left Leibniz algebra*, if it satisfies the left Leibniz identity $[[a,b],c] = [a,[b,c]] - [b,[a,c]]$ for all $a,b,c \in L$.

Leibniz algebras appeared first in the paper by A. Blokh [2], but the term "Leibniz algebra" was proposed in book [9] and article [10] by J.-L. Loday. In [11], J.-L. Loday and T. Pirashvili began the real study of the properties of Leibniz algebras. The theory of Leibniz algebras was developed very intensively in many different directions. Some of the results of this theory were presented in book [1] and in surveys [4,5,7]. Note that the Leibniz algebra is a natural generalization of a Lie algebra, namely a Leibniz algebra L, in which $[a,a] = 0$ for every element $a \in L$, is a Lie algebra.

As for Lie algebras, the derivations are significant linear transformations essentially defining the structure of Leibniz algebras. Their role is of importance in studies of the structure of specific types of Leibniz algebras. Recall that a linear transformation f of a Leibniz algebra L is called a *derivation*, if $f([a,b]) = [f(a),b] + [a, f(b)]$ for all $a,b \in L$.

Denote by $\mathrm{End}_F(L)$ the set of all linear transformations of L. Then $\mathrm{End}_F(L)$ is an associative algebra by the operations $+$ and $\circ$. As usual, $\mathrm{End}_F(L)$ is a Lie algebra by the operations $+$ and $[-,-]$, where $[f,g] = f \circ g - g \circ f$ for all $f,g \in \mathrm{End}_F(L)$.

УДК 512.554
2020 *Mathematics Subject Classification:* 17A32, 17A60, 17A99.

Let $\mathrm{Der}(L)$ be the subset of all derivations of L. It is possible to prove that $\mathrm{Der}(L)$ is a subalgebra of the Lie algebra $\mathrm{End}_F(L)$. $\mathrm{Der}(L)$ is called the *algebra of derivations* of L.

The derivations of Leibniz algebras were studied rather slightly. But their influence on the structure of Leibniz algebras is quite significant, which is indicated by the following result: if A is an ideal of a Leibniz algebra, then the factor-algebra of L by the annihilator of A is isomorphic to some subalgebra of $\mathrm{Der}(A)$ [6, Proposition 3.2].

It is clear that the study of algebras $\mathrm{Der}(L)$ should be started from cyclic Leibniz algebras. We note that the structure of cyclic Leibniz algebras was described in [3]. The derivations of free cyclic Leibniz algebra were analyzed in [8]. The rest types of cyclic Leibniz algebras are finite-dimensional. In this paper, we will describe the structure of the algebra of derivations of the nilpotent cyclic Leibniz algebra.

We now present the necessary information about the structure of the cyclic Leibniz algebra. Let L be a cyclic Leibniz algebra, $L = \langle a \rangle$, and let L be finite-dimensional over a field F. Then there exists a positive integer n such that L has a basis $a_1, \ldots, a_n$, where $a_1 = a$, $a_2 = [a_1, a_1], \ldots,$ $a_n = [a_1, a_{n-1}]$, $[a_1, a_n] = \alpha_2 a_2 + \ldots + \alpha_n a_n$. Moreover, $[L, L] = \mathrm{Leib}(L) = Fa_2 + \ldots + Fa_n$ (see [3]). We fix these designations. The first natural type of cyclic Leibniz algebras is as follows. It is a case, when $[a_1, a_n] = 0$. Then L is nilpotent.

The main result of this paper consists in the full description of an algebra of derivations of cyclic nilpotent Leibniz algebras.

Theorem. *Let L be a cyclic nilpotent Leibniz algebra. Then the following assertions hold.*

(i) *The algebra of derivations of L is a semidirect sum of an ideal $N(\mathrm{Der}(L))$, consisting of the derivations f such that $f(x) \in [L, L]$ for each element $x \in L$, and a cyclic subalgebra Fd_0, where a derivation d_0 is defined by*

$$d_0(a_1) = a_1, d_0(a_2) = 2a_2, \ldots, d_0(a_{n-1}) = (n-1)a_{n-1}, d_0(a_n) = na_n.$$

(ii) *The ideal $N(\mathrm{Der}(L))$ is abelian and has a basis $\{d_2, d_3, \ldots, d_n\}$, where*

$$\begin{aligned} d_2(a_1) &= a_2, d_2(a_2) = a_3, \ldots, d_2(a_{n-1}) = a_n, d_2(a_n) = 0, \\ d_3(a_1) &= a_3, d_3(a_2) = a_4, \ldots, d_3(a_{n-2}) = a_n, d_3(a_{n-1}) = d_3(a_n) = 0, \\ &\ldots \\ d_{n-1}(a_1) &= a_{n-1}, d_{n-1}(a_2) = a_n, d_{n-1}(a_3) = \ldots = d_{n-1}(a_{n-1}) = d_{n-1}(a_n) = 0, \\ d_n(a_1) &= a_n, d_n(a_2) = d_n(a_3) = \ldots = d_n(a_{n-1}) = d_n(a_n) = 0. \end{aligned}$$

(iii) *The ideal $N(\mathrm{Der}(L))$ is a direct sum of the ideals $Fd_2, Fd_3, \ldots, Fd_n$, moreover*

$$[d_0, d_2] = d_2, [d_0, d_3] = 2d_3, [d_0, d_4] = 3d_4, \ldots, [d_0, d_n] = (n-1)d_n.$$

In particular, $\mathrm{Der}(L)$ is a metabelian supersoluble Lie algebra, and

$$\dim_F(\mathrm{Der}(L)) = \dim_F(L).$$

1 Derivations of a cyclic Leibniz algebra

We recall some definitions.

Let L be a Leibniz algebra. Define the *lower central series* of L

$$L = \gamma_1(L) \geqslant \gamma_2(L) \geqslant \ldots \geqslant \gamma_\alpha(L) \geqslant \gamma_{\alpha+1}(L) \geqslant \ldots \geqslant \gamma_\delta(L)$$

by the following rule: $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, and, recursively, $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for all ordinals α and $\gamma_\lambda(L) = \bigcap_{\mu<\lambda} \gamma_\mu(L)$ for the limit ordinals λ. The last term $\gamma_\delta(L) = \gamma_\infty(L)$ is called the *lower hypocenter* of L. We have $\gamma_\delta(L) = [L, \gamma_\delta(L)]$.

If $\alpha = k$ is a positive integer, then $\gamma_k(L) = [L, [L, [L, \ldots] \ldots]]$ is the *left normed commutator* of k copies of L.

As usual, we say that a Leibniz algebra L is called *nilpotent*, if there exists a positive integer k such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be *nilpotent of nilpotency class* c, if $\gamma_{c+1}(L) = \langle 0 \rangle$, but $\gamma_c(L) \neq \langle 0 \rangle$.

The *left* (respectively, *right*) *center* $\zeta^{\text{left}}(L)$ (respectively, $\zeta^{\text{right}}(L)$) of a Leibniz algebra L is defined by

$$\zeta^{\text{left}}(L) = \{x \in L : [x, y] = 0 \text{ for each element } y \in L\}$$

(respectively,

$$\zeta^{\text{right}}(L) = \{x \in L : [y, x] = 0 \text{ for each element } y \in L\}).$$

It is not hard to prove that the left center of L is an ideal, but it is not true for the right center. Moreover, $\text{Leib}(L) \leqslant \zeta^{\text{left}}(L)$, so that $L/\zeta^{\text{left}}(L)$ is a Lie algebra. The right center is an subalgebra of L, and in general, the left and right centers are different. They even may have different dimensions (see [6]).

The center $\zeta(L)$ of L is defined by

$$\zeta(L) = \{x \in L : [x, y] = 0 = [y, x] \text{ for each element } y \in L\}.$$

The center is an ideal of L.

Define the *upper central series*

$$\langle 0 \rangle = \zeta_0(L) \leqslant \zeta_1(L) \leqslant \zeta_2(L) \leqslant \ldots \leqslant \zeta_\alpha(L) \leqslant \zeta_{\alpha+1}(L) \leqslant \ldots \leqslant \zeta_\gamma(L) = \zeta_\infty(L)$$

of a Leibniz algebra L by the following rule: $\zeta_1(L) = \zeta(L)$ is the center of L, and, recursively, $\zeta_{\alpha+1}(L)/\zeta_\alpha(L) = \zeta(L/\zeta_\alpha(L))$ for all ordinals α, and $\zeta_\lambda(L) = \bigcup_{\mu<\lambda} \zeta_\mu(L)$ for the limit ordinals λ. By definition, each term of this series is an ideal of L. The last term $\zeta_\infty(L)$ of this series is called the *upper hypercenter* of L. If $L = \zeta_\infty(L)$, then L is called a *hypercentral* Leibniz algebra.

We show here some basic elementary properties of derivations, which were proved in [8].

Lemma 1. *Let L be a Leibniz algebra over a field F and let f be a derivation of L. Then*

$$f(\zeta^{\text{left}}(L)) \leqslant \zeta^{\text{left}}(L), \quad f(\zeta^{\text{right}}(L)) \leqslant \zeta^{\text{right}}(L) \quad \textit{and} \quad f(\zeta(L)) \leqslant \zeta(L).$$

Corollary 1. *Let L be a Leibniz algebra over a field F and let f be a derivation of L. Then $f(\zeta_\alpha(L)) \leqslant \zeta_\alpha(L)$ for every ordinal α.*

Lemma 2. *Let L be a Leibniz algebra over a field F and let f be a derivation of L. Then $f(\gamma_\alpha(L)) \leqslant \gamma_\alpha(L)$ for all ordinals α, in particular, $f(\gamma_\infty(L)) \leqslant \gamma_\infty(L)$.*

Let L be a finite-dimensional cyclic Leibniz algebra of type (I): $L = Fa_1 \oplus Fa_2 \oplus \ldots \oplus Fa_n$, $[a_1, a_j] = a_{j+1}$ whenever $1 \leqslant j \leqslant n-1$, $[a_1, a_n] = 0$, $[a_j, a_k] = 0$ for all $j \geqslant 2$, $1 \leqslant k \leqslant n$. This algebra is nilpotent. Put $L_1 = L$, $L_2 = Fa_2 \oplus \ldots \oplus Fa_n, \ldots, L_{n-1} = Fa_{n-1} \oplus Fa_n$, $L_n = Fa_n$. We have

$$\begin{aligned} \gamma_1(L) &= L_1, & \zeta_1(L) &= L_n, \\ \gamma_2(L) &= L_2, & \zeta_2(L) &= L_{n-1}, \\ &\ldots & &\ldots \\ \gamma_{n-1}(L) &= L_{n-1}, & \zeta_{n-1}(L) &= L_2, \\ \gamma_n(L) &= L_n, & \zeta_n(L) &= L_1. \end{aligned}$$

Lemma 3. *Let L be a cyclic Leibniz algebra of type* (I)*. Denote by $N(\mathrm{Der}(L))$ the subset of* $\mathrm{Der}(L)$ *consisting of the derivations f such that $f(x) \in [L, L]$ for each element $x \in L$. Then $N(\mathrm{Der}(L))$ is an ideal of a Lie algebra* $\mathrm{Der}(L)$.

Proof. Let f, h be the arbitrary derivations from a subset $N(\mathrm{Der}(L))$, $\lambda \in F$. We have

$$\begin{gathered} (f-h)(x) = f(x) - h(x) \in [L, L], \\ (\lambda f)(x) = \lambda f(x) \in [L, L]. \end{gathered}$$

Let again $f \in N(\mathrm{Der}(L))$, and let h be an arbitrary derivation of L. Then

$$[f, h](x) = (f \circ h - h \circ f)(x) = f(h(x)) - h(f(x)).$$

By the definition of f, we obtain that $f(h(x)) \in [L, L]$. Since $f(x) \in [L, L]$, Lemma 2 shows that $h(f(x)) \in [L, L]$, so that $[f, h] \in N(\mathrm{Der}(L))$. It follows that $N(\mathrm{Der}(L))$ is an ideal of $\mathrm{Der}(L)$. □

Lemma 4. *Let L be a cyclic Leibniz algebra of type* (I)*. Denote by d_0 the linear transformation of L such that*

$$d_0(a_1) = a_1, d_0(a_2) = 2a_2, \ldots, d_0(a_{n-1}) = (n-1)a_{n-1}, d_0(a_n) = na_n.$$

Then d_0 is a derivation of L.

Proof. Let $x = \lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n$ and $y = \mu_1 a_1 + \mu_2 a_2 + \ldots + \mu_n a_n$ be an arbitrary elements of L. Then

$$\begin{aligned} d_0(x) &= d_0(\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n) \\ &= \lambda_1 d_0(a_1) + \lambda_2 d_0(a_2) + \ldots + \lambda_n d_0(a_n) \\ &= \lambda_1 a_1 + 2\lambda_2 a_2 + \ldots + n\lambda_n a_n, \\ d_0(y) &= d_0(\mu_1 a_1 + \mu_2 a_2 + \ldots + \mu_n a_n) \\ &= \mu_1 d_0(a_1) + \mu_2 d_0(a_2) + \ldots + \mu_n d_0(a_n) \\ &= \mu_1 a_1 + 2\mu_2 a_2 + \ldots + n\mu_n a_n. \end{aligned}$$

Suppose that a linear mapping f satisfies the above conditions. We have

$$\begin{aligned}
[x,y] &= [\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n, \mu_1 a_1 + \mu_2 a_2 + \ldots + \mu_n a_n] \\
&= [\lambda_1 a_1, \mu_1 a_1 + \mu_2 a_2 + \ldots + \mu_n a_n] \\
&= \lambda_1 \mu_1 a_2 + \lambda_1 \mu_2 a_3 + \ldots + \lambda_1 \mu_{n-1} a_n; \\
d_0([x,y]) &= d_0(\lambda_1 \mu_1 a_2 + \lambda_1 \mu_2 a_3 + \ldots + \lambda_1 \mu_{n-1} a_n) \\
&= \lambda_1 \mu_1 d_0(a_2) + \lambda_1 \mu_2 d_0(a_3) + \ldots + \lambda_1 \mu_{n-1} d_0(a_n) \\
&= 2\lambda_1 \mu_1 a_2 + 3\lambda_1 \mu_2 a_3 + \ldots + n\lambda_1 \mu_{n-1} a_n; \\
[d_0(x), y] &= [\lambda_1 a_1 + 2\lambda_2 a_2 + \ldots + n\lambda_n a_n, \mu_1 a_1 + \mu_2 a_2 + \ldots + \mu_n a_n] \\
&= [\lambda_1 a_1, \mu_1 a_1 + \mu_2 a_2 + \ldots + \mu_n a_n] \\
&= \lambda_1 \mu_1 a_2 + \lambda_1 \mu_2 a_3 + \ldots + \lambda_1 \mu_{n-1} a_n; \\
[x, d_0(y)] &= [\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n, \mu_1 a_1 + 2\mu_2 a_2 + \ldots + n\mu_n a_n] \\
&= [\lambda_1 a_1, \mu_1 a_1 + 2\mu_2 a_2 + \ldots + n\mu_n a_n] \\
&= \lambda_1 \mu_1 a_2 + 2\lambda_1 \mu_2 a_3 + \ldots + (n-1)\lambda_1 \mu_{n-1} a_n; \\
[d_0(x), y] + [x, d_0(y)] &= \lambda_1 \mu_1 a_2 + \lambda_1 \mu_2 a_3 + \ldots + \lambda_1 \mu_{n-1} a_n \\
&\quad + \lambda_1 \mu_1 a_2 + 2\lambda_1 \mu_2 a_3 + \ldots + (n-1)\lambda_1 \mu_{n-1} a_n \\
&= 2\lambda_1 \mu_1 a_2 + 3\lambda_1 \mu_2 a_3 + \ldots + n\lambda_1 \mu_{n-1} a_n.
\end{aligned}$$

Thus, $[d_0(x), y] + [x, d_0(y)] = d_0([x,y])$, which shows that d_0 is a derivation of L. □

Corollary 2. *Let L be a cyclic Leibniz algebra of type* (I). *Then*

$$\mathrm{Der}(L) = N(\mathrm{Der}(L)) \oplus F d_0.$$

Proof. Let f be an arbitrary derivation of L. We have $f(a_1) = \gamma a_1 + u$ for some scalar $\gamma \in F$ and some element $u \in [L, L]$. Put $g = f - \gamma d_0$. Then

$$g(a_1) = (f - \gamma d_0)(a_1) = f(a_1) - \gamma d_0(a_1) = \gamma a_1 + u - \gamma a_1 = u \in [L, L].$$

By Lemma 2, $g(a_j) \in [L, L]$ for all $j > 1$. It follows that $g(x) \in [L, L]$ for each element $x \in L$. Thus, $g \in N(\mathrm{Der}(L))$. It follows that $f \in N(\mathrm{Der}(L)) + F d_0$. It is not hard to show that $N(\mathrm{Der}(L)) \cap F d_0 = \langle 0 \rangle$. □

Lemma 5. *Let L be a cyclic Leibniz algebra of type* (I). *Then a linear mapping f belongs to $N(\mathrm{Der}(L))$ iff*

$$\begin{aligned}
f(a_1) &= \gamma_2 a_2 + \gamma_3 a_3 + \gamma_4 a_4 + \ldots + \gamma_{n-1} a_{n-1} + \gamma_n a_n, \\
f(a_2) &= \gamma_2 a_3 + \gamma_3 a_4 + \ldots + \gamma_{n-2} a_{n-1} + \gamma_{n-1} a_n, \\
f(a_3) &= \gamma_2 a_4 + \ldots + \gamma_{n-3} a_{n-1} + \gamma_{n-2} a_n, \\
&\ldots \\
f(a_{n-1}) &= \gamma_2 a_n, \\
f(a_n) &= 0.
\end{aligned}$$

Proof. Lemma 2 shows that $f(L_j) \leqslant L_j$ for all j, $1 \leqslant j \leqslant n$. We have

$$
\begin{aligned}
f(a_1) &= \gamma_2 a_2 + \gamma_3 a_3 + \ldots + \gamma_{n-1} a_{n-1} + \gamma_n a_n;\\
f(a_2) &= f([a_1, a_1]) = [f(a_1), a_1] + [a_1, f(a_1)]\\
&= [\gamma_2 a_2 + \gamma_3 a_3 + \ldots + \gamma_{n-1} a_{n-1} + \gamma_n a_n, a_1]\\
&\quad + [a_1, \gamma_2 a_2 + \gamma_3 a_3 + \ldots + \gamma_{n-1} a_{n-1} + \gamma_n a_n]\\
&= \gamma_2 [a_1, a_2] + \gamma_3 [a_1, a_3] + \ldots + \gamma_{n-1} [a_1, a_{n-1}] + \gamma_n [a_1, a_n]\\
&= \gamma_2 a_3 + \gamma_3 a_4 + \ldots + \gamma_{n-2} a_{n-1} + \gamma_{n-1} a_n;\\
f(a_3) &= f([a_1, a_2]) = [f(a_1), a_2] + [a_1, f(a_2)]\\
&= [\gamma_2 a_2 + \gamma_3 a_3 + \ldots + \gamma_{n-1} a_{n-1} + \gamma_n a_n, a_2]\\
&\quad + [a_1, \gamma_2 a_3 + \gamma_3 a_4 + \ldots + \gamma_{n-2} a_{n-1} + \gamma_{n-1} a_n]\\
&= \gamma_2 [a_1, a_3] + \gamma_3 [a_1, a_4] + \ldots + \gamma_{n-3} [a_1, a_{n-2}] + \gamma_{n-2} [a_1, a_{n-1}] + \gamma_{n-1} [a_1, a_n]\\
&= \gamma_2 a_4 + \gamma_3 a_5 + \ldots + \gamma_{n-3} a_{n-1} + \gamma_{n-2} a_n;\\
&\ldots\\
f(a_{n-1}) &= \gamma_2 a_n;\\
f(a_n) &= 0.
\end{aligned}
$$

Conversely, let $x = \lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n$, $y = \mu_1 a_1 + \mu_2 a_2 + \ldots + \mu_n a_n$ be an arbitrary elements of L. Suppose that a linear mapping f satisfies the above conditions. Then

$$
\begin{aligned}
[x, y] &= [\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n, \mu_1 a_1 + \mu_2 a_2 + \ldots + \mu_n a_n]\\
&= [\lambda_1 a_1, \mu_1 a_1 + \mu_2 a_2 + \ldots + \mu_n a_n]\\
&= \lambda_1 \mu_1 a_2 + \lambda_1 \mu_2 a_3 + \ldots + \lambda_1 \mu_{n-1} a_n;\\
f([x, y]) &= f(\lambda_1 \mu_1 a_2 + \lambda_1 \mu_2 a_3 + \ldots + \lambda_1 \mu_{n-1} a_n)\\
&= \lambda_1 \mu_1 f(a_2) + \lambda_1 \mu_2 f(a_3) + \ldots + \lambda_1 \mu_{n-1} f(a_n);\\
f(x) &= f(\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n)\\
&= \lambda_1 f(a_1) + \lambda_2 f(a_2) + \ldots + \lambda_n f(a_n);\\
f(y) &= f(\mu_1 a_1 + \mu_2 a_2 + \ldots + \lambda_n a_n)\\
&= \mu_1 f(a_1) + \mu_2 f(a_2) + \ldots + \mu_n f(a_n);\\
[f(x), y] &= [\lambda_1 f(a_1) + \lambda_2 f(a_2) + \ldots + \lambda_n f(a_n), \mu_1 a_1 + \mu_2 a_2 + \ldots + \mu_n a_n]\\
&= [\lambda_1 f(a_1), \mu_1 a_1 + \mu_2 a_2 + \ldots + \mu_n a_n]\\
&= \lambda_1 \mu_1 [f(a_1), a_1] + \lambda_1 \mu_2 [f(a_1), a_2] + \ldots + \lambda_1 \mu_n [f(a_1), a_n];\\
[x, f(y)] &= [\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n, \mu_1 f(a_1) + \mu_2 f(a_2) + \ldots + \mu_n f(a_n)]\\
&= [\lambda_1 a_1, \mu_1 f(a_1) + \mu_2 f(a_2) + \ldots + \mu_n f(a_n)]\\
&= \lambda_1 \mu_1 [a_1, f(a_1)] + \lambda_1 \mu_2 [a_1, f(a_2)] + \ldots + \lambda_1 \mu_n [a_1, f(a_n)];\\
[f(x), y] + [x, f(y)] &= \lambda_1 \mu_1 [f(a_1), a_1] + \lambda_1 \mu_2 [f(a_1), a_2] + \ldots + \lambda_1 \mu_n [f(a_1), a_n]\\
&\quad + \lambda_1 \mu_1 [a_1, f(a_1)] + \lambda_1 \mu_2 [a_1, f(a_2)] + \ldots + \lambda_1 \mu_n [a_1, f(a_n)]\\
&= \lambda_1 \mu_1 ([f(a_1), a_1] + [a_1, f(a_1)]) + \lambda_1 \mu_2 ([f(a_1), a_2] + [a_1, f(a_2)]) + \ldots\\
&\quad + \lambda_1 \mu_n ([f(a_1), a_n] + [a_1, f(a_n)]).
\end{aligned}
$$

The above-presented equalities indicate that

$$\begin{aligned}
[f(a_1), a_1] + [a_1, f(a_1)] &= f([a_1, a_1]) = f(a_2),\\
[f(a_1), a_2] + [a_1, f(a_2)] &= f([a_1, a_2]) = f(a_3),\\
&\dots\\
[f(a_1), a_{n-1}] + [a_1, f(a_{n-1})] &= f([a_1, a_{n-1}]) = f(a_n),\\
[f(a_1), a_n] + [a_1, f(a_n)] &= f([a_1, a_n]) = 0.
\end{aligned}$$

Thus, a linear transformation of L, satisfying the above equalities, is a derivation of L. □

Corollary 3. *Let L be a cyclic Leibniz algebra of type* (I)*. Then $N(\mathrm{Der}(L))$ is isomorphic to a Lie subalgebra of the matrix algebra $\mathrm{M}_n(F)$ consisting of the matrices having the form*

$$\begin{pmatrix}
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0\\
\gamma_2 & 0 & 0 & 0 & \dots & 0 & 0 & 0\\
\gamma_3 & \gamma_2 & 0 & 0 & \dots & 0 & 0 & 0\\
\gamma_4 & \gamma_3 & \gamma_2 & 0 & \dots & 0 & 0 & 0\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
\gamma_{n-2} & \gamma_{n-3} & \gamma_{n-4} & \gamma_{n-5} & \dots & 0 & 0 & 0\\
\gamma_{n-1} & \gamma_{n-2} & \gamma_{n-3} & \gamma_{n-4} & \dots & \gamma_2 & 0 & 0\\
\gamma_n & \gamma_{n-1} & \gamma_{n-2} & \gamma_{n-3} & \dots & \gamma_3 & \gamma_2 & 0
\end{pmatrix}.$$

Lemma 6. *Let L be a cyclic Leibniz algebra of type* (I)*. Then the ideal $N(\mathrm{Der}(L))$ is abelian and has a dimension $n-1$.*

Proof. We now use the isomorphism from Corollary 3. Let $X = \|\sigma_{j,m}\|$, $Y = \|\tau_{j,m}\| \in \mathrm{M}_n(F)$ be two matrices such that $\sigma_{j,m} = \tau_{j,m} = 0$ whenever $j \leqslant m$ and

$$\begin{aligned}
\sigma_{2,1} &= \sigma_{3,2} = \dots = \sigma_{n,n-1} = \sigma_2, & \tau_{2,1} &= \tau_{3,2} = \dots = \tau_{n,n-1} = \tau_2,\\
\sigma_{3,1} &= \sigma_{4,2} = \dots = \sigma_{n,n-2} = \sigma_3, & \tau_{3,1} &= \tau_{4,2} = \dots = \tau_{n,n-2} = \tau_3,\\
&\dots & &\dots\\
\sigma_{n-1,1} &= \sigma_{n,2} = \sigma_{n-1}, & \tau_{n-1,1} &= \tau_{n,2} = \tau_{n-1},\\
\sigma_{n,1} &= \sigma_n, & \tau_{n,1} &= \tau_n.
\end{aligned}$$

Let $Z = XY = \|\xi_{j,m}\|$. Clearly, $\xi_{j,m} = 0$ whenever $j \leqslant m$. We have

$$\begin{aligned}
&\xi_{1,j} = 0 \ \text{ for all } j, 1 \leqslant j \leqslant n, \xi_{2,j} = 0 \ \text{ for all } j, 1 \leqslant j \leqslant n,\\
&\xi_{3,1} = \sigma_{3,2}\tau_{2,1} = \sigma_2\tau_2, \xi_{3,j} = 0 \ \text{ for all } j, 2 \leqslant j \leqslant n,\\
&\xi_{4,1} = \sigma_{4,2}\tau_{2,1} + \sigma_{4,3}\tau_{3,1} = \sigma_3\tau_2 + \sigma_2\tau_3,\\
&\xi_{4,2} = \sigma_{4,3}\tau_{3,2} = \sigma_2\tau_2, \xi_{4,j} = 0 \ \text{ for all } j, 3 \leqslant j \leqslant n,\\
&\quad\dots\\
&\xi_{n,1} = \sigma_{n,2}\tau_{2,1} + \sigma_{n,3}\tau_{3,1} + \dots + \sigma_{n,n-1}\tau_{n-1,1} = \sigma_{n-1}\tau_2 + \sigma_{n-2}\tau_3 + \dots + \sigma_2\tau_{n-1},\\
&\xi_{n,2} = \sigma_{n,3}\tau_{3,2} + \sigma_{n,4}\tau_{4,2} + \dots + \sigma_{n,n-1}\tau_{n-1,2} = \sigma_{n-2}\tau_2 + \sigma_{n-3}\tau_3 + \dots + \sigma_2\tau_{n-2},\\
&\quad\dots\\
&\xi_{n,n-2} = \sigma_{n,n-1}\tau_{n-1,n-2} = \sigma_2\tau_2, \xi_{n,n-1} = 0, \xi_{n,n} = 0.
\end{aligned}$$

These equalities show that $XY = YX$. It follows that the ideal $N(\mathrm{Der}(L))$ is abelian as a Lie subalgebra. Furthermore, we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \gamma_2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \gamma_3 & \gamma_2 & 0 & 0 & \dots & 0 & 0 & 0 \\ \gamma_4 & \gamma_3 & \gamma_2 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \gamma_{n-2} & \gamma_{n-3} & \gamma_{n-4} & \gamma_{n-5} & \dots & 0 & 0 & 0 \\ \gamma_{n-1} & \gamma_{n-2} & \gamma_{n-3} & \gamma_{n-4} & \dots & \gamma_2 & 0 & 0 \\ \gamma_n & \gamma_{n-1} & \gamma_{n-2} & \gamma_{n-3} & \dots & \gamma_3 & \gamma_2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \gamma_2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \gamma_2 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \gamma_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \gamma_2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \gamma_3 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \gamma_3 & 0 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \gamma_3 & 0 & 0 \end{pmatrix}$$

$$+ \dots$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \gamma_{n-1} & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \gamma_{n-1} & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \gamma_n & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

Denote, by d_j, the linear transformation of L such that

$$\begin{aligned} d_2(a_1) &= a_2, d_2(a_2) = a_3, \dots, d_2(a_{n-1}) = a_n, d_2(a_n) = 0, \\ d_3(a_1) &= a_3, d_3(a_2) = a_4, \dots, d_3(a_{n-2}) = a_n, d_3(a_{n-1}) = d_3(a_n) = 0, \\ &\dots \\ d_{n-1}(a_1) &= a_{n-1}, d_{n-1}(a_2) = a_n, d_{n-1}(a_3) = \dots = d_{n-1}(a_{n-1}) = d_{n-1}(a_n) = 0, \\ d_n(a_1) &= a_n, d_n(a_2) = d_n(a_3) = \dots = d_n(a_{n-1}) = d_n(a_n) = 0. \end{aligned}$$

Lemma 5 shows that d_j is a derivation of L such that $d_j \in N(\mathrm{Der}(L))$. The above equalities show that these mappings generate $N(\mathrm{Der}(L))$.

Let $\lambda_2, \lambda_3, \dots, \lambda_n$ be the elements of F such that $\lambda_2 d_2 + \lambda_3 d_3 + \dots + \lambda_n d_n = 0$. We have

$$0 = (\lambda_2 d_2 + \lambda_3 d_3 + \dots + \lambda_n d_n)(a_1) = \lambda_2 a_2 + \lambda_3 a_3 + \dots + \lambda_{n-1} a_{n-1} + \lambda_n a_n.$$

Since a subset $\{a_2, a_3, \dots, a_{n-1}, a_n\}$ is a basis of $[L, L]$, $\lambda_2 = \lambda_3 = \dots = \lambda_n = 0$. This means that the derivations $d_2, d_3, \dots, d_n$ are linearly independent. □

2 Proof of the main theorem

Assertion (i) follows from Corollary 2. Assertion (ii) follows from Lemma 6. Furthermore, $(d_0 \circ d_2)(a_1) = d_0(d_2(a_1)) = d_0(a_2) = 2a_2$, $(d_2 \circ d_0)(a_1) = d_2(d_0(a_1)) = d_2(a_1) = a_2$ and

$[d_0, d_2](a_1) = (d_0 \circ d_2)(a_1) - (d_2 \circ d_0)(a_1) = a_1$. With regard for Lemma 5, we obtain

$$[d_0, d_2](a_2) = a_3, \ldots, [d_0, d_2](a_{n-1}) = a_n, [d_0, d_2](a_n) = 0.$$

These equalities prove that $[d_0, d_2] = d_2$. Using the similar arguments, we obtain that

$$[d_0, d_3] = 2d_3, [d_0, d_4] = 3d_4, \ldots, [d_0, d_n] = (n-1)d_n.$$

In particular, it follows that every subspace Fd_j is an ideal of $\mathrm{Der}(L)$. □

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Received 12.07.2021

Семко М.М., Скасків Л.В., Ярова О.А. *Про диференціювання циклічних алгебр Лейбніца* // Карпатські матем. публ. — 2022. — Т.14, №2. — C. 345–353.

Нехай L – алгебра над полем F. Тоді L називатимемо лівою алгеброю Лейбніца, якщо її операція множення $[-,-]$ додатково задовольняє так званій лівій тотожності Лейбніца: $[[a,b],c] = [a,[b,c]] - [b,[a,c]]$ для всіх елементів $a,b,c \in L$. Лінійне перетворення f алгебри Лейбніца L називатимемо диференціюванням алгебри L, якщо $f([a,b]) = [f(a),b] + [a,f(b)]$ для всіх елементів $a,b \in L$. Добре відомо, що множина усіх диференціювань $\mathrm{Der}(L)$ алгебри Лейбніца L є підалгеброю алгебри Лі $\mathrm{End}_F(L)$ усіх лінійних перетворень алгебри L. Алгебри диференціювань алгебр Лейбніца відіграють важливу роль у вивченні структури алгебр Лейбніца. Їх роль аналогічна тій, яку відіграють групи автоморфізмів при вивченні структури груп.

У цій роботі отримано повний опис алгебри диференціювань нільпотентної циклічної алгебри Лейбніца. Зокрема, було доведено, що ця алгебра є метабелевою та надрозв'язною алгеброю Лі, а її вимірність дорівнює вимірності алгебри L.

Ключові слова і фрази: (циклічна) алгебра Лейбніца, алгебра Лі, ідеал, диференціювання.