On generalized double almost statistical convergence of weight $g$

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The purpose of this paper is to introduce the concept of $\lambda$-double almost statistical convergence of weight $g$, which emerges naturally from the concept of the double almost convergence and $\lambda$-statistical convergence. Some interesting inclusion relations have been considered.

Key words and phrases: weight function $g$, double statistical convergence, double almost convergence, modulus function.

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Introduction

An extension of the usual concept of sequential limits, which is called statistical convergence, was first recognized by H. Fast [6] as follows.

A sequence $(x_k)$ of real numbers is said to be statistically convergent to $L$ if for an arbitrary $\varepsilon > 0$ we have

$$\lim_{n \to \infty} \frac{1}{n} \left| \left\{ k \leq n : |x_k - L| \geq \varepsilon \right\} \right| = 0.$$ 

Statistical convergence turned out to be one of the most active areas of research in summability theory after the works of J.A. Fridy [7], T. Šalát [18], J.S. Connor [5] and some others.

M. Mursaleen [13] defined $\lambda$-statistical convergence which is more general than statistical convergence as follows.

A sequence $(x_k)$ is said to be $\lambda$-statistically convergent if there is a complex number $L$ such that

$$\lim_{n \to \infty} \frac{1}{\lambda_n} \left| \left\{ k \in I_n : |x_k - L| \geq \varepsilon \right\} \right| = 0.$$ 

Later on E. Savaş [19] continued the study of the concept of $\lambda$-almost statistical convergence by using almost convergence. Recently, $\lambda$-statistical convergence of order $\alpha$, $0 < \alpha \leq 1$, was introduced and studied by R. Çolak and Ç.A. Bektaş [3]. This is a generalization of $\lambda$-statistical convergence.

In this paper, as new and more general approach, we introduce and study the concept of $\lambda$-double almost statistical convergence of weight $g$, where $g : [0, \infty] \times [0, \infty) \to [0, \infty)$, $g(x_{nm}) \to \infty$ for any sequence $(x_{nm})$ in $[0, \infty) \times [0, \infty)$ with $x_{nm} \to \infty$. Throughout the paper, the class of all such functions will be denoted by $G$.

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1 Basic facts and definitions

Let $w_2$ be the class of all real or complex double sequences. By the convergence of a double sequence we mean the convergence in Pringsheim’s sense, that is, double sequence $x = (x_{kl})$ has a Pringsheim limit $L$ denoted by $P$-lim $x$ provided that for a given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{kl} - L| < \epsilon$ whenever $k, l \geq N$. We call such an $x$ more briefly as “P-convergent” (see [15]). Also double sequences were introduced and studied by R.F. Patterson (see [16], [17]) and many others.

We use symbol $c_2$ to denote the class of P-convergent sequences. A double sequence $x = (x_{kl})$ is bounded if $\|x\| = \sup_{k,l \geq 0} |x_{kl}| < \infty$. Let $l_2^\infty$ and $c_2^\infty$ be the set of all real or complex bounded double sequences and the set of bounded and convergent double sequences, respectively.

Set

$$x_{kl} = \begin{cases} \max(k,l), & \text{if } \min(k,l) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to find that $\lim_{k,l} x_{kl} = 0$ but $\sup_{k,l} |x_{kl}| = \infty$. This shows that the convergence of a double sequence in Pringsheim’s sense does not imply the boundedness of its terms. Further J.D. Hill [8] studied the double sequences certain results obtained by G.G. Lorentz [9] for single sequences.

Following S. Banach [1] we can easily define the following.

A linear functional $\varphi$ on $l_2^\infty$ is said to be Banach limit if it has the following properties:

1) $\varphi (x) \geq 0$ if $x \geq 0$, i.e. $x_{kl} \geq 0$ for all $k,l$;

2) $\varphi (e) = 1$, where $e = (e_{kl})$ with $e_{kl} = 1$ for all $k,l$;

3) $\varphi (x) = \varphi (S_{10} x) = \varphi (S_{01} x) = \varphi (S_{11} x)$, where the shift operators $S_{10} x$, $S_{01} x$, $S_{11} x$ are defined by $S_{10} x = (x_{k+1,l})$, $S_{01} x = (x_{k,l+1})$, $S_{11} x = (x_{k+1,l+1})$.

Let $B_2$ be the set of all Banach limits on $l_2^\infty$. A double sequence $x = (x_{kl})$ is said to be almost convergent to a number $L$ if $\varphi (x) = L$ for all $\varphi \in B_2$ (see [8]).


A double sequence $x = (x_{kl})$ is said to be almost convergent to a number $L$ if

$$P \lim_{p,q \to \infty} \sup_{m,n \geq 0} \left| \frac{1}{(p+1)(q+1)} \sum_{k=m}^{m+p} \sum_{l=n}^{n+q} x_{kl} - L \right| = 0,$$

that is, the average value of $(x_{ij})$ taken over any rectangle

$$D = \{(i,j) : m \leq i \leq m+p, n \leq j \leq n+q\}$$

tends to $L$ as both $p$ and $q$ tend to $\infty$ and this convergence is uniform in $m$ and $n$. We denote the space of double almost convergent sequences by $\hat{c}_2$, namely

$$\hat{c}_2 = \left\{ x = (x_{kl}) : \lim_{kl \to \infty} |t_{klpq} (x) - L| = 0 \text{ uniformly in } p, q \right\},$$
where
\[ t_{klpq}(x) = \frac{1}{(k+1)(l+1)} \sum_{k=p}^{k+p-l+q} \sum_{l=q}^{l+q} x_{kl}. \]

M. Mursaleen and O.H. Edely [14] presented the notion of statistical convergence for double sequence \( x = (x_{kl}) \) as follows.

A real double sequence \( x = (x_{kl}) \) is said to be statistically convergent to \( L \), provided that for each \( \varepsilon > 0 \)
\[
P-lim_{mn} \frac{1}{mn} \left| \left\{ (k,l) : k \leq m \text{ and } l \leq n, |x_{kl} - L| \geq \varepsilon \right\} \right| = 0.
\]

More recent developments on double sequences can be found in [2, 4, 10, 12] and some others, where some more references can be found.

**Definition 1.** Let \( \lambda = (\lambda_n) \) and \( \mu = (\mu_m) \) be two non-decreasing sequences of positive real numbers both tending to \( \infty \) as \( n \) and \( m \) approach \( \infty \), respectively. Also let \( \lambda_{n+1} \leq \lambda_n + 1 \), \( \lambda_1 = 1 \) and \( \mu_{m+1} \leq \mu_m + 1 \), \( \mu_1 = 1 \). We write the generalized double de la Vallee-Poussin mean by
\[ t_{nm}(x) = \frac{1}{\lambda_n \mu_m \sum_{i \in I_n, j \in I_m} x_{kl}}. \]

A sequence \( x = (x_{kl}) \) is said to be \((V^2, \lambda, \mu)\)-summable to a number \( L \) if \( t_{nm}(x) \to L \) as \( n, m \to \infty \) in Pringsheim’s sense.

Throughout this paper, we shall denote \( \lambda_n \mu_m \) by \( \bar{\lambda}_{nm} \), and \( i \in I_n, j \in I_m \) by \((i,j) \in I_{nm}\).

**2 Main Results**

We now introduce our fundamental definition. Throughout this paper, for typographical convenience we shall use the notation \( x_{klpq} \) to denote \( x_{k+p,l+q} \).

**Definition 2.** Let the sequence \( \lambda = (\lambda_{nm}) \) of real numbers be defined as above and let \( g \in G \).
A sequence \( x = (x_{kl}) \) is said to be \( \lambda \)-double almost statistically convergent of weight \( g \) if there is a complex number \( L \) such that
\[
P-lim_{mn \to \infty} \frac{1}{g(\lambda_{nm})} \left| \left\{ (k,l) \in I_{nm} : |x_{klpq} - L| \geq \varepsilon \right\} \right| = 0
\]
uniformly in \( p, q \). In this case we write \( S^g_{x,\lambda} \)-lim \( x_{kl} = L \).

The set of all \( \lambda \)-double almost statistically convergent sequences of weight \( g \) will be denoted by \( S^g_{x,\lambda} \). For example, the sequence \( x = (x_{kl}) \) defined by
\[
x_{klpq} = \begin{cases} klpq, & klpq = (nm)^2, \\
0, & klpq \neq (nm)^2, \end{cases} \quad n, m = 1, 2, \ldots,
\]
is \( \lambda \)-double almost statistically convergent of weight \( g \) to 0 for any \( g \in G \), for which there exist \( M_1, M_2 > 0 \) and \((r,s) \in \mathbb{N} \times \mathbb{N}\) such that
\[ M_1 \leq \frac{(nm)^\alpha}{g(nm)} \leq M_2 \text{ for all } n \geq r \text{ and } m \geq s,
\]
where \( \frac{1}{2} < \alpha \leq 1 \) and \( \lambda = (nm) \).
Remark 1. In the above definition, if we consider $g(\lambda_{nm}) = (nm)$, $\alpha = 1$, we have the notion of double almost statistical convergence [8]. The set of all double almost statistically convergent sequences will be denoted by $\tilde{S}$.

This definition led to the following theorem.

Theorem 1. Let $g \in G$ and $x = (x_{kl})$, $y = (y_{kl})$ be sequences of complex numbers.

(i) If $\tilde{S}^g_\lambda$-lim $x_{kl} = x_0$ and $c \in C$, then $\tilde{S}^g_\lambda$-lim $cx_{kl} = cx_0$.

(ii) If $\tilde{S}^g_\lambda$-lim $x_{kl} = x_0$ and $\tilde{S}^g_\lambda$-lim $y_{kl} = y_0$, then $\tilde{S}^g_\lambda$-lim $(x_{kl} + y_{kl}) = x_0 + y_0$.

Proof. (i) For $c = 0$ the result is clear. Let $c \neq 0$. We find that

$$
\frac{1}{g(\lambda_{nm})} \left| \{(k,l) \in I_{nm} : |cx_{klpq} - cx_0| \geq \epsilon \} \right| = \frac{1}{g(\lambda_{nm})} \left| \{(k,l) \in I_{nm} : |x_{klpq} - x_0| \geq \epsilon |c| \} \right|
$$

and the result follows.

(ii) The result follows from the fact that

$$
\frac{1}{g(\lambda_{nm})} \left| \{(k,l) \in I_{nm} : |x_{klpq} + y_{klpq} - (x_0 + y_0)| \geq \epsilon \} \right|
\leq \frac{1}{g(\lambda_{nm})} \left| \{(k,l) \in I_{nm} : |x_{klpq} - x_0| \geq \frac{\epsilon}{2} \} \right|
+ \frac{1}{g(\lambda_{nm})} \left| \{(k,l) \in I_{nm} : |y_{klpq} - y_0| \geq \frac{\epsilon}{2} \} \right|.
$$

□

Definition 3. Let $\lambda = (\lambda_{nm})$ be as above and let $g \in G$. Let $t$ be a positive real number. A sequence $x = (x_{kl})$ is said to be strongly $(\tilde{V}, \lambda)$-double almost summable of weight $g$ if there is a complex number $L$ such that

$$
\lim_{n,m \to \infty} \frac{1}{g(\lambda_{nm})} \sum_{(k,l) \in I_{nm}} |x_{klpq} - L|^t = 0
$$

uniformly in $p, q$. The set of all strongly $(\tilde{V}, \lambda)$-double almost summable sequences of weight $g$ will be denoted by $[\tilde{V}^g_t, \lambda]$.

Remark 2. For $g(n) = (nm)^\alpha$, $0 < \alpha \leq 1$, this notion coincides with the notion of strong $(\tilde{V}, \lambda)$-double almost summability of order $\alpha$.

Theorem 2. Let $g_1, g_2 \in G$. If there exist $M > 0$ and $(r,s) \in \mathbb{N} \times \mathbb{N}$ such that $g_1(\lambda_{nm}) / g_2(\lambda_{nm}) \leq M$ for all $n \geq r$ and $m \geq s$, then $\tilde{S}^{g_1}_\lambda \subseteq \tilde{S}^{g_2}_\lambda$.

Proof. Write that,

$$
\frac{1}{g_2(\lambda_{nm})} \left| \{(k,l) \in I_{nm} : |x_{klpq} - L| \geq \epsilon \} \right| = \frac{g_1(\lambda_{nm})}{g_2(\lambda_{nm})} \cdot \frac{1}{g_1(\lambda_{nm})} \left| \{(k,l) \in I_{nm} : |x_{klpq} - L| \geq \epsilon \} \right|
\leq M \cdot \frac{1}{g_1(\lambda_{nm})} \left| \{(k,l) \in I_{nm} : |x_{klpq} - L| \geq \epsilon \} \right|
$$
for all \( n \geq r \) and \( m \geq s \). If \( x = (x_{kl}) \in \bar{S}_\lambda^{g_1} \), then the right hand side tends to zero uniformly in \( p, q \) for every \( \varepsilon > 0 \) and in this case

\[
\frac{1}{g_2(\lambda_{nm})} \left| \{(k, l) \in I_{nm} : |x_{klpq} - L| \geq \varepsilon\} \right| = 0
\]

uniformly in \( p, q \) and finally \( x \in \bar{S}_\lambda^{g_2} \). Hence \( \bar{S}_\lambda^{g_1} \subseteq \bar{S}_\lambda^{g_2} \).

\[\square\]

**Corollary 1.** In particular, let \( g \in G \) and if there exist \( M > 0 \) and \((r, s) \in \mathbb{N} \times \mathbb{N} \) such that \((nm)/g(\lambda_{nm}) \leq M \) for all \( n \geq r \) and \( m \geq s \), then \( \bar{S}_\lambda^{g_1} \subseteq \bar{S}_\lambda \).

**Theorem 3.** \( \bar{S} \subseteq \bar{S}_\lambda^{g} \) if \( \lim\inf_{nm \to \infty} \frac{g(\lambda_{nm})}{nm} > 0 \).

**Proof.** For any \( \varepsilon > 0 \), we write

\[
\{k \leq n \text{ and } l \leq m : |x_{klpq} - L| \geq \varepsilon\} \supseteq \{(k, l) \in I_{nm} : |x_{klpq} - L| \geq \varepsilon\}.
\]

Hence, it follows that for \( p, q \in \mathbb{N} \)

\[
\frac{1}{nm} \left| \{k \leq n \text{ and } l \leq m : |x_{klpq} - L| \geq \varepsilon\} \right| \geq \frac{1}{nm} \left| \{(k, l) \in I_{nm} : |x_{klpq} - L| \geq \varepsilon\} \right| \geq \frac{g(\lambda_{nm})}{nm} \cdot \frac{1}{g(\lambda_{nm})} \left| \{(k, l) \in I_{nm} : |x_{klpq} - L| \geq \varepsilon\} \right|.
\]

If \( x \to L(\bar{S}) \), then \( \frac{1}{nm} \left| \{k \leq n \text{ and } l \leq m : |x_{klpq} - L| \geq \varepsilon\} \right| \to 0 \) as \( n, m \to \infty \) and consequently we find

\[
\frac{1}{nm} \left| \{k \leq n \text{ and } l \leq m : |x_{klpq} - L| \geq \varepsilon\} \right| \to 0
\]

and so

\[
\frac{1}{g(\lambda_{nm})} \left| \{(k, l) \in I_{nm} : |x_{klpq} - L| \geq \varepsilon\} \right| \to 0
\]

as \( n, m \to \infty \). It is clear that \( x \to L(\bar{S}_\lambda^{g}) \).

\[\square\]

**Theorem 4.** Let \( g_1, g_2 \in G \). If there exist \( M > 0 \) and \((r, s) \in \mathbb{N} \times \mathbb{N} \) such that

\[
g_1(\lambda_{nm})/g_2(\lambda_{nm}) \leq M
\]

for all \( n \geq r \) and \( m \geq s \), then \([\bar{V}_t^{g_1}, \lambda] \subseteq [\bar{V}_t^{g_2}, \lambda]\).

**Proof.** The proof is similar to the proof of Theorem 3.6 and so is omitted.

\[\square\]

**Corollary 2.** Let \( g \in G \). If there exist \( M > 0 \) and \((r, s) \in \mathbb{N} \times \mathbb{N} \) such that \((nm)/g(\lambda_{nm}) \leq M \) for all \( n \geq r \) and \( m \geq s \), then \( \bar{S}_\lambda^{g_1} \subseteq \bar{S}_\lambda \).

**Theorem 5.** If \( 0 < t < u < \infty \) and \( g \in G \), then \([\bar{V}_u^{g}, \lambda] \subseteq [\bar{V}_t^{g}, \lambda]\).

The proof follows from Hölder’s inequality.
Theorem 6. Let \( g_1, g_2 \in G \) and there exist \( M > 0 \) and \((r, s) \in \mathbb{N} \times \mathbb{N}\) such that \( g_1(\lambda_{nm})/g_2(\lambda_{nm}) \leq M \) for all \( n \geq r \) and \( m \geq s \) and let \( 0 < p < \infty \). If a sequence \( x = (x_{kl}) \) is strongly \((V, \lambda)\)-almost double summable of weight \( g_1 \) to \( L \), then it is \( \lambda \)-almost double statistically convergent of weight \( g_2 \) to \( L \), i.e. \( [V_{g_1}^{\lambda}, \lambda] \subset \hat{S}^{g_2}_{\lambda} \).

Proof. Let \( x = (x_{kl}) \in [V_{g_1}^{\lambda}, \lambda] \) and let \( \varepsilon > 0 \) be given. Consider

\[
\sum_{(k,l) \in I_{nm}} |x_{klpq} - L|^t = \sum_{(k,l) \in I_{nm}} |x_{klpq} - L|^t + \sum_{(k,l) \in I_{nm}} |x_{klpq} - L|^t \geq \sum_{k \in I_{nm}} |x_{klpq} - L|^t \geq \{(k,l) \in I_{nm} : |x_{klpq} - L| \geq \varepsilon\} \cdot \varepsilon^t.
\]

Now it follows that

\[
\frac{1}{g_1(\lambda_{nm})} \sum_{(k,l) \in I_{nm}} |x_{klpq} - L|^t \geq \frac{1}{g_1(\lambda_{nm})} \{(k,l) \in I_{nm} : |x_{klpq} - L| \geq \varepsilon\} \cdot \varepsilon^t = \frac{g_2(\lambda_{nm})}{g_1(\lambda_{nm})} \cdot \frac{1}{g_2(\lambda_{nm})} \{|(k,l) \in I_{nm} : |x_{klpq} - L| \geq \varepsilon\} \cdot \varepsilon^t \geq \frac{1}{M} \cdot \frac{1}{g_2(\lambda_{nm})} \{|(k,l) \in I_{nm} : |x_{klpq} - L| \geq \varepsilon\} \cdot \varepsilon^t
\]

for all \( n \geq r \) and \( m \geq s \). If \( x \rightarrow L([V_{g_1}^{\lambda}, \lambda]) \) then the left hand side tends to zero and consequently the right hand side also tends to zero uniformly in \( p, q \). Hence \( x \rightarrow L(\hat{S}^{g_2}_{\lambda}) \).

\[\Box\]

Corollary 3. Let \( g \in G \). If there exist \( M > 0 \) and \((r, s) \in \mathbb{N} \times \mathbb{N}\) such that \( \frac{nm}{g(\lambda_{nm})} \leq M \) for all \( n \geq r, m \geq s \) and \( 0 < p < \infty \), then \( [V_{g}^{\lambda}, \lambda] \subset \hat{S}_{\lambda} \).

3 Conclusion

Recently, \( \lambda \)-statistical convergence has been considered as a better option than statistically convergence. It is found very interesting that some results on sequences, series and summability can be proved by replacing the statistical convergence by \( \lambda \)-statistical convergence. This concept has also been defined and studied in different setups. In this paper, we study the concept of \( \lambda \)-double almost statistical convergence of weight \( g \), which emerges naturally from the concepts of the double almost convergence and \( \lambda \)-double statistical convergence.

References


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Метою цiєї статтi є впровадити поняття λ-подвiйної майже статистичної збiжностi з вагою g, яка природним чином випливає з поняття подвiйної майже збiжностi та λ-статистичної збiжностi. У статтi розглянуто деякi цiкавi вiдношення включення.

Ключовi слова і фрази: вагова функцiя g, подвiйна статистична збiжнiсть, подвiйна майже збiжнiсть, модуль функцiї.