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Essentially iso-retractable modules and rings

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A.K. Chaturvedi et al. (2021) call a module M essentially iso-retractable if for every essential submodule N of M there exists an isomorphism $f : M \to N$. We characterize essentially iso-retractable modules, co-semisimple modules (V-rings), principal right ideal domains, simple modules and semisimple modules. Over a Noetherian ring, we prove that every essentially iso-retractable module is isomorphic to a direct sum of uniform submodules.

Key words and phrases: retractable module, iso-retractable module, essentially iso-retractable module, essentially compressible module.

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Introduction

In [3,4], first author introduced the notion of iso-retractable modules. A module M is called *iso-retractable* if for each nonzero submodule N of M there exists an isomorphism $\theta : M \to N$. A nonzero module is called simple if its every nonzero submodule is equal to the module; and a module is called semisimple if its every essential submodule is equal to the module. Observing that the class of iso-retractable modules is a generalization of simple modules, first and third authors in [5] introduced the notion of essentially iso-retractable modules which is a common generalization of semisimple modules and iso-retractable modules. They call a module M *essentially iso-retractable* if for every essential submodule N of M there exists an isomorphism $f : M \to N$.

The main aim of this work is to relate the classes of essentially iso-retractable modules and rings with other known classes in ring and module theory. Also, we provide some new properties and characterizations of the essentially iso-retractable modules and rings here.

This story begins by the idea of compressible modules. Following [2], an R-module M is *compressible* if for each nonzero submodule N of M there exists a monomorphism $\theta : M \to N$. In 1979, S.M. Khuri [8] defined the notion of retractable modules as a generalization of compressible modules. He called an R-module M *retractable* if for each nonzero submodule N of M there exists a nonzero homomorphism $\theta : M \to N$. In 2006, P.F. Smith et.al. [15] defined the notion of essentially compressible modules as a generalization of compressible mod-

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ules. They called an R-module M *essentially compressible* if for each essential submodule N of M there exists a monomorphism $\theta : M \to N$. In 2007, M.R. Vedadi [17] defined the notion of essentially retractable modules as a generalization of retractable modules and called an R-module M *essentially retractable* if for each essential submodule N of M there exists a nonzero homomorphism $\theta : M \to N$. In 2009, A. Ghorbani et.al. [7] defined the notion of epi-retractable modules as a dualization of compressible modules. They called an R-module M *epi-retractable* if for each nonzero submodule N of M there exists an epimorphism $\theta : M \to N$.

A brief outline of this paper is as follows. In Section 1, we provide a necessary and sufficient condition for a ring to be right essentially iso-retractable. We prove that the matrix ring of a right essentially iso-retractable ring is right essentially iso-retractable. In Proposition 3, we give a characterization of essentially iso-retractable modules in terms of essential monomorphisms. We characterize iso-retractable modules in terms of essentially iso-retractable and uniform modules (see Proposition 4). We do not know whether a submodule of an essentially iso-retractable module is always essentially iso-retractable. But, the quotient of an essentially iso-retractable module need not be essentially iso-retractable. In Proposition 5, we find some sufficient conditions for a submodule and a homomorphic image of an essentially iso-retractable module to be essentially iso-retractable. In case of iso-retractable modules, the direct sum of iso-retractable modules need not be iso-retractable. But in case of essentially iso-retractable modules, the direct sum is essentially iso-retractable (see Proposition 6).

In Section 2, we begin with a characterization of co-semisimple modules. As a consequence, we have a characterization of V-rings. Next, we characterize a semisimple ring. In Proposition 9, we characterize a semisimple ring which shows that if every essentially iso-retractable R-module is projective, then the ring R is semisimple. We show that every essentially iso-retractable module is an epi-retractable module (Proposition 10). In Theorem 2, we give a characterization of simple modules. In general, submodules of a projective (respectively, injective) module need not be projective (respectively, injective). In Proposition 11, we prove that an essentially iso-retractable module M is projective (respectively, injective) if and only if every submodule of M is projective (respectively, injective). We characterize principle right domains (see Theorem 4). In Propositions 3 and 17, we discuss the structure of essentially iso-retractable modules under certain conditions. In general, a projective module need not be essentially iso-retractable. We prove that if every projective right R-module is essentially iso-retractable then the ring R is right hereditary (see Proposition 12).

Throughout the paper, all rings are associative with identity and all modules are unital right modules, unless otherwise stated. The terminology not defined here may be found in [1,9].

1 Preliminaries of essentially iso-retractable modules

Recall [10], an element $x \in R$ is *right regular* if $xr = 0$ implies $r = 0$ for $r \in R$. The following are well known facts and the proof is routine.

Lemma 1. *Let I be a nonzero right ideal of a ring R.*

(1) There exists an epimorphism $\theta : R \to I$ if and only if there exists an element $a \in R$ such that $I = aR$.

(2) There exists an isomorphism $\theta : R \to I$ if and only if there exists a right regular element $a \in R$ such that $I = aR$.

Proposition 1. *A ring R is right essentially iso-retractable if and only if for every essential right ideal I of R there is a right regular element $a \in R$ such that $I = aR$.*

Lemma 2. *Every ring is a retractable ring.*

Proof. Let I be a nonzero right ideal of a ring R. Then there exists $0 \neq a \in I$ such that $0 \neq aR \subseteq I$. By Lemma 1 (1), there exists an epimorphism $\theta : R \to aR$ which implies that $\theta : R \to I$ is a nonzero homomorphism. □

Proposition 2. *Let M be a finitely generated quasi-projective essentially iso-retractable R-module such that $Hom_R(M, N) \neq 0$ for all nonzero submodule N of M. Then $End_R(M)$ is a right essentially iso-retractable ring.*

Proof. Let I be an essential right ideal of $End_R(M)$. Then IM is an essential submodule of M by [15, Lemma 5.4]. Since M is essentially iso-retractable, there exists an isomorphism $f \in Hom_R(M, IM) = I$. Clearly $f \in I$ is a right regular element in $End_R(M)$. Since M is quasi-projective, for any $g \in I$ there exists $h \in End_R(M)$ such that $g = fh$. Therefore $I = fEnd_R(M)$ and so $End_R(M)$ is a right essentially iso-retractable ring by Proposition 1. □

Corollary 1. *If R is a right essentially iso-retractable ring, then the matrix ring $M_n(R)$ is so for any $n \geq 1$.*

Proof. Apply Proposition 2 for $M = R^n$. □

Proposition 3. *The following are equivalent for a module M.*

(1) M is essentially iso-retractable.

(2) There is an essential monomorphism $f : M \to M'$ for some essentially iso-retractable module M'.

(3) M is isomorphic to an essentially iso-retractable module.

Example. *1. Every iso-retractable module is essentially iso-retractable, but the converse need not be true. For example, $\mathbb{Z}_p \oplus \mathbb{Z}_q$ as $\mathbb{Z}$-module is essentially iso-retractable but not iso-retractable, where p and q are prime integers.*

2. Let K be a field and $R = K[x, y]$. Then R_R is compressible and so essentially compressible. But, by Proposition 1, R_R is not essentially iso-retractable and so not iso-retractable as $I =< x, y >$ is a nonzero essential right ideal of R which is not principle.

3. Let $R = \mathbb{Z}[x]$. Then R_R is retractable by Lemma 2 and so R_R is essentially retractable. But, by Lemma 1 (1), R_R is not epi-retractable as $I =< 2, x >$ is not a principal right ideal.

4. $\mathbb{Z}_n$ is an essentially iso-retractable $\mathbb{Z}$-module if and only if $n = p_1 p_2 \cdots p_r$, where $r \geq 1$ and $p_1, p_2, \cdots, p_r$ are distinct primes.

5. $\mathbb{Z}_6$ is an essentially iso-retractable (essentially compressible) $\mathbb{Z}$-module but it is not compressible (not iso-retractable).

6. $\mathbb{Z}_4$ is an essentially retractable (epi-retractable) $\mathbb{Z}$-module but it is not essentially compressible and so not essentially iso-retractable (not compressible).

Essentially iso-retractable modules need not be uniform and vice-versa. For example, $\mathbb{Z}_6$ as $\mathbb{Z}$-module is essentially iso-retractable but not uniform and $\mathbb{Z}_4$ as $\mathbb{Z}$-module is uniform but not essentially iso-retractable. We give a characterization of iso-retractable modules in terms of uniform module.

Proposition 4. *An R-module M is iso-retractable if and only if M is essentially iso-retractable and uniform.*

Proof. It follows from [4, Theorem 1.12]. □

The following is an immediate consequence of the above result and a fact that in a prime ring every ideal is essential.

Corollary 2. *A prime ring is essentially iso-retractable if and only if it is iso-retractable.*

In general, we do not know whether a submodule of an essentially iso-retractable module is always essentially iso-retractable. But, quotients of an essentially iso-retractable module need not be essentially iso-retractable. For example, $\mathbb{Z}$ as $\mathbb{Z}$-module is essentially iso-retractable but the homomorphic image $\mathbb{Z}_4$ as $\mathbb{Z}$-module is not essentially iso-retractable. In the following, we give some sufficient conditions.

Proposition 5. *Let N be a submodule of an essentially iso-retractable module M.*

(1) If N is essential, then N is essentially iso-retractable.

(2) If $\phi(N) + \phi^{-1}(N) \subseteq N$ for every injective endomorphism ϕ of M, then M/N is an essentially iso-retractable module.

Proof. (1) It follows by Proposition 3.

(2) Let $L/N \leq_e M/N$. Then $L \leq_e M$ and by assumption there exists an isomorphism $\phi : M \to L$. Since $\phi(N) + \phi^{-1}(N) \subseteq N$, define a mapping $\overline{\phi} : M/N \to L/N$ by $\overline{\phi}(x + N) = \phi(x) + N$. Clearly, $\overline{\phi}$ is a monomorphism. Let $y + N \in L/N$. Then $y \in L$ and so there exists an unique $m \in M$ such that $\phi(m) = y$. Thus $\overline{\phi}$ is surjective. Hence, $\overline{\phi}$ is an isomorphism. □

In general, the direct sum of iso-retractable modules need not be iso-retractable. For example, consider $\mathbb{Z}_p \oplus \mathbb{Z}_q$ as $\mathbb{Z}$-module. Then $\mathbb{Z}_p$ and $\mathbb{Z}_q$ both are iso-retractable but $\mathbb{Z}_p \oplus \mathbb{Z}_q$ as $\mathbb{Z}$-module is not iso-retractable. But in case of essentially iso-retractable modules it holds under some conditions.

Proposition 6. *The direct sum of essentially iso-retractable modules is essentially iso-retractable, provided the direct sum is distributive.*

Proof. Let $M = \oplus_{i \in I} M_i$, where each M_i is an essentially iso-retractable module and I is an index set. Let $K \leq_e M$. Then $K \cap M_i$ is an essential submodule of M_i for each $i \in I$. Therefore, there exists an isomorphism $\phi_i : M_i \to K \cap M_i$. Clearly, the mapping $\phi = \sum \phi_i : M \to K$ is an isomorphism. □

Lemma 3 ([15, Lemma 1.7]). *Let a module $M = \oplus_{i \in I} M_i$ be a direct sum of uniform submodules $M_i, i \in I$, and N be any nonzero submodule of M. Then there exists a subset $I' \subseteq I$ and an essential monomorphism $\theta : N \to \oplus_{i \in I'} M_i$.*

Proposition 7. *Let $M = \oplus_{i\in I} M_i$, where each M_i is iso-retractable and M is distributive. Then every nonzero submodule of M is an essentially iso-retractable module.*

Proof. Let N be a nonzero submodule of M. Because every iso-retractable module is uniform, by Lemma 3, there is an essential monomorphism $f : N \to \oplus_{i\in I'} M_i$ for some subset I' of I. By Proposition 6, $\oplus_{i\in I'} M_i$ is essentially iso-retractable and so N is essentially iso-retractable by Proposition 3. □

2 Some characterizations

Recall [6], an R-module M is called a *co-semisimple module* (or a *V-module*) if every simple module in $\sigma[M_R]$ is M-injective. A ring R is called a right *V-ring* if the right R-module R is co-semisimple. An R-module is called *co-cyclic* provided it contains an essential simple submodule.

Theorem 1. *The following are equivalent for an R-module M.*

(1) M_R is co-semisimple.

(2) In $\sigma[M_R]$ every co-cyclic module is essentially iso-retractable.

(3) In $\sigma[M_R]$ every cyclic co-cyclic module is essentially iso-retractable.

Proof. *(1) $\Longrightarrow$ (2).* Let M be a co-cyclic right R-module in $\sigma[M_R]$. Then M has a simple essential submodule N in $\sigma[M_R]$. Since M_R is co-semisimple, N is M-injective and so N is a direct summand of M. But since N is essential, $N = M$. Thus, M is simple and so M is essentially iso-retractable.

(2) $\Longrightarrow$ (3). Clear.

(3) $\Longrightarrow$ (1). Let S be a simple module in $\sigma[M_R]$ with M-injective hull $\hat{S}$. Since S is simple, there exists $y \in S$ such that $S = yR$. Let $0 \neq x \in \hat{S}$. Then there exists $r \in R$ such that $0 \neq xr \in S$ as $S \leq_e \hat{S}$. Hence there exists $r' \in R$ such that $0 \neq yr' = xr \in xR$. This implies that $yr'R = yR = S \leq xR$ as S is simple. Thus xR is a cyclic co-cyclic module with essential sycle S. By (3), there exists an isomorphism $f : xR \to S$. It follows that xR is simple and so $xR = S$. Therefore $\hat{S} = S$ and S is M-injective. □

Corollary 3. *The following are equivalent for a ring R.*

(1) R is a right V-ring.

(2) Every co-cyclic right R-module is essentially iso-retractable.

(3) Every cyclic co-cyclic right R-module is essentially iso-retractable.

Every semisimple module is essentially iso-retractable but the converse need not be true. For example, $\mathbb{Z}$ as $\mathbb{Z}$-module is essentially iso-retractable but not semisimple. In the following, we provide some sufficient conditions for an essentially iso-retractable modules to be semisimple. Recall [6], an R-module M satisfies *C2-condition* if every submodule isomorphic to a direct summand of M is a direct summand.

Proposition 8. *A module M is semisimple if and only if it is essentially iso-retractable and satisfies C2-condition.*

In general, an essentially iso-retractable module need not be projective. For example, $\mathbb{Z}_6$ as $\mathbb{Z}$-module is essentially iso-retractable but not projective. However in the following, we characterize a semisimple ring and show that if every essentially iso-retractable R-module is projective, then the ring R is semisimple.

Proposition 9. *The following are equivalent for a ring R.*

(1) R is semisimple.

(2) Every iso-retractable right R-module is projective.

(3) Every essentially iso-retractable right R-module is projective.

Recall [11], a module M is called *d-Rickart* (or *dual Rickart*) if $Im(f)$ is a direct summand of M for every $f \in End_R(M)$.

Proposition 10. *Consider the following for an R-module M.*

(1) M is semisimple.

(2) M is essentially iso-retractable.

(3) M is epi-retractable.

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). If every essential submodule of M is projective or M is a d-Rickart module, then (3) $\Longrightarrow$ (1).

Proof. *(1) $\Longrightarrow$ (2).* Clear.

(2) $\Longrightarrow$ (3). Suppose that M is essentially iso-retractable and N be a submodule of M. Then there exists a submodule N' of M maximal with respect to the property that $N \cap N' = 0$ such that $N \oplus N' \leq_e M$. Since M is essentially iso-retractable, there is an isomorphism $f : M \to N \oplus N'$. Let $\pi : N \oplus N' \to N$ be a canonical projection. Then clearly $\pi \circ f : M \to N$ is an epimorphism. Therefore, M is epi-retractable.

(3) $\Longrightarrow$ (1) Suppose that M is epi-retractable and K be an essential submodule of M. Then there exists an epimorphism $h : M \to K$.

Case I. Suppose that every essential submodule of M is projective. Then K is projective and so there exists a homomorphism $f : K \to M$ such that $h \circ f = I_K$. This implies that K is a direct summand of M. Since K is essential in M, it follows that $K = M$. Thus M has no proper essential submodule, and so M is semisimple.

Case II. Suppose that M is a d-Rickart module. Since $\theta : M \to M$ is a nonzero homomorphism such that $Im(\theta) = K$, $Im(\theta) = K$ is a direct summand of M. Thus M is semisimple. □

Recall [16], a module M is said to satisfy $(**)$*-property* if every nonzero endomorphism of M is an epimorphism.

Theorem 2. *The following are equivalent for a module M.*

(1) M is a simple module.

(2) M is an essentially iso-retractable module with the $(**)$-property.

(3) M is an epi-retractable module with the $(**)$-property.

Proof. *(1)* $\Longrightarrow$ *(2)*. Clear.

(2) $\Longrightarrow$ *(3)*. It follows from Proposition 10.

(3) $\Longrightarrow$ *(1)*. It follows from [12, Proposition 3.5]. □

In general, submodules of a projective (respectively, injective) module need not be projective (respectively, injective). For example, $\mathbb{Z}_4$ as $\mathbb{Z}_4$-module is projective but $2\mathbb{Z}_4$ as $\mathbb{Z}_4$-module is not projective and $\mathbb{Q}$ as $\mathbb{Z}$-module is injective but $\mathbb{Z}$ as $\mathbb{Z}$-module is not injective. We show that in case of essentially iso-retractable projective module, the general assertion holds positively.

Proposition 11. *If M is an essentially iso-retractable module, then M is projective (respectively, injective) if and only if every submodule of M is projective (respectively, injective).*

Proof. Let N be a submodule of M. Then there exists a submodule N' of M maximal with respect to the property that $N \cap N' = 0$ such that $N \oplus N' \leq_e M$. Since M is essentially iso-retractable, $M \cong N \oplus N'$. Now M is projective (respectively, injective) implies that $N \oplus N'$ is projective (respectively, injective). Therefore, N is projective (respectively, injective), because N is a direct summand of $N \oplus N'$. □

In general, a projective module need not be essentially iso-retractable. For example, $\mathbb{Z}_4$ as $\mathbb{Z}_4$-module is projective but not essentially iso-retractable. In Proposition 9, we show that if every essentially iso-retractable R-module is projective, then the ring R is semisimple. However, we observe the following result.

Proposition 12. *If every projective right R-module is essentially iso-retractable, then the ring R is right hereditary.*

Proof. It follows from Proposition 11. □

Recall [12], a module M is *quasi-polysimple* if every submodule of M contains a uniform submodule.

Theorem 3. *Let M be an essentially iso-retractable R-module.*

(1) If M is finitely generated torsion free, then M is isomorphic to a free R-module.

(2) If M is nonsingular and quasi-polysimple, then M is isomorphic to the direct sum $\oplus_{i\in I}K_i$ of nonsingular uniform right ideals K_i of R such that K_i does not contain a nonzero nilpotent right ideal of R for each $i \in I$.

Proof. *(1)* Let M be a nonzero essentially iso-retractable R-module. Then by the Zorn's lemma, we can choose an index set I and nonzero elements $m_i \in M$, $i \in I$, such that $\bigoplus_i m_iR \leq_e M$. Since M is torsion free, the map $f_i : R \to m_iR$ defined by $f_i(r) = m_ir, \forall r \in R$, is an isomorphism. Therefore, $R \cong m_iR$ and hence $\oplus_{i\in I}m_iR \cong R^{(I)}$. Since M is essentially iso-retractable, $M \cong \oplus_{i\in I}m_iR \cong R^{(I)}$. Hence, M is a free module.

(2) Assume that M is quasi-polysimple. Let U_i be uniform cyclic submodules of M such that $\sum_{i\in I} U_i$ is direct. It follows by the Zorn's lemma that $\oplus_{i\in I}U_i \leq_e M$. Thus, $M \cong \oplus_{i\in I}U_i$. For a fix $i \in I$, U_i is cyclic so that $U_i = x_iR$ for some $x_i \in U_i$. Since U_i is nonsingular, $r.ann(x_i)$ is non essential right ideal of R. It gives a nonzero right ideal K_i of R such that $K_i \cap r.ann(x_i) = 0$. Clearly, $K_i \cong x_iK_i$ and hence K_i is uniform and nonsingular right ideal of R. Let J_i be right

ideals of R such that $J_i \subseteq K_i$ and $J_i^n = 0$ for some $n \in \mathbb{N}$. By [15, Proposition 1.4 (d)], $ann_R(M)$ is a semiprime ideal and $J_i^n = 0 \in ann_R(M)$ implies that $J_i = 0$. Thus, K_i does not contain any nonzero nilpotent right ideal. Now, $K_i \cong x_i K_i$ which embeds in $x_i R = U_i$. Let $f_i : K_i \to U_i$ be embeddings. Since U_i is uniform, $f_i(K_i) \leq_e U_i$. Therefore, $\oplus_{i \in I} K_i$ embeds in $\oplus_{i \in I} U_i \leq_e M$ and $\oplus_{i \in I} f_i(K_i) \leq_e \oplus_{i \in I} U_i$. This implies that $\oplus_{i \in I} K_i \cong \oplus_{i \in I} f_i(K_i)$ is isomorphic to an essential submodule of M. Due to M as essentially iso-retractable, we have $M \cong \oplus_{i \in I} K_i$. □

Proposition 13. *Let M be an essentially iso-retractable module. Then, M is semisimple if M satisfies any one of the following coditions:*

1) M is finite;

2) M is injective;

3) M is projective.

Proof. *1)* Clear.

2) Let K be an essential submodule of an injective essentially iso-retractable module M. Then $K \cong M$. Therefore, K is injective and by [14, Theorem 2.15], K is a direct summand of M. It follows that $K = M$. Thus M has no proper essential submodule, and so M is semisimple.

3) It follows from Proposition 11 and Proposition 10. □

Corollary 4. *Let $\widehat{M}$ be an essentially iso-retractable module. Then M is semisimple and injective.*

Proof. Let $\widehat{M}$ be essentially iso-retractable. Since M is essential in $\widehat{M}$, $\widehat{M} \cong M$. Therefore, M is injective and essentially iso-retractable by Proposition 3. Hence, by Theorem 13, M is semisimple. □

Proposition 14. *The following are equivalent for a nonzero module M.*

(1) M is not singular and iso-retractable.

(2) M is uniform, nonsingular and essentially iso-retractable.

(3) M is uniform, nonsingular and epi-retractable.

Proof. *(1) $\Longleftrightarrow$ (2)* It is easy to verify that every iso-retractable module is either singular or nonsingular. So, the result follows from Proposition 4.

(2) $\Longrightarrow$ (3) It follows from Proposition 10.

(3) $\Longrightarrow$ (1) Let N be a nonzero submodule of M. Since M is epi-retractable, there is a surjective homomorphism $f : M \to N$. Since M is nonsingular, $ker(f)$ is non essential. But since M is uniform, $ker(f) = 0$. Therefore $f : M \to N$ is an isomorphism. Thus, M is iso-retractable. □

Theorem 4. *The following are equivalent for a ring R.*

(1) R is a principal right ideal domain.

(2) R_R is uniform and there exists a uniform nonsingular essentially iso-retractable R-module.

(3) R_R is uniform and there exists a uniform nonsingular epi-retractable R-module.

Proof. *(1)* $\Longrightarrow$ *(2)* Since R is a principal right ideal domain, R_R is nonsingular and iso-retractable by Lemma 1. Hence, it follows from Proposition 14.

(2) $\Longrightarrow$ *(3)* It follows from Proposition 10.

(3) $\Longrightarrow$ *(1)* It follows from [7, Proposition 2.16]. □

The polynomial ring $R[x]$ of a right essentially compressible ring R is also a right essentially compressible ring (see [15, Proposition 5.1]). One may think an analogous result for a right essentially iso-retractable ring. But, the following example erase this possibility. The ring of integers $\mathbb{Z}$ is an essentially iso-retractable ring by Proposition 1. But $\mathbb{Z}[x]$ is not an essentially iso-retractable ring as $I = < 2, x >$ is an essential ideal which is not principal.

Proposition 15. *Let R be an algebra over an uncountable field F with $dim_F R < |F|$. If $R[x]$ is a right essentially iso-retractable ring then R is so.*

Proof. Let I be an essential right ideal of R and $0 \neq f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$. Since $0 \neq (a_0, a_1, \ldots, a_n) \in R^{n+1}$ and $I^{n+1} \leq_e R_R^{n+1}$, there exists $0 \neq r \in R$ such that $0 \neq (a_0, a_1, \cdots, a_n)r \in I^{n+1}$. This implies that $0 \neq f(x)r \in I[x]$. Hence $I[x]$ is an essential right ideal of $R[x]$. Now by Proposition 1, there exists a right regular element $p(x) \in R[x]$ such that $I[x] = p(x)R[x]$. By [15, Proposition 5.2], there exists $\lambda \in F$ such that $p(\lambda)$ is a right regular element of R. Clearly $p(\lambda) \in I$ and $I = p(\lambda)R$. Hence R is a right essentially iso-retractable ring by Proposition 1. □

Proposition 16. *Let M be an essentially iso-retractable module. Then either M is semisimple or M has an infinite descending chain $M_1 \geq M_2 \geq \ldots$ such that $M_i \cong M$.*

Proof. Let M be essentially iso-retractable. Suppose M is not semisimple. Then M has a proper essential submodule M_1 and $M \cong M_1$. Now, M_1 is not semisimple, it has a proper essential submodule M_2 and $M_1 \cong M_2$. Continuing in this manner, we get a descending chain $M_1 \geq M_2 \geq \ldots$ of submodules of M such that $M_i \cong M$. □

Proposition 17. *Over a Noetherian ring, every essentially iso-retractable module is isomorphic to a direct sum of uniform submodules.*

Proof. Let R be a Noetherian ring and M be an essentially iso-retractable R-module. By [13, Theorem 2.2, Lemma 2.1], M is quasi-polysimple and M contains an essential submodule $\oplus_{i \in I} U_i$, where each U_i is uniform. Since M is essentially iso-retractable, therefore $M \cong \oplus_{i \in I} U_i$. □

Recall [12], a ring R is said to be a *PRI-ring* if every right ideal of R is principal.

Proposition 18. *Every essentially iso-retractable ring is a PRI-ring.*

Proposition 19. *Being essentially iso-retractable is a morita invariant property.*

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А.К. Чатуверді та ін. (2021) називають модуль M суттєво ізо-ретракційним, якщо для кожного його суттєвого підмодуля N існує ізоморфізм $f : M \to N$. Ми характеризуємо суттєво ізо-ретракційні модулі, конапівпрості модулі (V-кільця), області правих головних ідеалів, прості модулі та напівпрості модулі. Ми доводимо, що над нетеровим кільцем кожен суттєво ізо-ретракційний модуль є ізоморфним до прямої суми одноріних підмодулів.

Ключові слова і фрази: ретракційний модуль, ізо-ретракційний модуль, суттєво ізо-ретракційний модуль, суттєво стисливий модуль.