ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2021, **13** (3), 790–804
doi:10.15330/cmp.13.3.790-804

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2021, Т.13, №3, С.790–804

# An efficient hybrid technique for the solution of fractional-order partial differential equations

**Jassim H.K.[1], Ahmad H.[2], Shamaoon A.[1], Cesarano C.[2,✉]**

In this paper, a hybrid technique called the homotopy analysis Sumudu transform method has been implemented solve fractional-order partial differential equations. This technique is the amalgamation of Sumudu transform method and the homotopy analysis method. Three examples are considered to validate and demonstrate the efficacy and accuracy of the present technique. It is also demonstrated that the results obtained from the suggested technique are in excellent agreement with the exact solution which shows that the proposed method is efficient, reliable and easy to implement for various related problems of science and engineering.

*Key words and phrases:* fractional differential equation, Sumudu transform, homotopy analysis method.

[1] University of Thi-Qar, Nasiriyah, 00964 Dhi Qar, Iraq
[2] International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186 Roma, Italy
✉ Corresponding author
E-mail: hassankamil@utq.edu.iq (Jassim H.K.), hijaz555@gmail.com (Ahmad H.), 19070010@lums.edu.pk (Shamaoon A.), c.cesarano@uninettunouniversity.net (Cesarano C.)

## Introduction

Newly, the fractional calculus (FC) and its various applications in mathematics, physics and engineering have received considerable attention. FC applications are found in many areas, such as dynamic device control theory, chemical mechanics, probability and statistics, electrical networks, corrosion electrochemistry, and optics and signal processing. Linear/nonlinear fractional-order differential equations may be successfully modeled. A fractional PDE is obtained from the classical diffusion equation of mathematical physics by replacing the $n$th order time derivative with a fractional-order derivative $\alpha$, which is now the area of increasing interest apparent in the literature study [10–12, 19].

In recent decades, many of the numerical and analytical techniques have been implemented to solve fractional-order PDEs, such as the fractional variational iteration method [23,34,42,44, 45], fractional differential transform method [25, 36, 46], fractional series expansion method [9, 29], fractional Sumudu variational iteration method [20, 31], fractional natural decomposition method [32, 38], fractional Sumudu decomposition method [17, 30, 33], fractional Sumudu homotopy perturbation method [28], fractional reduce differential transform method [24, 26, 41], fractional Adomian decomposition method [16, 21, 47], fractional Laplace decomposition method [27], fractional Laplace homotopy perturbation method [14], fractional Laplace variational iteration method [13, 15, 18, 35, 37], variational iteration method [4–8] and local mesh less

УДК 517.95
2020 *Mathematics Subject Classification:* 34K37, 45J99, 34A08.

method [1–3, 22, 40]. As the main aim of this work the homotopy analysis Sumudu transform method is implemented to solve FPDEs and nonlinear system of FPDEs.

## 1 Fractional calculus

In this section, we demonstrate some notations and definitions that will be used further in the study. FC theory is almost more than two decades old in the literature. Several definitions of fractional integrals and derivatives have been proposed but the first major contribution to give proper definition is due to Liouville as follows.

**Definition 1** ([30,39]). *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $\Psi(\tau) \in C_\varepsilon$, $\varepsilon \geq -1$, is defined as*

$$I_\tau^\alpha \Psi(\tau) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^\tau (\tau - s)^{\alpha-1} \Psi(s)\, ds, & \alpha > 0,\ \tau > 0, \\ \Psi(\tau), & \alpha = 0, \end{cases}$$

*where $\Gamma(\cdot)$ is the well-known Gamma function.*

**Definition 2** ([30,39]). *The Caputo fractional derivative (CFD) with order $\alpha > 0$ of $\Psi(\tau)$ is defined as follows*

$$D_\tau^\alpha \Psi(\mu) = \frac{1}{\Gamma(m-\alpha)} \int_0^\tau (\tau - s)^{m-\alpha-1} \Psi^{(m)}(s)\, ds$$

*for $m-1 < \alpha < m$, $m \in \mathbb{N}$, $\tau > 0$, and $\varphi \in C_{-1}^m$.*

The fundamental basic properties of the Caputo fractional derivative are given as:

(i)

$$D^\alpha I^\alpha \Psi(x,\tau) = \Psi(x,\tau);$$

(ii)

$$I^\alpha D^\alpha \Psi(x,\tau) = \Psi(x,\tau) - \sum_{k=0}^{m-1} \frac{\tau^k}{k!} \Psi^{(k)}(x,0);$$

(iii)

$$D^\alpha \tau^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\beta+1)} \tau^{\beta-\alpha}, \quad \alpha > 0.$$

**Definition 3** ([39]). *The Mittag–Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined as*

$$E_\alpha(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha+1)}.$$

**Definition 4** ([30,43]). *The Sumudu transform (ST) is defined by*

$$S[\Psi(\tau)] = \int_0^\infty e^{-\tau} \Psi(\omega\tau)\, d\tau, \quad \omega \in (-\omega_1, \omega_2).$$

Some properties of ST:

(i) $S[k] = k$ for any constant $k$;

(ii) $S[\tau^{n\alpha}/\Gamma(n\alpha+1)] = \omega^{n\alpha}$.

**Definition 5** ([43]). *The ST of the CFD is defined as*

$$S[D_\tau^\alpha \Psi(x,\tau)] = \omega^{-\alpha} S[\Psi(x,\tau)] - \sum_{k=0}^{m-1} \omega^{(-\alpha+k)} \Psi^{(k)}(x,0), \quad m-1<\alpha<m.$$

## 2 Analysis of FHASTM

Let us consider a general fractional nonlinear PDE of the form

$$D_\tau^\alpha \Psi(x,\tau) + R\Psi(x,\tau) + N\Psi(x,\tau) = G(x,\tau), \quad m-1<\alpha\le m, \quad x\in R, \quad \tau>0, \tag{1}$$

subject to the initial condition $\Psi(x,0) = \Psi^{(k)}(x,0)$, $k=1,2,\dots,m-1$, where $D_\tau^\alpha\Psi(x,\tau)$ is the CFD of the function $\Psi(x,\tau)$ defined as

$$D_\tau^\alpha \Psi(x,\tau) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^\tau (\tau-s)^{m-\alpha-1} \frac{\partial^m \Psi(x,s)}{\partial \tau^m}\, ds, & m-1<\alpha<m, \\ \dfrac{\partial^m \Psi(x,\tau)}{\partial \tau^m}, & \alpha = m \in \mathbb{N}, \end{cases}$$

and $R$ is the linear differential operator, $N$ represents the general nonlinear differential operator, and $G(x,\tau)$ is the source term. Now taking the ST of both sides of equation (1) we have

$$S[D_t^\alpha \Psi(x,\tau)] + S[R\Psi(x,\tau)] + S[N\Psi(x,\tau)] = S[G(x,\tau)].$$

Using the differentiation properties of the ST and above initial condition, we have

$$\omega^{-\alpha} S[\Psi(x,\tau)] - \sum_{k=0}^{m-1} \omega^{k-\alpha}\Psi^{(k)}(x,0) + S[R\Psi(x,\tau)] + S[N\Psi(x,\tau)] = S[G(x,\tau)],$$

or

$$S[\Psi(x,\tau)] - \sum_{k=0}^{m-1} \omega^{k}\Psi^{(k)}(x,0) + \omega^\alpha (S[R\Psi(x,\tau)] + S[N\Psi(x,\tau)] - S[G(x,\tau)]) = 0.$$

We define the nonlinear operator

$$\begin{aligned} N[\varnothing(x,\tau;q)] = S[\varnothing(x,\tau;q)] &- \sum_{k=0}^{m-1} \omega^k \varnothing^{(k)}(x,0) \\ &+ \omega^\alpha (S[R\varnothing(x,\tau;q)] + S[N\varnothing(x,\tau;q)] - S[G(x,\tau)]), \end{aligned} \tag{2}$$

where $q\in[0,1]$ and $\varnothing(x,\tau;q)$ is a real function of $x,\tau$ and $q$, the so-called zero order deformation equation of the equation (2) has the form

$$(1-q)S[\varnothing(x,\tau;q) - \Psi_0(x,\tau)] = qhH(x,\tau)N[\varphi(x,\tau;q)], \tag{3}$$

where $q\in[0,1]$ is the embedding parameter, $H(x,\tau)$ denotes a nonzero auxiliary function, $h\neq 0$ is an auxiliary parameter $\Psi_0(x,\tau)$ is an initial guess of $\Psi(x,\tau)$ and $\varnothing(x,\tau;q)$ is an unknown function. Obviously, when the parameter $q=0$ and $q=1$, it holds

$$\varnothing(x,\tau;0) = \Psi_0(x,\tau), \quad \varnothing(x,\tau;1) = \Psi(x,\tau),$$

respectively. Thus as $q$ increases from 0 to 1, the solution $\varnothing(x,\tau;q)$ varies from the initial guess $\Psi_0(x,\tau)$ to the solution $\Psi(x,\tau)$. Expanding $\varnothing(x,\tau;q)$ in Taylor's series with respect to $q$, we have

$$\varnothing(x,\tau;q) = \Psi_0(x,\tau) + \sum_{m=1}^{\infty} \Psi_m(x,\tau)q^m, \tag{4}$$

where

$$\Psi_m(x,\tau) = \frac{1}{m!}\frac{\partial^m \varnothing(x,\tau;q)}{\partial q^m}\Big|_{q=0}.$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $h$, and the auxiliary function are properly chosen. The series (4) converges at $q = 1$, then we has

$$\Psi(x,\tau) = \Psi_0(x,\tau) + \sum_{m=1}^{\infty} \Psi_m(x,\tau), \tag{5}$$

which must be one of the solution of the original nonlinear equation (1). According to the definition of equation (5), the governing equation can be deduced from the zero-order deformation equation (3).

Define the vectors $\vec{\Psi}_m(x,\tau) = \{\Psi_0(x,\tau), \Psi_1(x,\tau), \ldots, \Psi_m(x,\tau)\}$. Differentiating the zero-order deformation equation (12) $m$-times with respect to $q$ and then dividing by $m!$ and finally setting $q = 0$ we get the following $m$th order deformation equation

$$S[\Psi_m(x,\tau) - x_m\Psi_{m-1}(x,\tau)] = hH(x,\tau)R_m(\vec{\Psi}_{m-1}(x,\tau)).$$

Applying the inverse ST, we have

$$\Psi_m(x,\tau) = x_m\Psi_{m-1}(x,\tau) + S^{-1}[hH(x,\tau)R_m(\vec{\Psi}_{m-1}(x,\tau))],$$

where

$$R_m(\vec{\Psi}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\varnothing(x,\tau;q)]}{\partial q^{m-1}}\Big|_{q=0}, \quad X_m = \begin{cases} 0, & x \le 1, \\ 1, & x > 1. \end{cases}$$

In this way, it is easily to obtain $\Psi_m(x,\tau)$ for $m \ge 1$, at $m$th order, $h = -1$. We have

$$\Psi(x,\tau) = \sum_{m=0}^{\infty} \Psi_m(x,\tau).$$

## 3 Applications

**Example 1.** *Consider the following nonlinear FPDE*

$$D_\tau^\alpha \Psi + \Psi\Psi_x - \Psi_{xx} = 0, \quad 0 < \alpha \le 1, \tag{6}$$

*with* $\Psi(x,0) = x$. *Applying ST to equation* (6), *we have*

$$\frac{S[\Psi]}{\omega^\alpha} - \frac{\Psi(x,0)}{\omega^\alpha} + S[\Psi\Psi_x - \Psi_{xx}] = 0 \quad \text{or} \quad S[\Psi] - x + \omega^\alpha S[\Psi\Psi_x - \Psi_{xx}] = 0.$$

*We now define a nonlinear operator as*

$$N[\varnothing(x,\tau;q)] = S[\varnothing(x,\tau;q)] + \omega^\alpha S\left[\varnothing(x,\tau;q)\frac{\partial\varnothing(x,\tau;q)}{\partial x} - \frac{\partial^2\varnothing(x,\tau;q)}{\partial x^2}\right],$$

*and thus*

$$R_m(\vec{\Psi}_{m-1}) = S(\Psi_{m-1}) - (1 - x_m)(x) + \omega^a S\left[\left(\sum_{i=0}^{m-1} \Psi_i(\Psi_{m-1-i})_x\right) - (\Psi_{m-1})_{xx}\right].$$

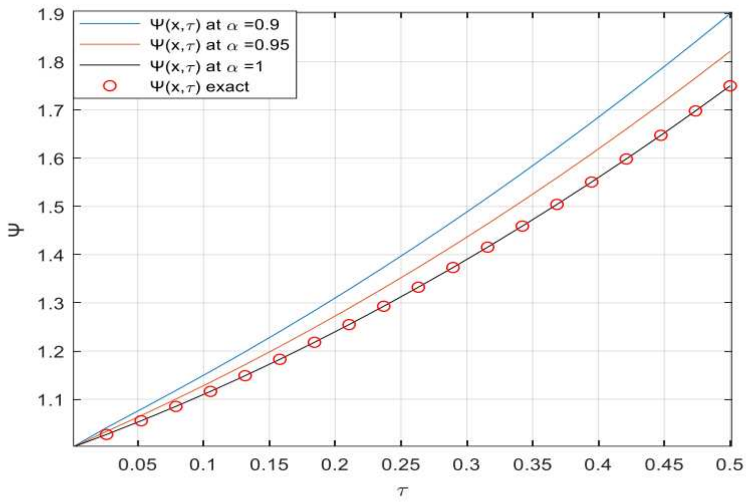


**Figure 1.** Plots of the exact and approximate solutions $\Psi(x,\tau)$ of (6) for values of $\alpha$ with the fixed value $x$.

*The* $m$*th order deformation equation is*

$$S[\Psi_m - x_w\Psi_{m-1}] = hH(x,\tau)R_m(\vec{\Psi}_{m-1}).$$

*Applying the inverse ST we have*

$$\Psi_m = x_m\Psi_{m-1} + hS^{-1}[H(x,\tau)R_m(\vec{\Psi}_{m-1})]. \tag{7}$$

*Solve above the equation* (7) *for* $m = 1,2,\ldots$ *choosing* $H(x,\tau) = 1$*. Let us take the initial conditions* $\Psi_0 = x$*,*

$$\begin{aligned}
\Psi_1 &= x_1\Psi_0 + hS^{-1}[R_1(\vec{\Psi}_0)] = (0)(x) + hS^{-1}[S(\Psi_0) - x + \omega^\alpha S(\Psi_0\Psi_{0_x} - \Psi_{0_{xx}})] \\
&= hS^{-1}[\omega^\alpha x] = \frac{hx\tau^\alpha}{\Gamma(\alpha+1)}, \\
\Psi_2 &= x_2\Psi_1 + hS^{-1}[R_2(\vec{\Psi}_1)] \\
&= (1)\left(\frac{hx\tau^\alpha}{\Gamma(\alpha+1)}\right) + hS^{-1}[S(\Psi_1) + \omega^\alpha S(\Psi_0\Psi_{1_x} + \Psi_1\Psi_{0_x} - \Psi_{1_{xx}})] \\
&= \frac{hx\tau^\alpha}{\Gamma(\alpha+1)} + hS^{-1}\left[hx\omega^\alpha + \omega^\alpha S\left(\frac{hx\tau^\alpha}{\Gamma(\alpha+1)} + \frac{hx\tau^\alpha}{\Gamma(\alpha+1)} - 0\right)\right] \\
&= \frac{hx\tau^\alpha}{\Gamma(\alpha+1)} + hS^{-1}[hx\omega^\alpha + 2hx\omega^{2\alpha}] = \frac{hx\tau^\alpha}{\Gamma(\alpha+1)} + \frac{h^2x\tau^\alpha}{\Gamma(\alpha+1)} + \frac{2h^2x\tau^{2\alpha}}{\Gamma(2\alpha+1)}, \\
&\cdots\cdots\cdots
\end{aligned}$$

*Setting then* $h = -1$*, the series solutions of equation* (6) *are given by*

$$\Psi(x,\tau) = x + \frac{x\tau^\alpha}{\Gamma(\alpha+1)} + \frac{2x\tau^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots.$$

*The exact result of Example 1 when* $\alpha = 1$ *is* $\Psi(x,\tau) = x/(1-\tau)$*.*

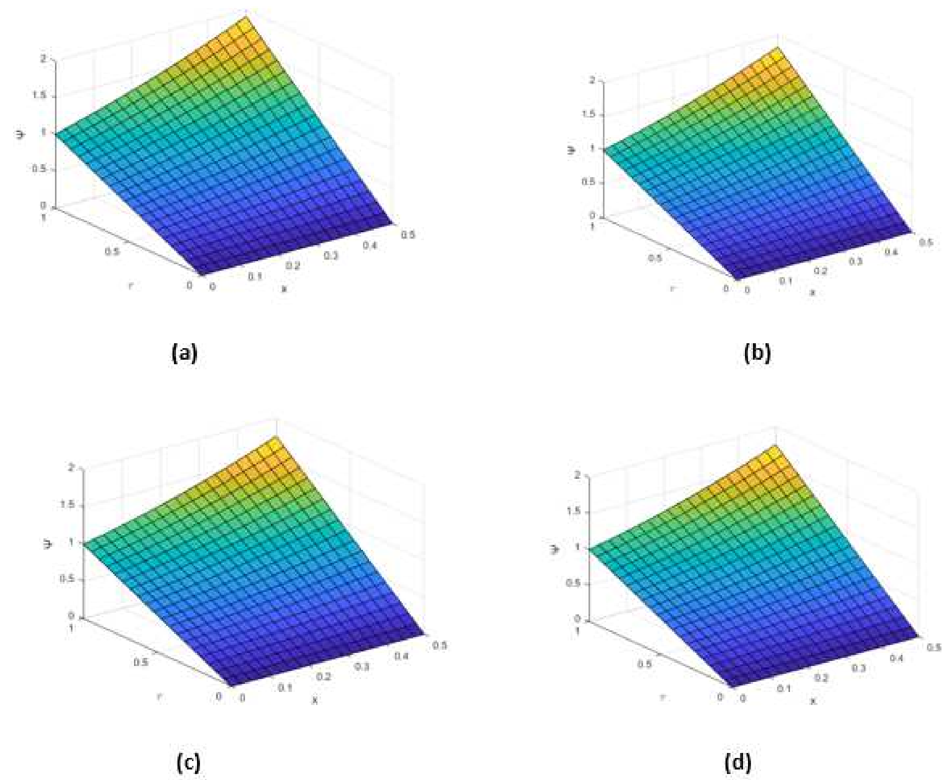


**Figure 2.** The surface graph of the approximate solutions $\Psi(x,\tau)$ of (6):
**(a)** $\Psi(x,\tau)$ when $\alpha = 0.9$; **(b)** $\Psi(x,\tau)$ when $\alpha = 0.95$;
**(c)** $\Psi(x,\tau)$ when $\alpha = 1$, **(d)** $\varphi(\mu,\tau)$ exact solution.

In Figure 1, we plot the graph of the exact and approximate solutions for (6) when $\alpha = 0.9, 0.95, 1$. In Figure 2, we plot 3D surface solution for (6) when $\alpha = 0.9, 0.95, 1$.

**Example 2.** *Consider the non-linear FPDE*

$$D_\tau^\alpha \Psi - \Psi_x^2 - \Psi\Psi_x = 0, \quad 0 < \alpha \leq 1, \tag{8}$$

*with the initial condition* $\Psi(x,0) = x^2$. *Applying ST to the equation* (8) *we obtain*

$$\frac{S[\Psi]}{\omega^\alpha} - \frac{\Psi(x,0)}{\omega^\alpha} - S[\Psi_x^2 + \Psi\Psi_{xx}] = 0. \tag{9}$$

*On simplifying and using the equation* (9) *we have*

$$S[\Psi] - x^2 - \omega^\alpha S[\Psi_x^2 + \Psi\Psi_{xx}] = 0.$$

*We now define a nonlinear operator as*

$$N[\varnothing(x,\tau;q)] = S[\varnothing(x,\tau;q)] - x^2 - \omega^\alpha S\Big[\Big(\frac{\partial\varnothing(x,\tau;q)}{\partial x}\Big)^2 + \varnothing(x,\tau;q)\frac{\partial^2\varnothing(x,\tau;q)}{\partial x^2}\Big],$$

*and thus*

$$R_m(\vec{\Psi}_{m-1}) = S(\Psi_{m-1}) - (1 - x_m)x^2 - \omega^\alpha S\Bigg[\sum_{i=0}^{m-1}(\Psi_i)_x(\Psi_{m-1-i})_x + \sum_{i=0}^{m-1}\Psi_i(\Psi_{m-1-i})_{xx}\Bigg].$$

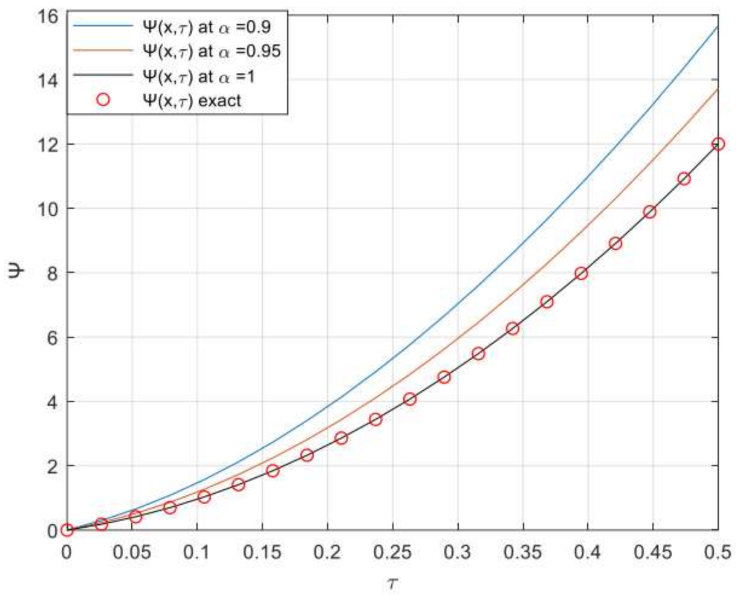


**Figure 3.** Plots of the exact and approximate solutions $\Psi(x,\tau)$ of (8) for different values of $\alpha$ with the fixed value $x$.

*The $m$th order deformation equation is*

$$S[\Psi_m - x_m\Psi_{m-1}] = hH(x,\tau)R_m(\vec{\Psi}_{m-1}).$$

*Applying the inverse ST we have*

$$\Psi_m = x_m\Psi_{m-1} + hS^{-1}[H(x,\tau)R_m(\vec{\Psi}_{m-1})]. \tag{10}$$

*Solve above the equation* (10) *for $m = 1, 2, \ldots$ choosing $H(x,\tau) = 1$. Let us take the initial conditions $\Psi_0 = x^2$,*

$$\begin{aligned}
\Psi_1 &= x_1\Psi_0 + hS^{-1}[R_1(\vec{\Psi}_0)] = (0)(x^2) + hS^{-1}[S(\Psi_0) - x^2 - \omega^\alpha S(\Psi_{0x}\Psi_{0_x} - \Psi_0\Psi_{0_{xx}})] \\
&= hS^{-1}[x^2 - x^2 - \omega^\alpha S(4x^2 + 2x^2)] = hS^{-1}[-\omega^\alpha(6x^2)] = \frac{-6hx^2\tau^\alpha}{\Gamma(\alpha+1)}, \\
\Psi_2 &= x_2\Psi_1 + hS^{-1}[R_2(\vec{\Psi}_1)] \\
&= (1)\Big(\frac{-6hx^2\tau^\alpha}{\Gamma(a+1)}\Big) + hS^{-1}[S(\Psi_1) - \omega^\alpha S(\Psi_{0_x}\Psi_{1_x} + \Psi_{1_x}\Psi_{0_x} + \Psi_0\Psi_{1_{xy}} + \Psi_1\Psi_{0_{xx}})] \\
&= \frac{-6hx^2\tau^\alpha}{\Gamma(\alpha+1)} + hS^{-1}\Big[-6hx^2\omega^\alpha - \omega^\alpha S\Big(\frac{-72hx^2\tau^\alpha}{\Gamma(\alpha+1)}\Big)\Big] \\
&= \frac{-6hx^2\tau^\alpha}{\Gamma(\alpha+1)} + hS^{-1}[-6hx^2\omega^\alpha + 2(6^2hx^2\omega^{2\alpha})] = \frac{-6hx^2\tau^\alpha}{\Gamma(a+1)} - \frac{6h^2x^2\tau^\alpha}{\Gamma(\alpha+1)} + \frac{2(6^2h^2x^2\tau^{2\alpha})}{\Gamma(2\alpha+1)},
\end{aligned}$$

$\ldots\ldots$

*Setting the $h = -1$, the series solutions of equation* (8) *are given by*

$$\Psi(x,\tau) = x^2 + \frac{6x^2\tau^\alpha}{\Gamma(\alpha+1)} + \frac{2(6^2x^2\tau^{2\pi})}{\Gamma(2\alpha+1)} + \cdots.$$

*The exact result of Example 2 when $\alpha = 1$ is $\Psi_{(x,\tau)} = x^2/(1-6\tau)$.*

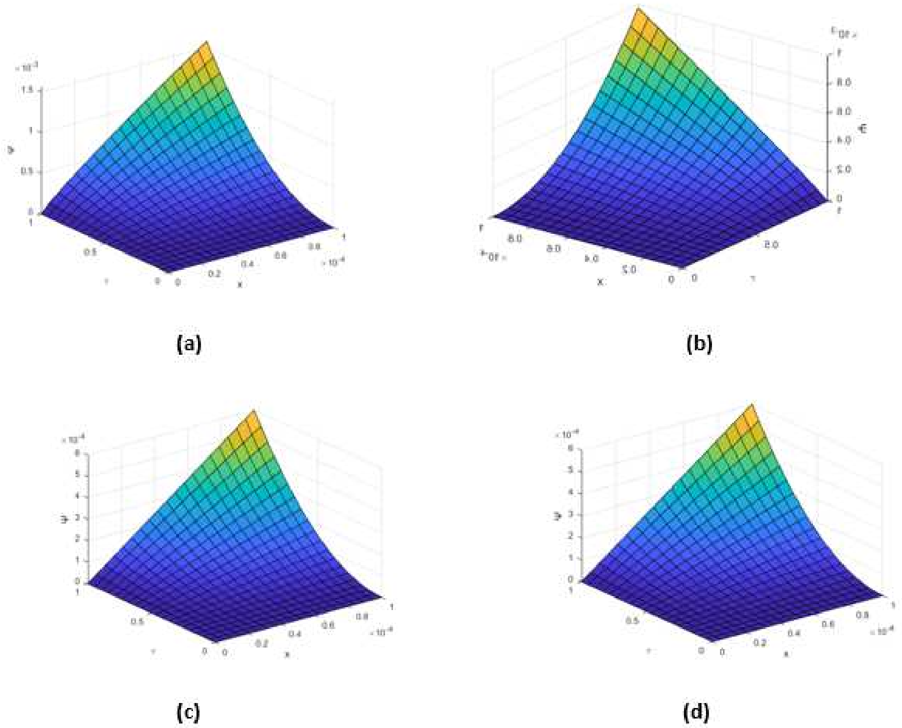


**Figure 4.** The surface graph of the approximate solutions $\Psi(x,\tau)$ of (8):
**(a)** $\Psi(x,\tau)$ when $\alpha = 0.9$; **(b)** $\Psi(x,\tau)$ when $\alpha = 0.95$;
**(c)** $\Psi(x,\tau)$ when $\alpha = 1$; **(d)** $\varphi(\mu,\tau)$ exact solution.

In Figure 3, we plot the graph of the exact and approximate solutions for (8) when $\alpha = 0.9, 0.95, 1$. In Figure 4, we plot 3D surface solution for (8) when $\alpha = 0.9, 0.95, 1$.

**Example 3.** *We consider the following non-linear system of equations*

$$\begin{aligned} &D_\tau^\alpha \Psi + u\Psi_x + \Psi - 1 = 0, \quad 0 < \alpha \leq 1, \\ &D_\tau^\beta u - \Psi u_x - u - 1 = 0, \quad 0 < \beta \leq 1, \end{aligned} \tag{11}$$

*with the initial conditions* $\Psi(x,0) = e^x$, $u(x,0) = e^{-x}$. *Applying ST on both sides in equations* (11), *we get*

$$\begin{aligned} &\frac{S[\Psi]}{\omega^\alpha} - \frac{\Psi(x,0)}{\omega^\alpha} + S[u\Psi_x + \Psi - 1] = 0, \\ &\frac{S[u]}{\omega^\alpha} - \frac{u(x,0)}{\omega^\alpha} + S[\Psi u_x - u - 1] = 0. \end{aligned} \tag{12}$$

*On simplifying and using the equations* (12) *we have*

$$\begin{aligned} &S[\Psi] - e^x + \omega^\alpha S[u\Psi_x + \Psi - 1] = 0, \\ &S[u] - e^{-x} - \omega^\beta S[\Psi u_x + u + 1] = 0. \end{aligned}$$

*We now define a nonlinear operator as*

$$\begin{aligned} N[\varnothing_1(x,\tau;q)] &= S[\varnothing_1] - e^x - \omega^\alpha + \omega^\alpha S\Big[\varnothing_2 \frac{\partial \varnothing_1}{\partial x} + \varnothing_1\Big], \\ N[\varnothing_2(x,\tau;q)] &= S[\varnothing_2] - e^{-x} - \omega^\beta - \omega^\beta S\Big[\varnothing_1 \frac{\partial \varnothing_2}{\partial x} + \varnothing_2\Big]. \end{aligned}$$

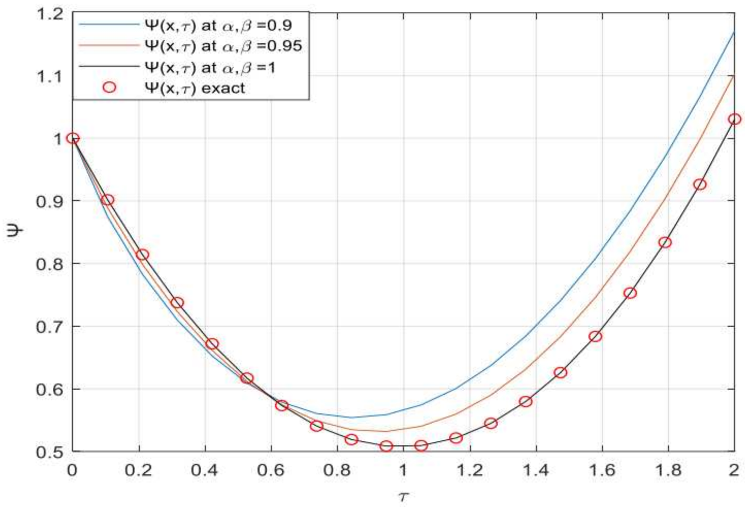


**Figure 5.** Plots of the exact and approximate solutions $\Psi(x,\tau)$ of (11) for different values of $\alpha$ with the fixed value $x$

*Thus,*

$$R_{1m}(\vec{\Psi}_{m-1}) = S(\Psi_{m-1}) - (1-x_m)(e^x+\omega^\alpha) + \omega^\alpha S\left[\left(\sum_{i=0}^{m-1} u_i(\Psi_{m-1-i})_x\right) + \Psi_{m-1}\right]$$

$$R_{2m}(\vec{u}_{m-1}) = S(u_{m-1}) - (1-x_m)(e^{-x}+\omega^\beta) - \omega^\beta S\left[\left(\sum_{i=0}^{m-1} \Psi_i(u_{m-1-i})_x\right) + u_{m-1}\right].$$

*Then $m$th order deformation equations are*

$$\begin{aligned} S[\Psi_m - x_m\Psi_{m-1}] &= hH(x,\tau)R_{1m}(\vec{\Psi}_{m-1}),\\ S[u_m - x_m u_{m-1}] &= hH(x,\tau)R_{2m}(\vec{u}_{m-1}). \end{aligned}$$

*Applying the inverse ST we have*

$$\begin{aligned} \Psi_m &= x_m\Psi_{m-1} + hS^{-1}[H(x,\tau)R_{1m}(\vec{\Psi}_{m-1})],\\ u_m &= x_m u_{m-1} + hS^{-1}[H(x,\tau)R_{2m}(u_{m-1})]. \end{aligned} \tag{13}$$

*Solve above the equations* (13) *for* $m = 1,2,\dots$ *choosing* $H(x,\tau) = 1$. *Let us take the initial conditions* $\Psi_0(x,\tau) = e^x$, $u_0(x,\tau) = e^{-x}$,

$$\begin{aligned} \Psi_1 &= x_1\Psi_0 + hS^{-1}[R_{11}(\vec{\Psi}_0)]\\ &= (0)(e^x) + hS^{-1}[S(\Psi_0) - (1-0)(e^x+\omega^\alpha) + \omega^\alpha S(u_0\Psi_{0_x} + \Psi_0)]\\ &= hS^{-1}[e^x - e^x - \omega^\alpha + \omega^\alpha S(1+e^x)] = hS^{-1}[-\omega^\alpha + \omega^\alpha + \omega^\alpha e^x] = \frac{he^x\tau^\alpha}{\Gamma(\alpha+1)},\\ u_1 &= x_1u_0 + hS^{-1}[R_{21}(\vec{u}_0)]\\ &= (0)(e^{-x}) + hS^{-1}[S(u_0) - (1-0)(e^{-x}+\omega^\beta) - \omega^\beta S(\Psi_0u_{0_x} + u_0)]\\ &= hS^{-1}[e^{-x} - e^{-x} - \omega^\beta - \omega^\beta S(-1+e^{-x})] = hS^{-1}[-\omega^\beta + \omega^\beta - \omega^\beta e^{-x}] = \frac{-he^{-x}\tau^\beta}{\Gamma(\beta+1)}, \end{aligned}$$

$$
\begin{aligned}
\Psi_2 &= x_2\Psi_1 + hS^{-1}[R_{12}(\vec{\Psi}_1)] \\
&= (1)\Big(\frac{-he^{-x}\tau^{\alpha}}{\Gamma(\alpha+1)}\Big) + hS^{-1}[S(\Psi_1) - \omega^{\alpha}S(u_0\Psi_{1_x} + u_1\Psi_{0_x} + \Psi_1)] \\
&= \frac{he^{x}\tau^{\alpha}}{\Gamma(\alpha+1)} + hS^{-1}\Big[he^{x}\omega^{\alpha} + \omega^{\alpha}S\Big(\frac{h\tau^{\alpha}}{\Gamma(\alpha+1)} - \frac{h\tau^{\beta}}{\Gamma(\beta+1)} + \frac{he^{x}\tau^{\alpha}}{\Gamma(\alpha+1)}\Big)\Big] \\
&= \frac{he^{x}\tau^{\alpha}}{\Gamma(\alpha+1)} + hS^{-1}[he^{x}\omega^{\alpha} + h\omega^{2\alpha} - h\omega^{\alpha+\beta} + he^{x}\omega^{2\alpha}] \\
&= \frac{he^{x}\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{h^2e^{x}\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{h^2\tau^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{h^2\tau^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{h^2e^{x}\tau^{2\alpha}}{\Gamma(2\alpha+1)}, \\
u_2 &= x_2u_1 + hS^{-1}[R_{22}(\vec{u}_1)] \\
&= (1)\Big(\frac{-he^{-x}\tau^{\beta}}{\Gamma(\beta+1)}\Big) + hS^{-1}[S(u_1) - \omega^{\beta}S(\Psi_0u_{1_x} + \Psi_1u_{0_x} + u_1)] \\
&= \frac{-he^{-x}\tau^{\beta}}{\Gamma(\beta+1)} + hS^{-1}\Big[-he^{-x}\omega^{\beta} - \omega^{\beta}S\Big(\frac{h\tau^{\beta}}{\Gamma(\beta+1)} - \frac{h\tau^{\alpha}}{\Gamma(\alpha+1)} - \frac{he^{-x}\tau^{\beta}}{\Gamma(\beta+1)}\Big)\Big] \\
&= \frac{-he^{-x}\tau^{\beta}}{\Gamma(\beta+1)} + hS^{-1}[-he^{-x}\omega^{\beta} - h\omega^{2\beta} + h\omega^{\alpha+\beta} + he^{-x}\omega^{2\beta}], \\
&= \frac{-he^{-x}\tau^{\beta}}{\Gamma(\beta+1)} - \frac{h^2e^{-x}\tau^{\beta}}{\Gamma(\beta+1)} - \frac{h^2\tau^{2\beta}}{\Gamma(2\beta+1)} + \frac{h^2\tau^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{h^2e^{-x}\tau^{2\beta}}{\Gamma(2\beta+1)},
\end{aligned}
$$

$\ldots\ldots$

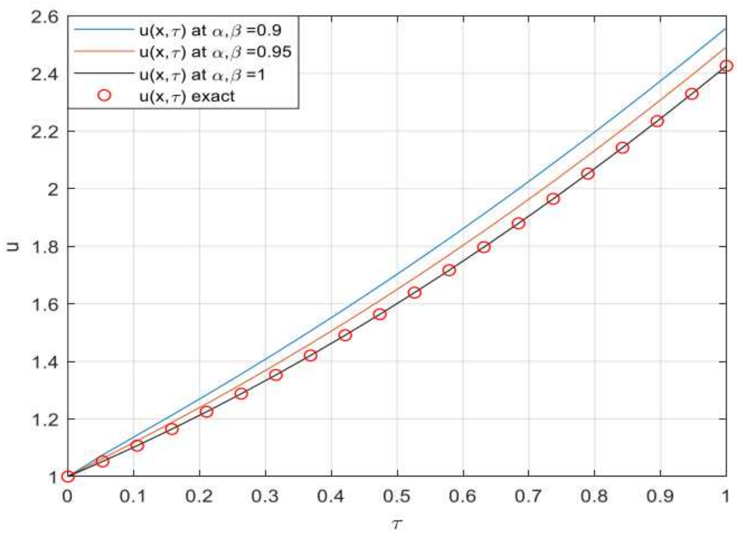


**Figure 6.** Plots of the exact and approximate solutions $u(x,\tau)$ of (11) for different values of $\alpha$ with the fixed value $x$

*Setting the* $h=-1$*, the series solutions of equations* (11) *are given by*

$$
\begin{aligned}
\Psi(x,\tau) &= e^{x} - \frac{e^{x}\tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\tau^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{e^{x}\tau^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots, \\
u(x,\tau) &= e^{-x} + \frac{e^{-x}\tau^{\beta}}{\Gamma(\beta+1)} - \frac{\tau^{2\beta}}{\Gamma(2\beta+1)} + \frac{\tau^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{e^{-x}\tau^{2\beta}}{\Gamma(2\beta+1)} + \cdots.
\end{aligned}
$$

*The exact result of Example 3 when* $\alpha=\beta=1$ *is* $\Psi(x,\tau)=e^{x-\tau}$, $u(x,\tau)=e^{-x+\tau}$.

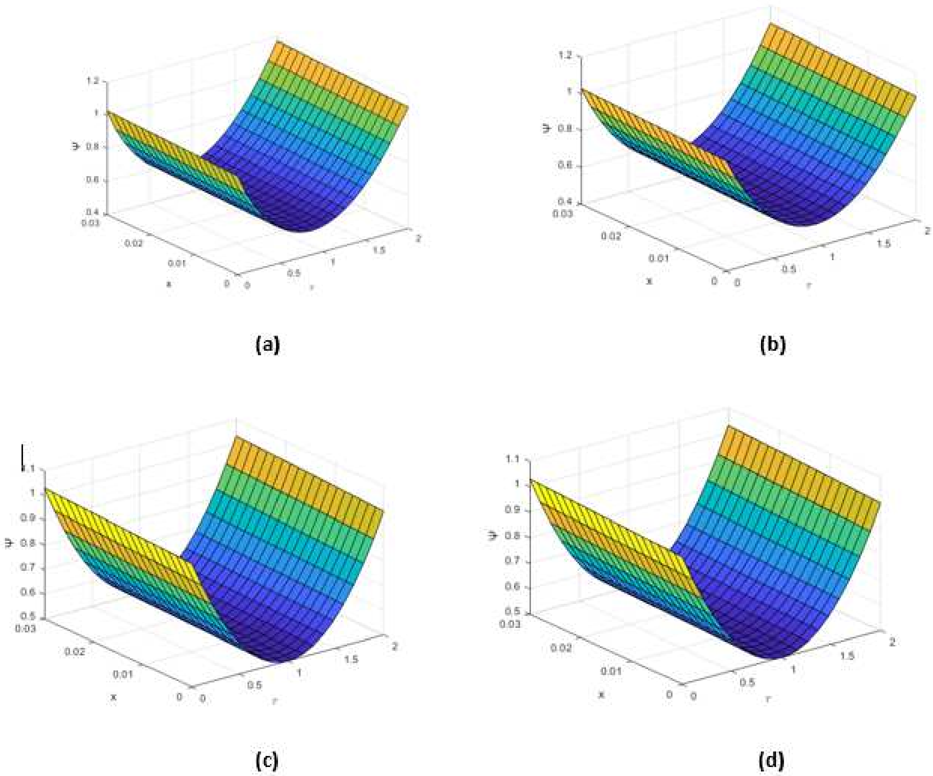


**Figure 7.** The surface graph of the approximate solutions $\Psi(x, \tau)$ of (11):
**(a)** $\Psi(x, \tau)$ when $\alpha = 0.9$; **(b)** $\Psi(x, \tau)$ when $\alpha = 0.95$;
**(c)** $\Psi(x, \tau)$ when $\alpha = 1$; **(d)** $\varphi(\mu, \tau)$ exact solution.

In Figures 5 and 6, we plot the graph of the exact and approximate solution for the equations (11) when $\alpha = 0.9, 0.95, 1$. In Figures 7 and 8, we plot 3D surface solutions for the equations (11) when $\alpha = 0.9, 0.95, 1$.

## 4 Conclusion

In this work, we utilized the HASTM to solve fractional-order PDEs and their approximate solutions were obtained. The HASTM was proved to be an effective approach for solving PDEs with CFD due to the excellent agreement between the obtained approximate solution and the exact solution. And it's rapid convergence shows that the procedure is reliable and introduces a significant improvement in solving linear and non-linear fractional-order PDEs.

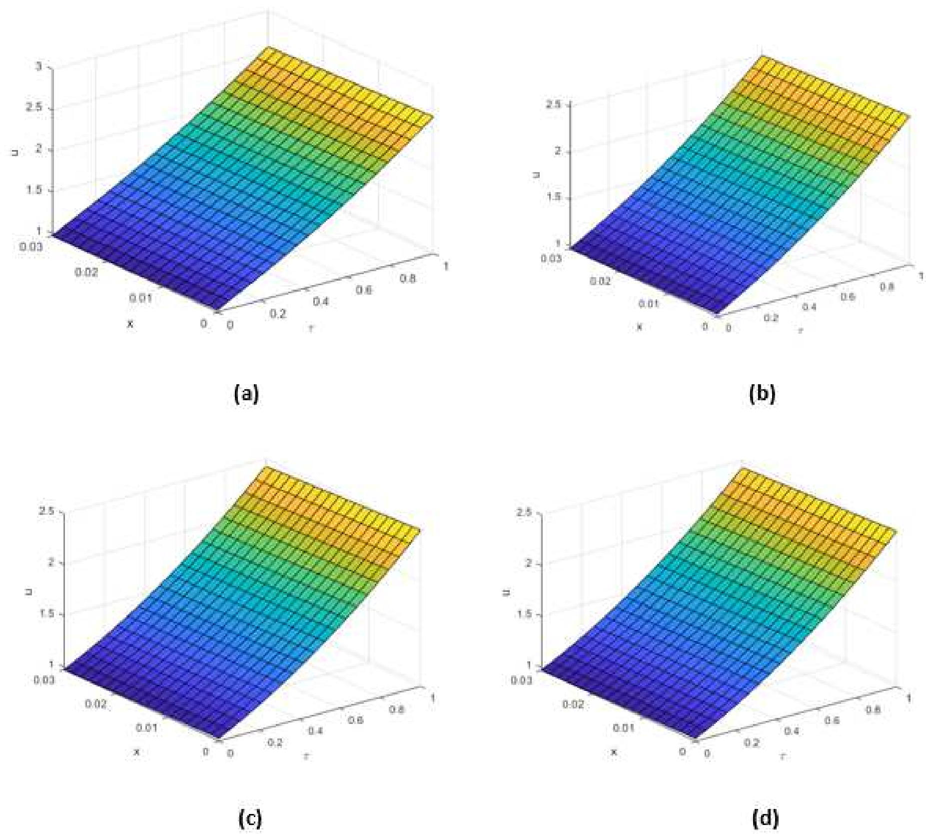


**Figure 8.** The surface graph of the approximate solutions $u(x,\tau)$ of (11):
**(a)** $u(x,\tau)$ when $\alpha = 0.9$; **(b)** $u(x,\tau)$ when $\alpha = 0.95$;
**(c)** $u(x,\tau)$ when $\alpha = 1$; **(d)** $\varphi(\mu,\tau)$ exact solution.

# References


[1] Ahmad I., Ahmad H., Abouelregal A.E., Thounthong P., Abdel-Atay M. *Numerical study of integer-order hyperbolic telegraph model arising in physical and related sciences.* Eur. Phys. J. Plus 2020, **135**, 759. doi:10.1140/epjp/s13360-020-00784-z

[2] Ahmad I., Ahmad H., Inc M., Yao S.W., Almohsen B. *Application of local meshless method for the solution of two term time fractional-order multi-dimensional PDE arising in heat and mass transfer.* Therm. Sci. 2020, **24** (1), 95–105. doi:10.2298/TSCI20S1095A.

[3] Ahmad I., Ahmad H., Thounthong P., Chu Y.M., Cesarano C. Solution of multi-term time-fractional PDE models arising in mathematical biology and physics by local meshless method. Symmetry 2020, **12** (7), 1195. doi:10.3390/sym12071195

[4] Ahmad H., Akgül A., Khan T.A., Stanimirović P.S., Chu Y.-M. *New perspective on the conventional solutions of the nonlinear time-fractional PDEs.* Complexity 2020, **2020**, 8829017. doi:10.1155/2020/8829017.

[5] Ahmad H., Khan T.A., Ahmad I., Chu Y.-M. *A new analyzing technique for nonlinear time fractional Cauchy reaction-diffusion model equations.* Results Phys. 2020, **19** 103462. doi:10.1016/j.rinp.2020.103462.

[6] Ahmad H., Khan T.A., Stanimirović P.S., Ahmad I. *Modified variational iteration technique for the numerical solution of fifth order KdV type equations.* Appl. Comput. Mech. 2020, **6**, 1220–1227. doi: 10.22055/jacm.2020.33305.2197

[7] Ahmad H., Khan T.A., Stanimirović P.S., Chu Y.-M., Ahmad I. *Modified variational iteration algorithm-II: convergence and applications to diffusion models.* Complexity 2020, **2020**, 8841718. doi:10.1155/2020/8841718

[8] Ahmad H., Seadawy A.R., Khan T.A., Thounthong P. *Analytic approximate solutions for some nonlinear parabolic dynamical wave equations.* J. Taibah Univ. Sci. 2020, **14** (1), 346–358. doi:10.1080/16583655.2020.1741943

[9] Alzaki L.K., Jassim H.K. *The approximate analytical solutions of nonlinear fractional ordinary differential equations.* Int. J. Nonlinear Anal. Appl. 2021, **12** (2), 527–535. doi:10.22075/ijnaa.2021.5094

[10] Amara A., Etemad S., Rezapour S. *Topological degree theory and Caputo-Hadamard fractional boundary value problems.* Adv. Differ. Equ. 2020, **2020**, 369. doi:10.1186/s13662-020-02833-4

[11] Baleanu D., Etemad S., Rezapour S. *A hybrid Caputo fractional modeling for thermostat with hybrid boundary value conditions.* Bound. Value Probl. 2020, **2020**, 64. 10.1186/s13661-020-01361-0

[12] Baleanu D., Jajarm A., Mohammadi H., Rezapour S. *A new study on the mathematical modeling of human liver with Caputo-Fabrizio fractional derivative.* Chaos Solitons Fractals 2020, **134** 109705. doi: 10.1016/j.chaos.2020.109705

[13] Baleanu D., Jassim H.K., Al Qurashi M. *Solving Helmholtz equation with local fractional derivative operators.* Fractal Fract. 2019, **3** (3), 43. doi:10.3390/fractalfract3030043

[14] Baleanu D., Jassim H.K. *A modification fractional homotopy perturbation method for solving Helmholtz and coupled Helmholtz equations on Cantor sets.* Fractal Fract. 2019, **3** (2), 30. doi:10.3390/fractalfract3020030

[15] Baleanu D., Jassim H.K. *Approximate solutions of the damped wave equation and dissipative wave equation in fractal strings.* Fractal Fract. 2019, **3** (2), 26. doi:10.3390/fractalfract3020026

[16] Baleanu D., Jassim H.K. *Approximate analytical solutions of Goursat problem within local fractional operators.* J. Nonlinear Sci. Appl. 2016, **9** (6), 4829–4837. doi;10.22436/jnsa.009.06.118

[17] Baleanu D., Jassim H.K. *Exact solution of two-dimensional fractional partial differential equations.* Fractal Fract. 2020, **4** (2), 21. doi:10.3390/fractalfract4020021

[18] Baleanu D., Jassim H.K., Khan H. *A modification fractional variational iteration method for solving nonlinear gas dynamic and coupled KdV equations involving local fractional operators.* Therm. Sci. 2018, **22** (1), 165–175. doi:10.2298/TSCI170804283B

[19] Baleanu D., Mousalou A., Rezapour S. *On the existence of solutions for some infinite coefficient-symmetric Caputo-Fabrizio fractional integro-differential equations.* Bound. Value Probl. 2017, **2017**, 145. doi:10.1186/s13661-017-0867-9

[20] Eaued H.A., Jassim H.K., Mohammed M.G. *A novel method for the analytical solution of partial differential equations arising in mathematical physics.* IOP Conf. Ser.: Mater. Sci. Eng. 2020, **928**, 042037. doi:10.1088/1757-899X/928/4/042037

[21] Fan Z.P., Jassim H.K., Rainna R.K., Yang X.J. *Adomian decomposition method for three-dimensional diffusion model in fractal heat transfer involving local fractional derivatives.* Therm. Sci. 2015, **19** (1), 137–141. doi:10.2298/TSCI15S1S37F

[22] Inc M., Khan M.N., Ahmad I., Yao S.W., Ahmad H., Thounthong P. *Analysing time-fractional exotic options via efficient local meshless method.* Results Phys. 2020, **19**, 103385. doi:10.1016/j.rinp.2020.103385.

[23] Jafari H., Jassim H.K. *Local fractional variational iteration method for nonlinear partial differential equations within local fractional operators.* Appl. Appl. Math. 2015, **10** (2), 1055–1065.

[24] Jafari H., Jassim H.K., Moshokoa S.P., Ariyan V.M., Tchier F. *Reduced differential transform method for partial differential equations within local fractional derivative operators.* Adv. Mech. Eng. 2016, **8** (4) 1–6. doi:10.1177/1687814016633013

[25] Jafari H., Jassim H.K., Tchier F., Baleanu D. *On the approximate solutions of local fractional differential equations with local fractional operator.* Entropy 2016, **18** (4), 150. doi:10.3390/e18040150

[26] Jafari H., Jassim H.K., Vahidi J. *Reduced differential transform and variational iteration methods for 3D diffusion model in fractal heat transfer within local fractional operators.* Therm. Sci. 2018, **22** (1), 301–307. doi:10.2298/TSCI170707033J

[27] Jassim H.K. *Analytical approximate solutions for local fractional wave equations.* Math. Methods Appl. Sci. 2020, **43** (2), 939–947. doi:10.1002/mma.5975

[28] Jassim H.K. *A new approach to find approximate solutions of Burger's and coupled Burger's equations of fractional order.* TWMS J. App. and Eng. Math. 2021, **11** (2), 415–423.

[29] Jassim H.K., Baleanu D. *A novel approach for Korteweg-de Vries equation of fractional order.* J. Appl. Comput. Mech. 2019, **5** (2), 192–198. doi:10.22055/jacm.2018.25732.1292

[30] Jassim H.K., Kadhim H.A. *Fractional Sumudu decomposition method for solving PDEs of fractional order.* J. Appl. Comput. Mech. 2021, **7** (1), 302–311. doi:10.22055/jacm.2020.31776.1920

[31] Jassim H.K., Khafif S.A. *SVIM for solving Burger's and coupled Burger's equations of fractional order.* Prog. Fract. Differ. Appl. 2021, **7** (1), 7. doi:10.18576/pfda/070107

[32] Jassim H.K., Mohammed M.G. *Natural homotopy perturbation method for solving nonlinear fractional gas dynamics equations.* Int. J. Nonlinear Anal. Appl. 2021, **12** (1), 812–820. doi:10.22075/ijnaa.2021.4936

[33] Jassim H.K., Shareef M.A. *On approximate solutions for fractional system of differential equations with Caputo-Fabrizio fractional operator.* J. Math. Comput. Sci. 2021, **23** (1) 58–66. doi:10.22436/jmcs.023.01.06

[34] Jassim H.K., Shahab W.A. *Fractional variational iteration method to solve one dimensional second order hyperbolic telegraph equations.* J. Phys.: Conf. Ser. 2018, **1032**, 012015. doi:10.1088/1742-6596/1032/1/012015

[35] Jassim H.K., Ünlü C., Moshokoa S.P., Khalique C.M. *Local fractional Laplace variational iteration method for solving diffusion and wave equations on Cantor sets within local fractional operators.* Math. Probl. Eng. 2015, **2015**, 309870. doi:10.1155/2015/309870

[36] Jassim H.K., Vahidi J., Ariyan V.M. *Solving Laplace equation within local fractional operators by using local fractional differential transform and Laplace variational iteration methods.* Nonlinear Dyn. Syst. Theory 2020, **20** (4), 388–396.

[37] Li Y., Wang L.F., Yuan S.J. *Reconstructive schemes for variational iteration method within Yang-Laplace transform with application to fractal heat conduction problem.* Therm. Sci. 2013, **17** (3), 715–721. doi:10.2298/tsci120826075l

[38] Mohsin N.H., Jassim H.K., Azeez A.D. *A new analytical method for solving nonlinear Burger's and coupled Burger's equations.* Mater. Today: Proc. 2021. doi:10.1016/j.matpr.2021.07.194 (in press)

[39] Podlubny I. Fractional Differential Equations. Academic Press, San Diego, CA, 1999.

[40] Shakeel M., Hussain I., Ahmad H., Ahmad I., Thounthong P., Zhan Y.-F. *Meshless technique for the solution of time-fractional partial differential equations having real-world applications.* J. Funct. Spaces 2020, **2020**, 8898309. doi:10.1155/2020/8898309.

[41] Singh J., Jassim H.K., Kumar D. *An efficient computational technique for local fractional Fokker-Planck equation.* Phys. A: Stat. Mech. Appl. 2020, **555**, 124525. doi;10.1016/j.physa.2020.124525

[42] Su W.H., Baleanu D., Yang X.-J., Jafari H. *Damped wave equation and dissipative wave equation in fractal strings within the local fractional variational iteration method.* Fixed Point Theory Appl. 2013, **2013**, 89. doi:10.1186/1687-1812-2013-89

[43] Wang K., Liu S. *A new Sumudu transform iterative method for time-fractional Cauchy reaction-diffusion equation.* Springer Plus 2016, **5**, 865. doi:10.1186/s40064-016-2426-8

[44] Xu S., Ling X., Zhao Y., Jassim H.K. *A novel schedule for solving the two-dimensional diffusion in fractal heat transfer.* Therm. Sci. 2015, **19** (1), 99–103. doi:10.2298/TSCI15S1S99X

[45] Yang X.J. Local Fractional Functional Analysis and Its Applications. Asian Academic, Hong Kong, China, 2011.

[46] Yang X.J., Machad J.A., Srivastava H.M. *A new numerical technique for solving the local fractional diffusion equation: two-dimensional extended differential transform approach.* Appl. Math. Comput. 2016, **274**, 143–151. doi:10.1016/j.amc.2015.10.072

[47] Yan S.P., Jafari H., Jassim H.K. *Local fractional Adomian decomposition and function decomposition methods for solving Laplace equation within local fractional operators.* Adv. Math. Phys. 2014, **2014**, 161580. doi: 10.1155/2014/161580

*Received 17.05.2021*
*Revised 25.11.2021*

Джассім Х.К., Ахмад Г., Шамаун А., Чезарано К. *Ефективний гібридний метод розв'язування диференціальних рівнянь з частинними похідними дробового порядку* // Карпатські матем. публ. — 2021. — Т.13, №3. — С. 790–804.

У цій роботі реалізовано гібридний метод, який називається гомотопічним аналізом за допомогою методу перетворення Сумуду, що дозволяє розв'язувати диференціальні рівнянння з частинними похідними дробового порядку. Цей метод є об'єднанням методу перетворення Сумуду та методу гомотопічного аналізу. Розглянуто три приклади для підтвердження і демонстрації ефективності та точності цієї методики. Також показано, що результати, отримані за допомогою запропонованої методики, чудово узгоджуються з точним розв'язком, що свідчить про ефективність, надійність та простоту реалізації запропонованого методу для різних суміжних проблем науки та техніки.

*Ключові слова і фрази:* диференціальне рівнянння з частинними похідними дробового порядку, перетворення Сумуду, метод гомотопічного аналізу.