ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2023, 15 (1), 236–245
doi:10.15330/cmp.15.1.236-245

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2023, Т.15, №1, С.236–245

Applications of uniform boundedness principle to matrix transformations

Sarıgöl M.A.

Using the uniform boundedness principle of Maddox, we characterize matrix transformations from the space $(\ell_p)_T$ to the spaces $m(\phi)$ and $n(\phi)$ for the case $1 \leq p \leq \infty$, which correspond to bounded linear operators. Here $(\ell_p)_T$ is the domain of an arbitrary triangle matrix T in the space ℓ_p, and the spaces $m(\phi)$ and $n(\phi)$ are introduced by W.L.C. Sargent. In special cases, we get some well known results of W.L.C. Sargent, M. Stieglitz and H. Tietz, E. Malkowsky and E. Savaş. Also we give other applications including some important new classes.

Key words and phrases: uniform boundedness principle, matrix domain, sequence space, dual space, matrix mapping, linear operator.

Pamukkale University, 20160, Denizli, Türkey
E-mail: msarigol@pau.edu.tr

Introduction

The set w of all real or complex valued sequences is a vector space under point-wise addition and scalar multiplication, and its any subspace is called as *a sequence space*. The sets ℓ_∞, c and ℓ_p, $1 \leq p < \infty$, of all bounded, convergent and absolutely p-summable sequences are well-known sequence spaces, respectively. We write ℓ instead of ℓ_1 for short.

For any $x \in w$, by x_v, $v \in \mathbb{N}$, we denote the coordinates of x, so $x = (x_v)$. The operator $\Delta x : w \to w$ is defined by $\Delta x = (\Delta x_v) = (x_v - x_{v-1})$, $v \geq 1$, where $x \in w$ and $x_0 = 0$. We denote the set of all rearrangements of x by $S(x)$ for any sequence x. Further, let

$$\Phi = \left\{ \phi \in \omega : \ 0 < \phi_v \leq \phi_{v+1} \leq \frac{v+1}{v} \phi_v, \ v \in \mathbb{N} \right\}.$$

For each $s \in \mathbb{N}$, let H_s be the class of all subsets of $\mathbb{N}$ which contain at most s elements. Then, for each $\phi \in \Phi$, the sequence spaces $m(\phi)$ and $n(\phi)$ were introduced and studied by W.L.C. Sargent [13] as

$$m(\phi) = \left\{ x \in \omega : \ \sup_{s \in \mathbb{N}} \sup_{\sigma \in H_s} \left\{ \frac{1}{\phi_s} \sum_{v \in \sigma} |x_v| \right\} < \infty \right\}$$

and

$$n(\phi) = \left\{ x \in \omega : \ \sup_{u \in S(x)} \left\{ \sum_{v=1}^{\infty} \Delta \phi_v |u_v| \right\} < \infty \right\},$$

УДК 517.982.276
2020 *Mathematics Subject Classification:* 40C05, 46A45, 46B45,47B37.

which are BK-spaces, i.e. a Banach sequence space with continuous coordinates, with respect to their natural norms

$$\|x\|_{m(\phi)} = \sup_{s\in\mathbb{N}} \sup_{\sigma\in H_s} \frac{1}{\phi_s} \sum_{v\in\sigma} |x_v| \qquad \text{and} \qquad \|x\|_{n(\phi)} = \sup_{u\in S(x)} \sum_{v=1}^{\infty} \Delta\phi_v |u_v|.$$

Let X, Y be any two sequence spaces and $A = (a_{nv})$ be an infinite matrix of real or complex numbers. Then we define $A(x) = ((A_n(x))$, the A -transform of $x \in X$, as

$$A_n(x) = \sum_{v=1}^{\infty} a_{nv} x_v, \quad n \in \mathbb{N},$$

if the series in the right hand side converges for each $n \in \mathbb{N}$. If $A(x)$ is well defined and belongs to Y for every $x \in X$, then A defines a matrix transformation from X to Y, which is denoted by the same letter $A : X \to Y$. By (X, Y), we mean the class of all infinite matrices A such that $A : X \to Y$.

The domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x \in w : \ A(x) \in X\},$$

which is *a sequence space*. Although in the most cases the new sequence space X_A generated by a matrix A from a sequence space X is the expansion or the contraction of the original space X, it may be observed in some cases that these spaces overlap. In fact, one can easily see that the inclusion $\ell_p \subset (\ell_p)_A$ strictly holds for $1 \leq p < \infty$, if A is given by $a_{n,n} = 1, a_{n,n+1} = -1$, and zero otherwise. Also, the inclusion $(\ell_p)_A \subset \ell_p$ strictly holds if A is given by $a_{n,v} = 1$ for $1 \leq v \leq n$, and zero otherwise.

For any sequence space X, the dual X^β is defined by

$$X^\beta = \left\{ a \in w : \ \sum_{v=1}^{\infty} a_v x_v \text{ converges for all } x \in X \right\}.$$

A matrix $T = (t_{nv})$ is called triangle if $t_{nn} \neq 0$ and $t_{nv} = 0$ for all $k > n$, $n \in \mathbb{N}$.

Throughout the paper, T will denote a triangle matrix which has unique triangle inverse S (see [14]), and also, for an infinite matrix $A = (a_{nv})$, we define the infinite matrices $\widehat{A} = (\widehat{a}_{nv}) = AS$ and $\overline{A} = (\overline{a}_{nv}) = TA$ by

$$\widehat{a}_{nv} = \sum_{r=v}^{\infty} a_{nr} s_{rv}, \quad n, v \in \mathbb{N},$$

and

$$\overline{a}_{nv} = \sum_{r=1}^{n} t_{nr} a_{rv}, \quad n, v \in \mathbb{N}, \tag{1}$$

where $\widehat{A}$ and $\overline{A}$ are the product of the matrices $S = (s_{rv})$ and $T = (t_{nv})$ by the matrix A from the left and right, respectively.

The sequence spaces play important roles in summability theory, a wide field of mathematics, which has several applications in linear algebra, approximations theory, calculus, and essentially in functional analysis. The classical theory deals with the generalization of the concept of convergence for sequences and series. The aim is to assign a limit for diververgent sequences and series by making use of an operator defined by an infinite matrix. The reason why matrices are used for a general linear operator is that a linear operator from a sequence space to another one can be given by an infinite matrix. In recent times, a large literature has grown up, concerned with characterising completely all matrices which transform one given sequence space into another, and also, many sequence spaces and related matrix transformations have been studied by several authors (see, e.g., [1–3, 10]).

In this paper, we characterize the matrix operators from the space $(\ell_p)_T$ to the spaces $m(\phi)$ and $n(\phi)$, for the case $1 \leq p \leq \infty$, using the uniform boundedness principle of Maddox (cf. [4, Theorem 25, p. 67]), and show that each matrix from these classes corresponds to a bounded linear operator. In special cases, some known results of W.L.C. Sargent [13], M. Stieglitz and H. Tietz [14], and E. Malkowsky and E. Savaş [5], and also other applications including new classes are obtained.

We require the following lemmas for the proof of our theorem.

Lemma 1 ([14, pp. 3–4]). *Let $A = (a_{nv})$ be an infinite matrix with complex numbers such that $\lim_n a_{nv}$ exists for $v \geq 1$. Then the following statements hold:*

(i) for $p = 1$, $A \in (\ell, c)$ if and only if $\sup_{n,v} |a_{nv}| < \infty$;

(ii) for $1 < p < \infty$, $A \in (\ell_p, c)$ if and only if $\sup_n \sum_{v=1}^{\infty} |a_{nv}|^{p^*} < \infty$, where p^* is the conjugate of p, i.e. $1/k + 1/p^* = 1$;

(iii) for $p = \infty$, $A \in (\ell_\infty, c)$ if and only if $\sum_{v=1}^{\infty} |a_{nv}|$ converges uniformly in n.

Lemma 2 ([4, p. 67]). *Let X be a second category p-normed space, where $0 < p \leq 1$. Suppose F is a family of lower semicontinuous seminorms q such that $q(x) \leq M(x) < \infty$ for all $x \in X$ and $q \in F$, where $M(x)$ is a number depending on x. Then there exists a constant K, independent of x and q such that*

$$q(x) \leq K\|x\|^{1/p} < \infty$$

for all $x \in X$ and all $q \in F$.

Lemma 3 ([13, Lemma 14]). *Let $1 \leq p \leq \infty$. Then $A \in (\ell_p, m(\phi))$ if and only if $A^t \in (n(\phi), \ell_{p^*})$, where A^t is the transpose of the matrix A.*

Lemma 4 ([8, Lemma 6.11]). *Let X and Y be arbitrary subsets of w and T be a triangle matrix. Then we have $A \in (X, Y_T)$ if and only if $\overline{A} = TA \in (X, Y)$.*

1 Main Results

The set of all bounded (continuous) linear operators from a normed space U to another normed space V is denoted by $\mathcal{B}(U, V)$, and the norm of a bounded linear operator $L \in \mathcal{B}(U, V)$ is defined by

$$\|L\| = \sup\left\{\frac{\|L(x)\|}{\|x\|} : \theta \neq x \in U\right\}.$$

Also, $U' = B(U, \mathbb{C})$ is the set of all bounded linear functionals on U, where $\mathbb{C}$ is the set of all complex numbers. Now, we begin with proving an auxiliary theorem which plays an important role in basic results.

Theorem 1. *Let* $1 \le p \le \infty$, *T be a triangle matrix and S be the inverse of T. If* $f \in (\ell_k)'_T$ *is defined by* $f(x) = \sum_{v=1}^{\infty} a_v x_v$, *then we have*

$$\|f\| = \begin{cases} \|b\|_{\ell_{p^*}}, & 1 < p < \infty, \\ \|b\|_{\ell_\infty}, & p = 1, \\ \|b\|_{\ell}, & p = \infty, \end{cases}$$

where

$$b_v = \sum_{r=v}^{\infty} a_r s_{rv}, \quad v \ge 0.$$

Proof. For $1 \le p < \infty$, it is well known that the space ℓ_p is the BK-space with respect to its natural norm. We note that, since T is a triangle matrix, it is immediate by [15, Theorem 4.3.2] that the space$(\ell_p)_T$ is also a BK-space with respect to the norm

$$\|x\|_{(\ell_p)_T} = \left(\sum_{n=1}^{\infty} |T_n(x)|^p \right)^{1/p},$$

where

$$T_n(x) = \sum_{v=1}^{n} t_{nv} x_v, \quad n = 0, 1, \ldots. \tag{2}$$

Also, if we define $T : (\ell_p)_T \to \ell_p$ by $T(x) = (T_n(x))$ for all $x \in (\ell_p)_T$, then it is easily seen that it is an isometrical isomorphism and so $(\ell_p)_T \cong \ell_p$. Let $S = T^{-1}$ and $y = T(x)$. Now, it can be written from the inversion of (2) that

$$\sum_{v=1}^{m} a_v x_v = \sum_{v=1}^{m} a_v \sum_{r=1}^{v} s_{vr} y_r = \sum_{r=1}^{m} \left(\sum_{v=r}^{m} a_v s_{vr} \right) y_r = \sum_{r=1}^{m} b_{mr} y_r,$$

where

$$b_{mr} = \begin{cases} \sum_{v=r}^{m} a_v s_{vr}, & 1 \le r \le m, \\ 0, & r > m. \end{cases}$$

This gives that the series $\sum_{v=1}^{\infty} a_v x_v$ converges for all $x \in (\ell_k)_T$ if and only if $B \in (\ell_k, c)$. On the other hand, for any matrix $R = (r_{mn}) \in (\ell_k, c)$, the remaining term tends to zero in uniformly in m, because

$$\left| \sum_{v=N}^{\infty} r_{mv} y_v \right| \le \sup_m \left(\sum_{v=1}^{\infty} |r_{mv}|^{p^*} \right)^{1/p^*} \left(\sum_{v=N}^{\infty} |y_v|^p \right)^{1/p} \to 0$$

by Lemma 1. So $\sum_{v=1}^{\infty} r_{mv} y_v$ converges uniformly in m, which gives

$$\lim_m \sum_{v=1}^{\infty} r_{mv} y_v = \sum_{v=1}^{\infty} \lim_m r_{mv} y_v. \tag{3}$$

By applying (3), we have

$$f(x) = \sum_{v=1}^{\infty} a_v x_v = \sum_{v=1}^{\infty} b_v y_v$$

for all $x \in (\ell_p)_T$, or equivalently, for all $y \in \ell_p$.

Thus, by considering

$$\ell_p' \cong \begin{cases} \ell_{p^*}, & 1 < p < \infty, \\ \ell_\infty, & p = 1, \\ \ell, & p = \infty, \end{cases}$$

it follows that

$$\|f\| = \sup_{\|x\|=1} |f(x)| = \sup_{\|y\|=1} |\sum_{v=1}^{\infty} b_v y_v| = \|b\|_{\ell_{p^*}}$$

and

$$\|f\| = \|b\|_{\ell_\infty}, \quad \text{and} \quad \|f\| = \|b\|_{\ell},$$

respectively. □

Now we are ready to give the basic theorems.

Theorem 2. *Let $\phi \in \Phi$ and T be a triangular matrix. Then the following statements hold:*

(i) for $1 < p < \infty$, each matrix $A \in ((\ell_p)_T, m(\phi))$ defines $L_A \in \mathcal{B}((\ell_p)_T, m(\phi))$ such that $L_A(x) = A(x)$ for each $x \in (\ell_p)_T$ and $A \in ((\ell_p)_T, m(\phi))$ if and only if

$$\mu_p(m(\phi)) = \sup_s \sup_{\sigma \in H_s} \sum_{v=1}^{\infty} \left| \sum_{n\in\sigma} \frac{\widehat{a}_{nv}}{\phi_s} \right|^{p^*} < \infty; \tag{4}$$

(ii) for $p = 1$, each matrix $A \in (\ell_T, m(\phi))$ defines $L_A \in \mathcal{B}(\ell_T, m(\phi))$ such that $L_A(x) = A(x)$ and $A \in (\ell_T, m(\phi))$ if and only if

$$\mu_1(m(\phi)) = \sup_s \sup_{\sigma \in H_s} \sup_v \left| \sum_{n\in\sigma} \frac{\widehat{a}_{nv}}{\phi_s} \right| < \infty; \tag{5}$$

(iii) for $p = \infty$, each matrix $A \in ((\ell_\infty)_T, m(\phi))$ defines $L_A \in \mathcal{B}((\ell_\infty)_T, m(\phi))$ such that $L_A(x) = A(x)$ for each $x \in (\ell_\infty)_T$ and $A \in ((\ell_\infty)_T, m(\phi))$ if and only if

$$\mu_\infty(m(\phi)) = \sup_s \sup_{\sigma \in H_s} \sum_{v=1}^{\infty} \left| \sum_{n\in\sigma} \frac{\widehat{a}_{nv}}{\phi_s} \right| < \infty. \tag{6}$$

Proof. The linearity of L_A is clear by the definition of a matrix operator. Further, since $(\ell_p)_T$ and $m(\phi)$ are BK spaces, it follows from [15, Theorem 4.2.8] that L_A is a bounded linear operator.

For the proof of second part, assume $A \in ((\ell_p)_T, m(\phi))$. Then, since $A_n \in (\ell_p)_T'$, for any $s \in \mathbb{N}$ and $\sigma \in H_s$, we get

$$f_{s,\sigma} = \frac{1}{\phi_s} \sum_{n\in\sigma} A_n \in (\ell_p)_T'.$$

If $F = \{f_{s,\sigma} : s \in \mathbb{N}, \sigma \in H_s\}$, then we have

$$|f_{s,\sigma}(x)| \le \frac{1}{\phi_s} \sum_{n\in\sigma} |A_n(x)| \le \|A(x)\|_{m(\phi)} < \infty$$

for each $s \in \mathbb{N}$, $\sigma \in H_s$ and all $x \in F$. Since $(\ell_k)_T$ is a complete metric space, it is of second category, and so it follows from Lemma 2 that there exists a constant M such that

$$\|f_{s,\sigma}\| \le M \tag{7}$$

for all $s \in \mathbb{N}$ and $\sigma \in H_s$. On the other hand, since

$$f_{s,\sigma}(x) = \frac{1}{\phi_s}\sum_{n\in\sigma} A_n(x) = \frac{1}{\phi_s}\sum_{v=1}^{\infty}(\sum_{n\in\sigma} a_{nv})x_v,$$

it follows from Theorem 1 that

$$\|f_{s,\sigma}\| = \frac{1}{\phi_s}\left\{\sum_{v=1}^{\infty}\left|\sum_{n\in\sigma}\left(\sum_{r=v}^{\infty} a_{nr}s_{rv}\right)\right|^{p^*}\right\}^{1/p^*} = \frac{1}{\phi_s}\left\{\sum_{v=1}^{\infty}\left|\sum_{n\in\sigma}\widehat{a}_{nv}\right|^{p^*}\right\}^{1/p^*}, \tag{8}$$

which implies (4) by (7).

Conversely, suppose that (4) holds. Given $x \in (\ell_k)_T$. We should show that $A(x) = (A_n(x)) \in m(\phi)$. Now, if we take $\sigma = \{n\}$, then $A_n(x)$ exists for all $n \in \mathbb{N}$. Let $s \in \mathbb{N}$ and $\sigma \in H_s$. By using the inequality of Peyerimhoff [9], we get, by (4) and (8),

$$\frac{1}{\phi_s}\sum_{n\in\sigma}|A_n(x)| \le 4\max_{\sigma'\subset\sigma}\left(\frac{1}{\phi_s}\left|\sum_{n\in\sigma'} A_n(x)\right|\right) \le 4\sup_s\sup_{\sigma\in H_s}|f_{s,\sigma}(x)| \le 4\{\mu_p(m(\phi))\}^{p^*}\|x\|_{(\ell_p)_T}.$$

This shows that

$$\|A(x)\|_{m(\phi)} \le 4\{\mu_p(m(\phi))\}^{p^*}\|x\|_{(\ell_p)_T}$$

for all $x \in (\ell_p)_T$.

Since (ii) and (iii) are proved as in (i), so they are omitted. □

Let A be an arbitrary infinite matrix. By $S(A)$ and $F(\mathbb{N})$, we denote the set of all matrices B which consist of rearrangements of the rows of A and the set of all finite subsets of $\mathbb{N}$, respectively.

Theorem 3. *Let $\phi \in \Phi$ and T be a triangular matrix. The following statements hold:*

(i) for $1 < p < \infty$, each matrix $A \in ((\ell_p)_T, n(\phi))$ defines $L_A \in \mathcal{B}((\ell_p)_T, n(\phi))$ such that $L_A(x) = A(x)$ for each $x \in (\ell_p)_T$ and $A \in ((\ell_p)_T, n(\phi))$ if and only if

$$\mu_p(n(\phi)) = \sup_{B\in S(A)}\sup_{\sigma\in F(\mathbb{N})}\sum_{v=1}^{\infty}\left|\sum_{n\in\sigma}\widehat{b}_{nv}\Delta\phi_n\right|^{p^*} < \infty; \tag{9}$$

(ii) for $p = 1$, each matrix $A \in (\ell_T, n(\phi))$ defines $L_A \in \mathcal{B}(\ell_T, n(\phi))$ such that $L_A(x) = A(x)$ for each $x \in \ell_T$ and $A \in (\ell_T, n(\phi))$ if and only if

$$\mu_1(n(\phi)) = \sup_{B\in S(A)}\sup_{\sigma\in F(\mathbb{N})}\sup_v\left|\sum_{n\in\sigma}\widehat{b}_{nv}\Delta\phi_n\right| < \infty; \tag{10}$$

(iii) for $p = \infty$, each matrix $A \in ((\ell_\infty)_T, n(\phi))$ defines $L_A \in \mathcal{B}((\ell_\infty)_T, n(\phi))$ such that $L_A(x) = A(x)$ for each $x \in (\ell_\infty)_T$ and $A \in ((\ell_\infty)_T, n(\phi))$ if and only if

$$\mu_\infty(n(\phi)) = \sup_{B\in S(A)}\sup_{\sigma\in F(\mathbb{N})}\sum_{v=1}^{\infty}\left|\sum_{n\in\sigma}\widehat{b}_{nv}\Delta\phi_n\right| < \infty. \tag{11}$$

Proof. Since the proof of first part is similar to the proof of corresponding part of Theorem 2, we omit it. Now let $A \in ((\ell_p)_T, n(\phi))$, $B \in S(A)$ and let $\sigma \subset \mathbb{N}$ be a finite set. As in the proof of Theorem 2, we get

$$f_{B,\sigma} = \sum_{n\in\sigma} B_n \Delta\phi_n \in (\ell_k)'_T.$$

Also, it follows by applying Theorem 1 that there exists a constant M such that

$$\|f_{B,\sigma}\| \leq M$$

for all $B \in S(A)$ and finite $\sigma \subset \mathbb{N}$. So, by considering

$$f_{B,\sigma}(x) = \sum_{n\in\sigma} B_n(x)\Delta\phi_n = \sum_{v=1}^{\infty}(\sum_{n\in\sigma} b_{nv}\Delta\phi_n)x_v,$$

we have

$$\|f_{B,\sigma}\| = \left\{ \sum_{v=1}^{\infty} \left| \sum_{n\in\sigma} \left(\sum_{r=v}^{\infty} b_{nr}s_{rv}\Delta\phi_n \right) \right|^{p^*} \right\}^{1/p^*} = \left\{ \sum_{v=1}^{\infty} \left| \sum_{n\in\sigma} \widehat{b}_{nv}\Delta\phi_n \right|^{p^*} \right\}^{1/p^*}, \tag{12}$$

which implies $\mu_p(n(\phi)) < \infty$.

Conversely, if $\mu_p(n(\phi)) < \infty$, as in the proof of Theorem 2, it is easily seen that $A_n(x)$ exists for all $n \in \mathbb{N}$ and all $x \in (\ell_p)_T$. Now, let $x \in (\ell_p)_T$, $B \in S(A)$, and $n_0 \in \mathbb{N}$ be given. Then, by Peyerimhoff's inequality, Theorem 1, and (12), we obtain

$$\begin{aligned}\sum_{n=1}^{n_0} |B_n(x)|\Delta\phi_n &\leq 4 \max_{\sigma\subset\{1,2,\dots,n_0\}} \left| \sum_{n\in\sigma} B_n(x)\Delta\phi_n \right| \\ &\leq 4 \sup_{B\in S(A)} \sup_{\sigma\in F(\mathbb{N})} \|f_{B,\sigma}\| \|x\|_{(\ell_k)_T} = 4\{\mu_p(n(\phi))\}^{p^*} \|x\|_{(\ell_p)_T},\end{aligned}$$

which gives

$$\|A(x)\|_{n(\phi)} \leq 4\{\mu_p(n(\phi))\}^{p^*} \|x\|_{(\ell_p)_T}.$$

The proofs of (ii) and (iii) are as in the above lines. □

2 Applications

Theorem 2 and Theorem 3 include the characterizations of some well known matrix classes and also other new classes. In this section, we give some of them. In fact, if we take $T = I$, identity matrix, then, since $(\ell_p)_T = \ell_k$ and $\widehat{A} = \overline{A} = A$, we immediately obtain the following results.

Corollary 1. *Let* $1 < p < \infty$ *and* $\phi \in \Phi$. *Then, the following statements hold:*

(i) $A \in (\ell_p, m(\phi))$ *if and only if* (4) *holds with* a_{nv} *instead of* $\widehat{a}_{nv}$;

(ii) $A \in (\ell_p, n(\phi))$ *if and only if* (9) *holds with* b_{nv} *instead of* $\widehat{b}_{nv}$;

(iii) $A \in (\ell, m(\phi))$ *if and only if* (5) *holds with* a_{nv} *instead of* $\widehat{a}_{nv}$;

(iv) $A \in (\ell, n(\phi))$ *if and only if* (10) *holds with* b_{nv} *instead of* $\widehat{b}_{nv}$;

(v) $A \in (\ell_\infty, m(\phi))$ *if and only if* (6) *holds with* a_{nv} *instead of* $\widehat{a}_{nv}$*;*

(vi) $A \in (\ell_\infty, n(\phi))$ *if and only if* (11) *holds with* b_{nv} *instead of* $\widehat{b}_{nv}$.

Note that the parts *(i)*, *(iii)* and *(v)* of this result are given by W.L.C. Sargent [13], which also includes the results of M. Stieglitz and H. Tietz [14] for $m(\phi) = \ell$ and $m(\phi) = \ell_\infty$, where $\phi_n = 1, \phi_n = n, n \geq 1$. Further, since each of the spaces $m(\phi)$ and $n(\phi)$ is the dual of the other (see [13]), it follows from Lemma 3 and Lemma 4 that $A \in (n(\phi), (\ell_p)_T) \Leftrightarrow \overline{A} \in (n(\phi), \ell_p) \Leftrightarrow \overline{A}^t \in (\ell_{p^*}, m(\phi)), 1 \leq p \leq \infty$, where the matrix $\overline{A} = (\overline{a}_{nv})$ is given by (1). So, by Theorem 2 and Corollary 1, we can state the following result giving characterizations of the converse of the matrix class in Theorem 3.

Corollary 2. *Let* $1 \leq p < \infty$ *and* $\phi \in \Phi$. *Further, let* T *be a triangular matrix and the infinite matrix* $\overline{A} = (\overline{a}_{nv})$ *be given by* (1). *Then, the following statements hold:*

(i) $A \in (n(\phi), (\ell_p)_T)$ *if and only if*

$$\sup_s \sup_{\sigma \in H_s} \sum_{n=1}^{\infty} \Big| \sum_{v \in \sigma} \frac{\overline{a}_{nv}}{\phi_s} \Big|^p < \infty;$$

(ii) $A \in (n(\phi), (\ell_\infty)_T)$ *if and only if*

$$\sup_s \sup_{\sigma \in H_s} \sup_n \Big| \sum_{v \in \sigma} \frac{\overline{a}_{nv}}{\phi_s} \Big| < \infty.$$

Applying Theorem 3 to the other known spaces we characterize some new classes as follows. Choose the matrix T as $t_{nv} = s_{n-v} t_v / r_n$ for $0 \leq v \leq n$, and $t_{nv} = 0$ for $v > n$, which is a general matrix and includes numerious matrices, then we have $(\ell_p)_T = \overline{\ell}_p$, $1 \leq p < \infty$, studied by M. Mursaleen and A.K. Noman [7]. Also the inverse S of the matrix T is given by $s_{nv} = (-1)^{n-v} D^{(s)}_{n-v} r_v / t_n$ for $0 \leq v \leq n$, and $s_{nv} = 0$ for $v > n$, where

$$D^{(s)}_{nv} = \frac{1}{s_0^{n+1}} \begin{vmatrix} cs_1 & s_0 & 0 & 0 & \dots & 0 \\ s_2 & s_1 & s_0 & 0 & \dots & 0 \\ s_3 & s_2 & s_1 & s_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots\vdots & \vdots & \\ s_{n-1} & s_{n-2} & s_{n-3} & s_{n-4} & \dots & s_0 \\ s_n & s_{n-1} & s_{n-2} & s_{n-3} & \dots & s_1 \end{vmatrix}.$$

This gives

$$\widehat{a}_{nv} = \sum_{i=v}^{\infty} (-1)^{i-v} \frac{r_v a_{ni} D^{(s)}_{i-v}}{i}, \qquad n, v \in \mathbb{N}. \tag{13}$$

So, we get characterizations of the general classes $(\overline{\ell}_p, m(\phi))$ and $(\overline{\ell}_p, n(\phi))$ for $k \geq 1$.

Corollary 3. *Let* $1 < p < \infty$ *and* $\phi \in \Phi$. *Then the following statements hold:*

(i) $A \in (\overline{\ell}_p, m(\phi))$ *if and only if* (4) *holds with* (13)*;*

(ii) $A \in (\overline{\ell}, m(\phi))$ *if and only if* (5) *holds with* (13)*;*

(iii) $A \in (\overline{\ell}_p, n(\phi))$ *if and only if* (9) *holds with* (13)*;*

(iv) $A \in (\overline{\ell}, n(\phi))$ *if and only if* (10) *holds with* (13).

Also, for $1 \leq p < \infty$, if we choose the matrix T as $t_{nv} = \gamma_n^{1/p^*} r_n R_{v-1}/R_n R_{n-1}$ for $1 \leq v \leq n$, and $t_{nv} = 0$ for $v > n$, then $(\ell_p)_T = |\overline{N}_r^{\gamma}|_k$, investigated by R.N. Mohapatra and M.A Sarıgöl [6] and M.A Sarıgöl [11, 12], where (r_n) and (γ_n) are sequences of positive numbers such that $P_{-1} = 0$ and $R_n = r_0 + r_1 + \cdots + r_n \to \infty$ as $n \to \infty$. Further, it is easily seen that the inverse S of this matrix is stated by $s_{nn} = R_n/(\gamma_n^{1/k^*} r_n)$, $s_{n,n-1} = R_{n-2}/(\gamma_{n-1}^{1/k^*} r_{n-1})$, and zero otherwise, which implies

$$\widehat{a}_{nv} = \sum_{r=v}^{\infty} a_{nr} s_{rv} = \frac{a_{nv} R_v - a_{n,v+1} R_{v-1}}{\gamma_v^{1/k^*} r_v}. \tag{14}$$

Therefore, Theorem 3 are reduced to the following result.

Corollary 4. *Let* $1 < p < \infty$ *and* $\phi \in \Phi$. *Then the following statements hold:*

(i) $A \in (|\overline{N}_r^{\gamma}|_p, m(\phi))$ *if and only if* (4) *holds with* (14)*;*

(ii) $A \in (|\overline{N}_r^{\gamma}|_p, n(\phi))$ *if and only if* (9) *holds with* (14)*;*

(iii) $A \in (|\overline{N}_r|, m(\phi))$ *if and only if* (5) *holds with* (14)*;*

(iv) $A \in (|\overline{N}_r|, n(\phi))$ *if and only if* (10) *holds with* (14).

Finally, we conclude this section with the result due to E. Malkowsky and E. Savaş [5] as follows. Choose T as the matrix of generalized weighted means, i.e. $t_{nr} = u_n v_r$ for $0 \leq r \leq n$, and zero otherwise, where (u_n) and (v_n) are sequences of nonzero numbers. Then $(\ell_p)_T = Z(u, v, \ell_p)$, $1 \leq p < \infty$, and, for $\phi_s = 1$, $\phi_s = s$ $(s \geq 1)$, $m(\phi) = \ell$ and $m(\phi) = \ell_\infty$, respectively. Further, it follows that $s_{rr} = (u_r v_r)^{-1}$, $s_{r,r-1} = (u_{r-1} v_r)^{-1}$, and zero otherwise, which gives

$$\widehat{a}_{nr} = \frac{1}{u_r}\Big(\frac{a_{nr}}{v_r} - \frac{a_{n,r+1}}{v_{r+1}}\Big) \quad \text{for all} \quad r, n \in \mathbb{N}. \tag{15}$$

Thus, by Theorem 2, we have the following result.

Corollary 5. *Let* $1 < p < \infty$ *and the matrix* $\widehat{A}$ *be defined by* (15). *Then we have*

(i) $A \in (Z(u, v, \ell_p), \ell)$ *if and only if*

$$\sup_N \sum_{r=1}^{\infty} \Big| \sum_{n \in N} \widehat{a}_{nr} \Big|^{p^*} < \infty,$$

where the supremum is taken through all finite subsets N *of* $\mathbb{N}$*;*

(ii) $A \in (Z(u, v, \ell_p), \ell_\infty)$ *if and only if*

$$\sup_n \sum_{r=1}^{\infty} |\widehat{a}_{nr}|^{p^*} < \infty.$$

References

[1] Grosseerdmann K.G. *Matrix transformations between the sequence spaces of Maddox*. J. Math. Anal. Appl. 1993, **180** (1), 223–238. doi:10.1006/jmaa.1993.1398

[2] Hazar G.C., Sarıgöl M.A. *Absolute Cesàro series spaces and matrix operators*. Acta Appl. Math. 2018, **154**, 153–165. doi:10.1007/s10440-017-0138-x

[3] Lascarides C.G., Maddox I.J. *Matrix transformations between some classes of sequences*. Math. Proc. Cambridge Philos. Soc. 1970, **68** (1), 99–104. doi:10.1017/S0305004100001109

[4] Maddox I.J. Elements of Functinal Analysis. Cambridge University Press, New York, 1970.

[5] Malkowsky E., Savaş E. *Matrix transformations between sequence spaces of generalized weighted means*. Appl. Math. Comput. 2004, **147** (2), 333–345. doi:10.1016/S0096-3003(02)00670-7

[6] Mohapatra R.N., Sarıgöl M.A. *On matrix operators on the series spaces* $|\overline{N}_p^{\gamma}|_k$. Ukrainian Math. J. 2018, **69** (11), 1772–1783. doi:10.1007/s11253-018-1469-0

[7] Mursaleen M., Noman A.K. *On generalized means and some related sequence spaces*. Comput. Math. Appl. 2011, **61** (4), 988–999. doi:10.1016/j.camwa.2010.12.047

[8] Mursaleen M., Noman A.K. *On some new difference sequence spaces of non-absolute type*. Math. Comput. Model. 2010, **52** (3-4), 603–617. doi: 10.1016/j.mcm.2010.04.006

[9] Peyerimhoff A. *Über ein Lemma von Herrn H.C. Cow*. J. Lond. Math. Soc. (2) 1957, **s1-32** (1), 33–36. doi: 10.1112/jlms/s1-32.1.33

[10] Sarıgöl M.A. *On difference sequence spaces*. J. Karadeniz Tech. Univ. Fac. Art. Sci. Ser. Math. Phys. 1987, **10**, 63–71.

[11] Sarıgöl M.A. *Matrix transformations on fields of absolute weighted mean summability*. Studia Sci. Math. Hungar. 2011, **48** (3), 331–341. doi:10.1556/SScMath.2011.1175

[12] Sarıgöl M.A., Agarwal R.P. *Banach spaces of absolutely k-summable series*. Georgian Math. J. 2021, **28** (6), 945–956. doi:10.1515/gmj-2021-2099.

[13] Sargent W.L.C. *Some sequence spaces related to the* ℓ^p *spaces*. J. Lond. Math. Soc. (2) 1960, **s1-35** (2), 161–171. doi:10.1112/jlms/s1-35.2.161

[14] Stieglitz M., Tietz H. *Matrixtransformationen von Folgenräumen Eine Ergebnisüberischt*. Math. Z. 1977, **154**, 1–16. doi:10.1007/BF01215107

[15] Wilansky A. Summability Through Functional Analysis. Elsevier Science Publisher North-Holland, Amsterdam, 1984.

Received 25.08.2021
Revised 04.02.2022

Сарігол М.А. *Застосування принципу рівномірної обмеженості до матричних перетворень* // Карпатські матем. публ. — 2023. — Т.15, №1. — С. 236–245.

Із використанням принципу рівномірної обмеженості Меддокса охарактеризовано матричні перетворення з простору $(\ell_p)_T$ у простори $m(\phi)$ та $n(\phi)$ у випадку $1 \leq p \leq \infty$, які відповідають обмеженим лінійним операторам. Тут $(\ell_p)_T$ — це область визначення довільної трикутної матриці T у просторі ℓ_p, а простори $m(\phi)$ та $n(\phi)$ введені В.Л.К. Сарджент. У спеціальних випадках отримано деякі добре відомі результати В.Л.К. Сарджент, М. Штігліца і Х. Тітца, Е. Малковського і Е. Саваша. Також нами подано інші застосування, включаючи деякі важливі нові класи.

Ключові слова і фрази: принцип рівномірної обмеженості, матрична область, простір послідовностей, подвійний простір, матричне відображення, лінійний оператор.