ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2025, **17** (1), 128–136
doi:10.15330/cmp.17.1.128-136

https://journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2025, Т.17, №1, С.128–136

# Statistical approximation properties of Lupaş $q$-analogue of $\lambda$-Bernstein operators

**Mursaleen M.[1,2], Naaz A.[2]**

In this paper, we introduce a new type of Lupaş-Bernstein operators with shape parameter $\lambda$ and establish a Korovkin type approximation theorem. We also find the rate of statistical convergence for these operators. Further, we give some graphs and numerical examples for the convergence of new operators and show that in some cases the errors are less than ordinary one.

*Key words and phrases:* Lupaş operator, $\lambda$-Bernstein polynomial, Korovkin theorem, statistical approximation.

[1] Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602105, Tamilnadu, India
[2] Aligarh Muslim University, Aligarh 202002, India
E-mail: mursaleenm@gmail.com (Mursaleen M.), naaz.ambreen11@gmail.com (Naaz A.)

## Introduction

In 1912, S.N. Bernstein [3] proposed the proof of the famous Weierstrass approximation theorem based on probabilistic method. The aim of such generalization is to provide appropriate tools for studying various problems of analysis, geometry, statistical inference and computer science.

In 1987, A. Lupaş [10] introduced a $q$-analogue of famous Bernstein operators and studied their approximating and shape-preserving properties. In 1997, G.M. Phillips [12] introduced another $q$-analogue of Bernstein operators. The $q$-analogue of operators given by A. Lupaş are less known. However, these operators have advantages over the generalization given by G.M. Phillips. The $q$-Bernstein polynomials possess remarkable properties which attracts many authors.

In this paper, we study the Lupaş $q$-analogue of the $\lambda$-Bernstein operators and obtain some approximation and numerical results.

For $q > 0$ and any positive integer $\mathfrak{r}$, define $q$-integers

$$[\mathfrak{r}]_q = \begin{cases} \dfrac{1-q^{\mathfrak{r}}}{1-q}, & q \neq 1, \\ \mathfrak{r}, & q = 1. \end{cases}$$

For the integers $0 \le k \le \mathfrak{r}$, $q$-binomial coefficients are defined by

$$\binom{\mathfrak{r}}{k}_q = \frac{[\mathfrak{r}]_q!}{[k]_q![\mathfrak{r}-k]_q!},$$

where

$$[\mathfrak{r}]_q! = [\mathfrak{r}]_q[\mathfrak{r}-1]_q \dots [1]_q.$$

УДК 517.98
2020 *Mathematics Subject Classification:* 41A10, 41A17, 40A30.

**Definition 1** ([10]). *Let $F \in \mathfrak{C}[0,1]$ (set of all continuous functions defined on interval $[0,1]$) and $0 < q \leq 1$. The linear operator $\mathcal{R}_m : \mathfrak{C}[0,1] \to \mathfrak{C}[0,1]$, defined by*

$$(\mathcal{R}_m F)(x) = \sum_{i=0}^{m} F\left(\frac{[i]_q}{[m]_q}\right) r_{m,i}(x),$$

*is known as the Lupaş $q$-analogue of Bernstein operators, where Lupaş basis is*

$$r_{m,i}(x) = \binom{m}{i}_q \frac{q^{\frac{i(i-1)}{2}} x^i (1-x)^{m-i}}{\prod_{j=1}^{m-1}(1-x+q^j x)}. \tag{1}$$

In 2010, Z. Ye et al. [13] published a paper in which they defined the new Bernstein basis with shape parameter $\lambda$. They discuss some properties of curve modeling with the help of new basis. Recently, Q.B. Cai et al. [4] introduced the $\lambda$-Bernstein operators. For more recent works on $\lambda$-Bernstein basis we refer the reader to [1,2,5,8,11].

In Section 1, we construct and calculate moments for Lupaş-Bernstein operators using $\lambda$-Bernstein basis (see [13]) and show that our operators possess endpoint interpolation property with the help of graph. In Section 2, Korovkin type statistical approximation and rate of statistical convergence are obtained. Further we give an example with graph showing that not every statistically convergent sequence is convergent. In the last section we give some graphical and numerical results supporting our approximation results.

## 1 Construction of operators

In this section, we define the $\lambda$-Lupaş-Bernstein operators and calculate moments.

For $F \in \mathfrak{C}[0,1]$, $\lambda$-Lupaş-Bernstein operators are defined as

$$\mathcal{L}^{m,\lambda}(F;x) = \sum_{i=0}^{m} F\left(\frac{[i]_q}{[m]_q}\right) \bar{l}_{m,i}(\lambda;x), \tag{2}$$

where the new type of basis are

$$\begin{cases} \bar{l}_{m,0}(\lambda;x) = r_{m,0}(x) - \dfrac{\lambda}{[m]_q+1} r_{m+1,1}(x), \\ \bar{l}_{m,k}(\lambda;x) = r_{m,k}(x) - \lambda\left(\frac{[m]_q-2[k]_q+1}{[m]_q^2-1} r_{m+1,k}(x) - \frac{[m]_q-2[k]_q-1}{[m]_q^2-1} r_{m+1,k+1}(x)\right), \\ \bar{l}_{m,m}(\lambda;x) = r_{m,m}(x) - \dfrac{\lambda}{[m]_q+1} r_{m+1,m}(x), \end{cases}$$

with $x \in [0,1]$, $\lambda \in [-1,1]$, $0 < q < 1$ and $0 < k < m$.

**Lemma 1.** *Let $e_\mu(x) = x^\mu$, $\mu = 0,1,2$. We have the following equalities*

$$\mathcal{L}^{m,\lambda}(1;x) = 1, \tag{3}$$

$$\begin{aligned} \mathcal{L}^{m,\lambda}(e_1;x) = x + \frac{\lambda}{[m]_q([m]_q^2-1)}\Big[ & -2[m+1]_q x + [m]_q + 1 \\ & + \frac{q^{\frac{m(m-1)}{2}} x^{m+1}}{\prod_{j=1}^{m-1}(1-x+q^j x)}\left(2[m+1]_q - \frac{([m]_q+1)q^m}{(1-x+q^m x)}\right)\Big], \end{aligned} \tag{4}$$

$$
\begin{aligned}
\mathcal{L}^{m,\lambda}\left(e_2;x\right) = x^2 &+ \frac{x(1-x)}{[m]_q} - \frac{x^2(1-x)(1-q)}{1-x+qx}\left(1-\frac{1}{[m]_q}\right) \\
&+ \frac{\lambda x}{[m]_q^2\left([m]_q^2-1\right)}\left[2[m+1]_q(1-2qx) + \frac{x^{m+1}}{\prod_{j=0}^{m-2}\left(1-x+q^jx\right)}\right. \\
&\left.\times\left(4[m+1]_q q^{\left(m^2-3m+3\right)} - \frac{2[m+1]_q\left([m]_q+2\right)q^{\frac{m(m-1)}{2}}}{(1-x+q^{m-1}x)+\frac{([m]_q+1)q^{m(m+1)/2}}{(1-x+q^{m-1}x)(1-x+q^mx)}}\right)\right].
\end{aligned}
\tag{5}
$$

*Proof.* From (1), we see $\sum_{i=0}^{m}\bar{l}_{m,i}(\lambda,x)=1$, so we can obtain (3). Next

$$
\begin{aligned}
\mathcal{L}^{m,\lambda}\left(e_1;x\right) &= \sum_{i=0}^{m}\frac{[i]_q}{[m]_q}\bar{l}_{m,i}(\lambda;x) = \sum_{i=0}^{m}\frac{[i]_q}{[m]_q}r_{m,i}(x) \\
&+\lambda\left[\sum_{i=0}^{m}\frac{[i]_q}{[m]_q}\frac{[m]_q-2[i]_q+1}{[m]_q^2-1}r_{m+1,i}(x) - \sum_{i=1}^{m-1}\frac{[i]_q}{[m]_q}\frac{[m]_q-2[i]_q-1}{[m]_q^2-1}r_{m+1,i+1}(x)\right],
\end{aligned}
$$

as the Lupaş-Bernstein operators defined in (2) preserve linear functions, that is we have $\mathcal{L}^m\left(ae_1+be_0;x\right)=ax+b$. We denote the latter parts above by $\triangle_1\ (m;x)$ and $\triangle_2\ (m;x)$, then we get

$$
\mathcal{L}^{m,\lambda}\left(e_1;x\right) = x+\lambda\Big(\triangle_1\ (m;x)+\triangle_2\ (m;x)\Big). \tag{6}
$$

Now we compute $\triangle_1\ (m;x)$ and $\triangle_2\ (m;x)$. We obtain

$$
\begin{aligned}
\triangle_1\ (m;x) &= \frac{[m+1]_q x}{[m]_q\left([m]_q-1\right)}\sum_{i=0}^{m-1}r_{m,i}(x) - \frac{2[m+1]_q x^2 q}{\left([m]_q^2-1\right)}\sum_{i=0}^{m-2}r_{m-1,i}(x) \\
&\quad - \frac{2[m+1]_q x}{[m]_q\left([m]_q^2-1\right)}\sum_{i=0}^{m-1}r_{m,i}(x).
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\triangle_2\ (m;x) &= \frac{[m+1]_q x}{[m]_q\left([m]_q+1\right)}\sum_{i=1}^{m-1}r_{m,i}(x) - \frac{1}{[m]_q\left([m]_q+1\right)}\sum_{i=1}^{m-1}r_{m+1,i+1}(x) \\
&+ \frac{2[m+1]_q x^2 q}{\left([m]_q^2-1\right)}\sum_{i=0}^{m-2}r_{m-1,i}(x) + \frac{2}{[m]_q\left([m]_q^2-1\right)}\sum_{i=1}^{m-1}r_{m+1,i+1}(x) \\
&- \frac{2[m+1]_q x}{[m]_q\left([m]_q^2-1\right)}\sum_{i=1}^{m-1}r_{m,i}(x).
\end{aligned}
\tag{8}
$$

Using (6), (7) and (8) we get (4).

We have

$$
\begin{aligned}
\mathcal{L}^{m,\lambda}\left(e_2;x\right) &= \sum_{i=0}^{m}\frac{[i]_q^2}{[m]_q^2}r_{m,i}(x) \\
&+\lambda\left[\sum_{i=0}^{m}\frac{[i]_q^2}{[m]_q^2}\frac{[m]_q-2[i]_q+1}{[m]_q^2-1}r_{m+1,i}(x) - \sum_{i=1}^{m-1}\frac{[i]_q^2}{[m]_q^2}\frac{[m]_q-2[i]_q-1}{[m]_q^2-1}r_{m+1,i+1}(x)\right].
\end{aligned}
$$

Let us denote the latter parts above in the square brackets of last formula by $\triangle_3\ (m;x)$ and $\triangle_4\ (m;x)$, we get

$$
\mathcal{L}^{m,\lambda}\left(e_2;x\right) = x^2+\frac{x(1-x)}{[m]_q} - \frac{x^2(1-x)(1-q)}{1-x+qx}\left(1-\frac{1}{[m]_q}\right)+\lambda\Big(\triangle_3\ (m;x)+\triangle_4\ (m;x)\Big). \tag{9}
$$

We calculate $\triangle_3\ (m;x)$ and $\triangle_4\ (m;x)$ and obtain

$$\begin{aligned}\triangle_3\ (m;x) = & \frac{[m+1]_q x}{[m]_q^2([m]_q-1)}\sum_{i=0}^{m-1} r_{m,i}(x) + \frac{[m+1]_q x^2 q}{[m]_q([m]_q-1)}\sum_{i=0}^{m-2} r_{m-1,i}(x) \\ & - \frac{2[m+1]_q x}{[m]_q^2([m]_q^2-1)}\sum_{i=0}^{m-1} r_{m,i}(x) - \frac{6[m+1]_q x^2 q}{[m]_q([m]_q^2-1)}\sum_{i=0}^{m-2} r_{m-1,i}(x) \\ & - \frac{2[m+1]_q[m-1]_q x^3 q^3}{[m]_q([m]_q^2-1)}\sum_{i=0}^{m-3} r_{m-2,i}(x),\end{aligned} \tag{10}$$

$$\begin{aligned}\triangle_4\ (m;x) = & -\frac{[m+1]_q x^2 q}{([m]_q+1)[m]_q}\sum_{i=0}^{m-2} r_{m-1,i}(x) + \frac{[m+1]_q x}{([m]_q+1)[m]^2}\sum_{i=1}^{m-1} r_{m,i}(x) \\ & - \frac{1}{([m]_q+1)[m]^2}\sum_{i=1}^{m-1} r_{m+1,i+1}(x) + \frac{2[m+1]_q[m-1]_q x^3 q^3}{([m]_q^2-1)[m]_q}\sum_{i=1}^{m-3} r_{m-2,i}(x) \\ & + \frac{2[m+1]_q x}{([m]_q^2-1)[m]_q^2}\sum_{i=1}^{m-1} r_{m,i}(x) - \frac{2}{([m]_q^2-1)[m]_q^2}\sum_{i=1}^{m-1} r_{m+1,i+1}(x).\end{aligned} \tag{11}$$

Using (9), (10) and (11), we get (5). Thus Lemma 1 is proved. □

**Remark 1.** *For $\lambda \in [-1,1]$, $x \in [0,1]$, $q > 0$, $\lambda$-Lupaş-Bernstein operators possess the endpoint interpolation property, that is*

$$\mathcal{L}^{m,\lambda}(F;0) = 0, \qquad \mathcal{L}^{m,\lambda}(F;1) = 1.$$

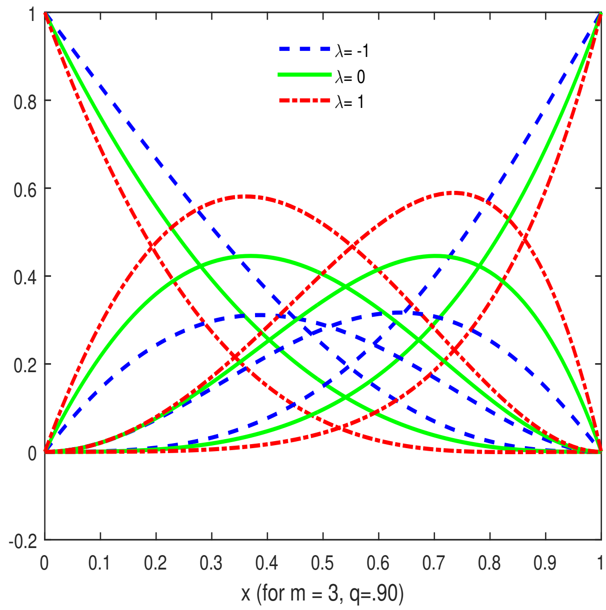



**Figure 1.** The graph of $\bar{l}_{3i}(\lambda;x)$ showing interpolation

## 2 Korovkin type statistical approximation

Here we study statistical approximation properties of operators $\mathcal{L}^{m,\lambda}$.

Around fifty years ago, H. Fast [6] gave an idea of statistical convergence and it has become an active research area. In [7], A.D. Gadjiev and C. Orhan gave Korovkin theorem (see [9]) by replacing ordinary convergence by statistical convergence.

Let $\mathcal{K} \subseteq \mathbb{N}$ (the set of natural numbers). Then $\delta(\mathcal{K}) = \lim_r \frac{1}{r}\complement\{k \leq r : k \in \mathcal{K}\}$ is called the asymptotic density of $\mathcal{K}$, where $\complement$ denotes the cardinality of the set. A sequence $\eta = \{\eta_k\}$ said to be statistically convergent to the number $s$ if $\delta(\mathcal{K}_\varepsilon) = 0$ for each $\varepsilon > 0$, where $\mathcal{K}_\varepsilon = \{k \leq r : |\eta_k - s| > \varepsilon\}$ (see [6]), i.e.

$$\lim_r \frac{1}{r}\complement\{k \leq r : |\eta_k - s| \geq \varepsilon\} = 0.$$

In this case, we write $st\text{-}\lim \eta = s$.

Here we give an example in which every convergent sequence is statistically convergent but not conversely.

Let the sequence $\{S_k\}$ be defined by

$$S_k = \begin{cases} 0, & k = m^2, \\ 1, & k \neq m^2,\ m \in \mathbb{N}. \end{cases}$$

This sequence is statistically convergent to 1 but not convergent.

Our aim is to study statistical approximation properties of the operators (2) and to obtain the rate of statistical convergence.

**Remark 2.** *For $q \in (0,1)$, note that $\lim_{m\to\infty}[m]_q = 0$ or $\frac{1}{q}$. We take a sequence $q_m \in (0,1)$ such that $\lim_{m\to\infty} q_m^m = 1$, so that $\lim_{m\to\infty}[m]_{q_m} = \infty$.*

We use the following famous Korovkin's theorem.

**Theorem 1** ([9]). *Let $\{T_m\}_{m\geq 1}$ be a sequence of positive linear operators from $\mathfrak{C}[0,1]$ into $\mathfrak{C}[0,1]$ such that $\lim_{m\to\infty} T_m(g) = g$ uniformly on $[0,1]$. Then, for every $\phi \in \mathfrak{C}[0,1]$, $\lim_{m\to\infty} T_m(\phi) = \phi$ uniformly on $[0,1]$, where $g = x^i$, $i = 0,1,2$.*

**Theorem 2** ([7]). *Let $\{\mathcal{L}_m\}_{m\geq 1}$ be a sequence of linear positive operators from $\mathfrak{C}[0,1]$ into $\mathfrak{C}[0,1]$ satisfying: $st\text{-}\lim_m \left\|\mathcal{L}^{m,\lambda}(e_\mu; x) - x^\mu\right\|_\infty = 0$, where $e_\mu(x) = x^\mu$, $\mu = 0,1,2$. Then for any function $F \in \mathfrak{C}[0,1]$, we have $st\text{-}\lim_m \left\|\mathcal{L}^{m,\lambda}(F) - F\right\|_\infty = 0$.*

**Theorem 3.** *Let $\left\{\mathcal{L}^{m,\lambda}\right\}_{m\geq 1}$ be a sequence of operators defined in (2) and the sequence $q = q_m$ be same as in Remark 2. Then for $t \in \mathfrak{C}[0,1]$*

$$st\text{-}\lim_m \left\| \mathcal{L}^{m,\lambda}(F,.) - F \right\|_\infty = 0.$$

*Proof.* Clearly for $\mu = 0$, the equality $\mathcal{L}^{m,\lambda}(1,x) = 1$, implies

$$st\text{-}\lim_m \left\|\mathcal{L}^{m,\lambda}(1;x) - 1\right\|_\infty = 0.$$

For $\mu = 1$, we have

$$\left|\mathcal{L}^{m,\lambda}(e_1;x) - x\right| \leq \left|\lambda\frac{-2[m+1]_q x + [m]_q + 1}{[m]_q\left([m]_q^2 - 1\right)}\right| + \left|\lambda\frac{q^{\frac{m(m-1)}{2}} x^{m+1}}{[m]_q\left([m]_q^2-1\right)\prod_{j=0}^{m-1}\left(1-x+q^j x\right)}\left(2[m+1]_q - \frac{\left([m]_q+1\right)q^m}{(1-x+q^m x)}\right)\right|.$$

For a given $\epsilon > 0$, let us define the following sets

$$L = \left\{m : \left|\mathcal{L}^{m,\lambda}(e_1;x) - x\right| \geq \epsilon\right\}, \qquad L' = \left\{m : \left|\lambda\frac{-2[m+1]_q x + [m]_q + 1}{[m]_q\left([m]_q^2-1\right)}\right| \geq \epsilon\right\},$$

$$L'' = \left\{m : \left|\lambda\frac{q^{m(m-1)/2} x^{m+1}}{[m]_q\left([m]_q^2-1\right)\prod_{j=0}^{m-1}\left(1-x+q^j x\right)}\left(2[m+1]_q - \frac{\left([m]_q+1\right)q^m}{(1-x+q^m x)}\right)\right| \geq \epsilon\right\}.$$

So we get

$$st\text{-}\lim_n \left\|\mathcal{L}^{m,\lambda}(e_1;x) - x\right\|_\infty = 0.$$

Lastly for $\mu = 2$, we have

$$\begin{aligned}\left|\mathcal{L}^{m,\lambda}(e_2;x) - x^2\right| \leq{}& \left|\frac{x(1-x)}{[m]_q} - \frac{x^2(1-x)(1-q)}{1-x+qx}\left(1-\frac{1}{[m]_q}\right)\right| \\ &+ \left|\frac{4[m+1]_q x}{[m]_q^2\left([m]_q^2-1\right)}(1-qx)\right| + \left|\frac{2[m+1]_q x(1-2qx)}{[m]_q^2([m]-1)}\right| \\ &+ \left|\frac{x^{m+1}}{[m]_q^2\left([m]_q^2-1\right)\prod_{j=0}^{m-2}\left(1-x+q^j x\right)}\right. \\ &\times\left(4[m+1]_q q^{\frac{m^2-3m+3}{2}} - \frac{2[n+1]_q\left([m]_q+2\right)q^{m(m+1)/2}}{(1-x+q^{m-1}x)}\right. \\ &\left.\left.+\frac{\left([m]_q+1\right)q^{m(m+1)/2}}{(1-x+q^{m-1}x)\,(1-x+q^m x)}\right)\right|.\end{aligned}$$

If we choose

$$\alpha_{1m} = \left|\frac{x(1-x)}{[m]_q} - \frac{x^2(1-x)(1-q)}{1-x+qx}\left(1-\frac{1}{[m]_q}\right)\right|,$$

$$\alpha_{2m} = \left|\frac{4[m+1]_q x}{[m]_q^2\left([m]_q^2-1\right)}(1-qx)\right|, \qquad \alpha_{3m} = \left|\frac{2[m+1]_q x(1-2qx)}{[m]_q^2([m]-1)}\right|,$$

$$\begin{aligned}\alpha_{4m} = {}&\left|\frac{x^{m+1}}{[m]_q^2\left([m]_q^2-1\right)\prod_{j=0}^{m-2}\left(1-x+q^j x\right)}\left(4[m+1]_q q^{\frac{m^2-3m+3}{2}} - \frac{2[n+1]_q\left([m]_q+2\right)q^{m(m+1)/2}}{(1-x+q^{m-1}x)}\right.\right. \\ &\left.\left.+\frac{\left([m]_q+1\right)q^{m(m+1)/2}}{(1-x+q^{m-1}x)(1-x+q^m x)}\right)\right|,\end{aligned}$$

then $st\text{-}\lim_m \alpha_{1m} = st\text{-}\lim_m \alpha_{2m} = st\text{-}\lim_m \alpha_{3m} = st\text{-}\lim_m \alpha_{4m} = 0$.

Now given $\epsilon > 0$, we define the following five sets:

$$L = \left\{ m : \left| \mathcal{L}^{m,\lambda}(e_2; x) - x^2 \right| \geq \epsilon \right\},$$

$$L_1 = \left\{ m : \alpha_{1m} \geq \frac{\epsilon}{4} \right\}, \qquad L_2 = \left\{ m : \alpha_{2m} \geq \frac{\epsilon}{4} \right\},$$

$$L_3 = \left\{ m : \alpha_{3m} \geq \frac{\epsilon}{4} \right\}, \qquad L_4 = \left\{ m : \alpha_{4m} \geq \frac{\epsilon}{4} \right\}.$$

It is obvious that $L \subseteq L_1 \bigcup L_2 \bigcup L_3 \bigcup L_4$. Thus we obtain

$$\delta\Big(\{K \leq m : \|\mathcal{L}^{m,\lambda}(e_2; x) - x^2\|_\infty \geq \epsilon\}\Big) \leq \delta\Big(\{K \leq m : \alpha_{1m} \geq \frac{\epsilon}{4}\}\Big) + \delta\Big(\{K \leq m : \alpha_{2m} \geq \frac{\epsilon}{4}\}\Big) + \delta\Big(\{K \leq m : \alpha_{3m} \geq \frac{\epsilon}{4}\}\Big) + \delta\Big(\{K \leq m : \alpha_{4m} \geq \frac{\epsilon}{4}\}\Big).$$

The right hand side of the inequalities is zero. Thus the equality $st\text{-}\lim_m \|\mathcal{L}^{m,\lambda}(e_2; x) - x^2\|_\infty = 0$ holds. The proof is completed. $\square$

## 3 Rate of statistical convergence

In this part, we obtain the rate of statistical convergence of operators (2) by means of modulus of continuity. The modulus of continuity for functions $t \in \mathfrak{C}[0,1]$ is defined by

$$\omega(F; \varrho) = \sup_{y_1, y_2 \in [0,1],\ |y_1 - y_2| < \varrho} \Big| F(y_1) - F(y_2) \Big|,$$

where $\omega(F; \varrho)$ satisfies the following conditions:

$$\lim_{\varrho \to 0} \omega(F; \varrho) = 0$$

for all $F \in \mathfrak{C}[0,1]$ and

$$\Big| F(y_1) - F(y_2) \Big| \leq \omega(F; \varrho) \left( \frac{|y_1 - y_2|}{\varrho} + 1 \right).$$

The next theorem gives the pointwise rate of statistical convergence.

**Theorem 4.** *Let the sequence $q = q_m$ with $0 < q_m \leq 1$ be as defined in Remark 2. Then we have*

$$\Big| \mathcal{L}^{m,\lambda}(e_1; x) - F(x) \Big| \leq 2\omega\left( F; \sqrt{\varrho_m(x)} \right),$$

*where*

$$\begin{aligned}
\varrho_m(x) = {} & \frac{x(1-x)}{[m]_q} - \frac{x^2(1-x)(1-q)}{1-x+qx} \left( 1 - \frac{1}{[m]_q} \right) \\
& + \frac{\lambda x}{[m]_q \left( [m]_q^2 - 1 \right)} \Bigg[ -2\left( -2[m+1]_q x + [m]_q + 1 \right) \\
& - \frac{2q^{\frac{m(m-1)}{2}} x^{m+1}}{\prod_{j=0}^{m-1} (1 - x + q^j x)} \left( 2[m+1]_q - \frac{\left([m]_q + 1\right) q^m}{(1 - x + q^m x)} \right) \\
& + \frac{1}{[m]_q} \Bigg[ 2[m+1]_q (1 - 2qx) + \frac{x^{m+1}}{\prod_{j=0}^{m-2} (1 - x + q^j x)} \\
& \times \left( 4[m+1]_q q^{(m^2 - 3m + 3)} - \frac{2[m+1]_q \left( [m]_q + 2 \right) q^{\frac{m(m-1)}{2}}}{(1 - x + q^{m-1}x) + \frac{\left([m]_q + 1\right) q^{\frac{m(m+1)}{2}}}{(1 - x + q^{m-1}x)(1 - x + q^m x)}} \right) \Bigg] \Bigg].
\end{aligned} \tag{12}$$

*Proof.* Let $F \in \mathfrak{C}[0,1]$ be a non decreasing function. Using linearity and positivity of the operators $\mathcal{L}^{m,\lambda}$, for $\varrho > 0$ we get

$$\left|\mathcal{L}^{m,\lambda}\left(e_1;x\right) - F\left(x\right)\right| \leq \mathcal{L}^{m,\lambda}\Big(\left|F\left(y\right) - F\left(x\right)\right|;x\Big),$$

$$\left|\mathcal{L}^{m,\lambda}\left(e_1;x\right) - F\left(x\right)\right| \leq \omega(F;\varrho)\left\{\mathcal{L}^{m,\lambda}(1;x) + \frac{1}{\varrho}\mathcal{L}^{m,\lambda}\Big(\left|y-x\right|;x\Big)\right\}.$$

By using Cauchy-Schwarz inequality, we obtain

$$\left|\mathcal{L}^{m,\lambda}\left(e_1;x\right) - F\left(x\right)\right| \leq \omega\left(F;\varrho_m\right)\left(1 + \frac{1}{\varrho_m}\left[\mathcal{L}^{m,\lambda}\left(\left(e_1-x\right)^2;x\right)\right]^{\frac{1}{2}}\ \left[\mathcal{L}^{m,\lambda}(1;x)\right]^{\frac{1}{2}}\right).$$

Choosing $\varrho_m$ as in (12), the theorem is proved. □

By the condition $st\text{-}\lim_m \varrho_m = 0$, we get

$$st\text{-}\lim_m \omega(F;\varrho) = 0.$$

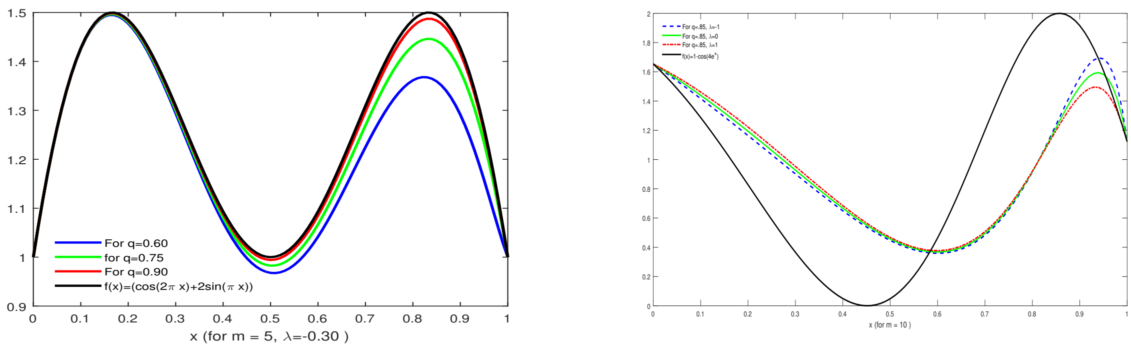
This gives the pointwise rate of statistical convergence.

## 4 Graphical examples

In this section, we present our results graphically.

**Example 1.** *Let $\lambda = -0.3$ for the operator $\mathcal{L}^{m,\lambda}(F;x)$. Figure 1 (a) explains that our operators show better approximation towards the function $F(x) = cos(2\pi x) + 2sin(\pi x)$ as $q \to 1$. Also, it shows for less iteration $m = 5$ the operators give better error estimation.*

**Example 2.** *Let $F(x) = 1 - cos\left(4e^x\right)$ and $m = 10$. In Figure 1 (b), the graph of function $F(x)$ and operator $\mathcal{L}^{m,\lambda}(F;x)$ for $\lambda = -1, 0, 1$ are showed. One can observe here the approximation is better for $\lambda = -1$.*




(a) for different values of $q$ (b) for different values of $\lambda$

**Figure 2.** Approximation by $\mathcal{L}^{m,\lambda}(F;x)$ operators to the function $F(x)$

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*Received 11.09.2021*
*Revised 25.02.2024*


Мурсалін М., Нааз А. *Статистичні апроксимаційні властивості $q$-аналогів операторів Лупаша-Бернштейна з параметром $\lambda$* // Карпатські матем. публ. — 2025. — Т.17, №1. — С. 128–136.

У цій статті запропоновано новий тип операторів Лупаша-Бернштейна з параметром форми $\lambda$ та доведено теорему наближення типу Коровкіна. Досліджено також швидкість статистичної збіжності цих операторів. Крім того, наведено графіки та числові приклади, що ілюструють збіжність нових операторів, і показано, що в окремих випадках похибки є меншими, ніж у звичайних операторів.

*Ключові слова і фрази:* оператор Лупаша, поліном Бернштейна з параметром $\lambda$, теорема Коровкіна, статистична апроксимація.