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More on the extension of linear operators on Riesz spaces

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The classical Kantorovich theorem asserts the existence and uniqueness of a linear extension of a positive additive mapping, defined on the positive cone E^+ of a Riesz space E taking values in an Archimedean Riesz space F, to the entire space E. We prove that, if E has the principal projection property and F is Dedekind σ-complete then for every $e \in E^+$ every positive finitely additive F-valued measure defined on the Boolean algebra $\mathfrak{F}_e$ of fragments of e has a unique positive linear extension to the ideal E_e of E generated by e. If, moreover, the measure is τ-continuous then the linear extension is order continuous.

Key words and phrases: positive operator, linear extension, Riesz space, vector lattice.

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Introduction

We use the standard terminology and notation as in [3]. Our special terminology (as well as the main technical tools) concerns the lateral order on Riesz spaces. The *lateral order* $\sqsubseteq$ on a Riesz space E is defined by setting $x \sqsubseteq y$ $(x, y \in E)$ if and only if x is a *fragment*[1] of y, that is, $x \perp (y - x)$ (see [6] for a detailed study of the lateral order). Given elements $x, y, z \in E$, the notation $x = y \sqcup z$ means that $x = y + z$ and $y \perp z$.

The classical extension problem for positive linear operators deals with the extension of an operator from a linear subspace or the positive cone of a Riesz space E, see e.g. [2], [3, Section 1.2], [4]. Here we consider the problem of extension of a positive linear operator from nonlinear sets. More precisely, we consider extension of a linear operator $\nu \colon \mathfrak{F}_e \to F^+$ defined on the set $\mathfrak{F}_e$ of all fragments of an element $e \in E^+$ and taking values in a Dedekind σ-complete Riesz space F. An obvious necessary condition for the function ν to have a linear extension from $\mathfrak{F}_e$ to the ideal E_e of E generated by e is that ν is a finitely additive F-valued measure, that is, for every $x, y \in \mathfrak{F}_e$ with $x \perp y$ one has $\nu(x + y) = \nu(x) + \nu(y)$, because for any $x, y \in \mathfrak{F}_e$ the condition $x + y \in \mathfrak{F}_e$ holds if and only if $x \perp y$. We prove, in particular, that the above necessary condition is sufficient.

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[1] *component* in the terminology of [3]

We consider the order convergence in the strong sense. A net $(x_\alpha)_{\alpha\in A}$ in a Riesz space E *order converges* to a limit $x \in E$ if there is a net $(y_\alpha)_{\alpha\in A}$ in E such that $y_\alpha \downarrow 0$ and $|x_\alpha - x| \le y_\alpha$ for some $\alpha_0 \in A$ and all $\alpha \ge \alpha_0$ (write $x_\alpha \xrightarrow{o} x$). See e.g. [1] and [8] for more details on the order convergence.

For the proof of the main result, we need some lemmas. The first one is an analogue of the Riesz decomposition property for the lateral order recently obtained by M. Pliev.

Lemma 1 (Proposition 3.11 of [7])**.** *Let E be a Riesz space, $u_1, \dots, u_m, v_1, \dots v_n \in E$ and $\bigsqcup_{i=1}^m u_i = \bigsqcup_{k=1}^n v_k$. Then there exists a disjoint family $(w_{i,k})$ of elements of E, where $i \in \{1,\dots,m\}$ and $k \in \{1,\dots,n\}$ such that*

(i) $u_i = \bigsqcup_{k=1}^n w_{i,k}$ for any $i \in \{1,\dots,m\}$;

(ii) $v_k = \bigsqcup_{i=1}^m w_{i,k}$ for any $k \in \{1,\dots,n\}$.

The following notion brings a convenient simple tool in addition to the Freudenthal spectral theorem to verify the uniqueness of a linear operator defined by using the just mentioned theorem. Let E, F be Riesz spaces. A function $f\colon E \to F$ is said to be *vertically order σ-continuous* if for every $w \in E^+$, every $x \in E$ and every increasing sequence $(x_n)_{n=1}^\infty$ in E such that $0 \le x - x_n \le \frac{1}{n} w$ one has $f(x_n) \xrightarrow{o} f(x)$.

Lemma 2. *Let E, F be Riesz spaces with F Archimedean. Then every regular linear operator $T\colon E \to F$ is vertically order σ-continuous on E.*

Proof. Obviously, the difference of any two vertically order σ-continuous linear operators is vertically order σ-continuous. So, with no loss of generality we assume $T \ge 0$. Let $w \in E^+$, $x \in E_w$ and $(x_n)_{n=1}^\infty$ be an increasing sequence in E_w such that $0 \le x - x_n \le \frac{1}{n} w$ for all $n \in \mathbb{N}$. Then

$$0 \le Tx - Tx_n = T(x - x_n) \le \frac{1}{n} Tw$$

for all $n \in \mathbb{N}$, which implies $Tx_n \xrightarrow{o} Tx$. □

While the vertical order continuity of a linear operator is an automatic property, the horizontal order continuity is the main partial continuity which implies the entire order continuity.

Let E, F be Riesz spaces. A linear operator $T\colon E \to F$ is said to be

- *horizontally order continuous*[2] if for every $e \in E^+$ and every laterally increasing net (e_α) in $\mathfrak{F}_e$ the condition $\sup_\alpha e_\alpha = e$ implies $Te_\alpha \xrightarrow{o} Te$;
- *horizontally order σ-continuous* if for every $e \in E^+$ and every laterally increasing sequence (e_n) in $\mathfrak{F}_e$ the condition $\sup_n e_n = e$ implies $Te_n \xrightarrow{o} Te$.

Lemma 3 (Proposition 3.9 of [5])**.** *Let E be a Riesz space with the principal projection property, F be a Dedekind complete Riesz space and $T \in \mathcal{L}_r(E, F)$. Then the following assertions hold:*

1) if T is horizontally order continuous then T is order continuous;

2) if T is horizontally order σ-continuous then T is order σ-continuous.

[2]up-laterally-to-order continuous in terminology of [6], and disjointly continuous in terminology of [5]

1 Main result

Given a Riesz space E and $e \in E$, by $\mathfrak{F}_e$ we denote the Boolean algebra of all fragments of e, and by E_e the ideal of E generated by e, that is,

$$\mathfrak{F}_e = \{x \in E : x \sqsubseteq e\} \quad \text{and} \quad E_e = \{x \in E : (\exists \lambda > 0)\ |x| \le \lambda |e|\}.$$

Let $\mathcal{B}$ be a Boolean algebra and F be a Riesz space. A mapping $\nu \colon \mathcal{B} \to F^+$ is called a *positive finitely additive vector measure* if $\nu(x \sqcup y) = \nu(x) + \nu(y)$ for all disjoint $x, y \in \mathcal{B}$. A positive finitely additive vector measure $\nu \colon \mathcal{B} \to F$ is said to be:

- τ-continuous provided for every nonempty upward directed set $\mathcal{A} \subseteq \mathcal{B}$ for which $\sup \mathcal{A}$ exists in $\mathcal{B}$ one has that $\sup \nu(\mathcal{A})$ exists in F and $\nu(\sup \mathcal{A}) = \sup \nu(\mathcal{A})$;
- σ-continuous provided for every increasing sequence (x_n) in $\mathcal{B}$ for which $\sup_n x_n$ exists in $\mathcal{B}$ one has that $\sup_n \nu(x_n)$ exists in F and $\nu(\sup_n x_n) = \sup_n \nu(x_n)$.

Theorem 1. *Let E be a Riesz space with the principal projection property, $0 < e \in E$ and F be a Dedekind σ-complete Riesz space. Then for every positive finitely additive vector measure $\nu \colon \mathfrak{F}_e \to F$ there exists a unique positive linear operator $T \colon E_e \to F$, which extends ν, that is, $Tx = \nu(x)$ for all $x \in E_e$. Moreover, if ν is τ-continuous (or σ-continuous) then T is order continuous (respectively, order σ-continuous).*

Proof. Fix any positive additive mapping $\nu \colon \mathfrak{F}_e \to F$. Let X denote the set of all e-step functions in E, that is,

$$X := \Big\{ \sum_{k=1}^{m} a_k e_k : m \in \mathbb{N},\, e = \bigsqcup_{k=1}^{m} e_k,\, a_k \in \mathbb{R} \Big\}.$$

Observe that X is a linear subspace of E_e including $\mathfrak{F}_e$. First we define a linear operator $\widetilde{T} \colon X \to F$ by setting

$$\widetilde{T}\Big(\sum_{k=1}^{m} a_k e_k\Big) = \sum_{k=1}^{m} a_k \nu(e_k) \tag{1}$$

for every $x = \sum_{k=1}^{m} a_k e_k \in X$, where $m \in \mathbb{N}$, $e = \bigsqcup_{k=1}^{m} e_k$, $a_k \in \mathbb{R}$. Using Lemma 1, one can easily show that the value of $\widetilde{T}$ at a point $x \in X$ defined by (1) does not depend on the representation of x and $\widetilde{T}$ is a linear operator. By (1), $\widetilde{T}x = \nu(x)$ for all $x \in E_e$. By the positivity of ν we have $\widetilde{T} \ge 0$.

Now we extend $\widetilde{T}$ from X to E_e. Fix any $x \in E_e^+$ and define an extension $T \colon E_e \to F$ of $\widetilde{T}$. Using Freudenthal's spectral theorem [3, Theorem 2.8], choose a sequence (x_n) in X such that $0 \le x_n \uparrow x$ and $x - x_k \le \frac{1}{k} e$ for all $k \in \mathbb{N}$. Say, $x_n = \sum_{k=1}^{m_n} a_k^{(n)} e_k^{(n)}$, where $m_n \in \mathbb{N}$, $e = \bigsqcup_{k=1}^{m_n} e_k^{(n)}$ and $a_k^{(n)} \in \mathbb{R}$ for $n = 1, 2, \ldots$. Choose $\lambda > 0$ so that $x \le \lambda e$. Then $0 \le a_k^{(n)} \le \lambda$ for all $n \in \mathbb{N}$ and $k \in \{1, \ldots, m_n\}$ and hence

$$\widetilde{T}x_n = \sum_{k=1}^{m_n} a_k^{(n)} \nu(e_k^{(n)}) \le \sum_{k=1}^{m_n} \lambda \nu(e_k^{(n)}) = \lambda \nu(e).$$

By the positivity of $\widetilde{T}$, $(\widetilde{T}x_n)_{n=1}^{\infty}$ is an increasing sequence in F order bounded by $\lambda \nu(e)$. By the Dedekind σ-completeness of F, there exists $f \in F^+$ such that $\widetilde{T}x_n \uparrow f$ in F. Show that f is independent on the choice of the sequence (x_n) and hence is uniquely determined by x.

Indeed, let (y_n) be another sequence in X such that $0 \le y_n \uparrow x$ and $x - y_k \le \frac{1}{k}e$ for all $k \in \mathbb{N}$. Suppose $\widetilde{T}y_n \uparrow g$ and prove that $g = f$. By the assumptions, $-\frac{1}{k}e \le x_k - y_k \le \frac{1}{k}e$ and hence, $|\widetilde{T}x_k - \widetilde{T}y_k| \le \frac{1}{k}\nu(e)$ for all $k \in \mathbb{N}$ which implies that $(\widetilde{T}x_k - \widetilde{T}y_k) \xrightarrow{o} 0$. On the other hand, $(\widetilde{T}x_k - \widetilde{T}y_k) \xrightarrow{o} f - g$. This yields $f = g$. Then we set $Tx = f$. The additivity of T on E_e^+ follows from the additivity of order limits. By the Kantorovich theorem, there exists a unique positive linear extension $T\colon E_e \to F$ (which we denote using the same letter T). The existence of a positive linear extension of ν from $\mathfrak{F}_e$ to E_e is proved.

To prove the uniqueness, observe that any linear extension $\widetilde{T}\colon X \to F$ of ν must satisfy (1) and then use Lemma 2 together with Freudenthal's spectral theorem.

Now assume that ν is τ-continuous and prove the order continuity of T. By Lemma 3, it is enough to prove the horizontal order continuity of T. Let $x \in E_e^+$, (x_α) be a net in $\mathfrak{F}_x$ with $x_\alpha \xrightarrow{h} x$. For every α, set $e_\alpha := P_{x_\alpha}e$ and $e^* := P_x e$. By (3) of [3, Theorem 1.44], $P_u v \sqsubseteq v$ for all $u, v \in E$, hence $e^*, e_\alpha \in \mathfrak{F}_e$ for all α. By (2) of [3, Theorem 1.48], $e_\alpha \uparrow e^*$. The later two observations imply that $e_\alpha \xrightarrow{h} e^*$. By the τ-continuity of ν, one has

$$Te_\alpha = \nu(e_\alpha) \uparrow \nu(e^*) = Te^*. \tag{2}$$

Since $x = x_\alpha \sqcup (x - x_\alpha)$, by (3) of [3, Theorem 1.45], $P_x = P_{x_\alpha} + P_{x - x_\alpha}$ and hence,

$$P_{x-x_\alpha}e = P_x e - P_{x_\alpha}e = e^* - e_\alpha \ \text{ for all } \ \alpha. \tag{3}$$

Choose $\lambda > 0$ so that $x \le \lambda e$. Since $x - x_\alpha \sqsubseteq x$, we obtain $x - x_\alpha \le x \le \lambda e$, and hence

$$0 \le x - x_\alpha = P_{x-x_\alpha}(x - x_\alpha) \le P_{x-x_\alpha}x \le \lambda P_{x-x_\alpha}e \overset{(3)}{=} \lambda(e^* - e_\alpha)$$

for all λ. Thus, by the positivity of T and (2)

$$0 \le Tx - Tx_\alpha = T(x - x_\alpha) \le \lambda T(e^* - e_\alpha) = \lambda(Te^* - Te_\alpha) \downarrow 0.$$

So, we have proved the order continuity of T at every positive point x of E_e, which is enough by the linearity of T. The case of the σ-continuity of ν is considered similarly. □

The following simple example shows that an extension to the band B_e generated by e need not exist in quite natural cases.

Example 1. *Set $E = L_p := L_p[0,1]$ with $0 \le p < \infty$, $F = L_\infty$, $e = \mathbf{1}_{[0,1]}$ (the characteristic function of $[0,1]$). Then $B_e = L_p$, and the measure $\nu\colon \mathfrak{F}_e \to F$ defined by setting $\nu(x) = x$ for all $x \in \mathfrak{F}_e$ has no positive linear extension $T\colon L_p \to L_\infty$.*

Indeed, if such an extension T existed then it would satisfy (1) in place of $\widetilde{T}$, which implies $Tx = x$ for all e-step functions x. Then by Lemma 2 and Freudenthal's spectral theorem, $Tx = x$ for all $x \in L_\infty$. It follows that T is a linear bounded projection (bounded, by [3, Theorem 4.3]) of L_p onto the non-closed linear subspace L_∞ of L_p, which contradicts the boundedness of T.

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Класична теорема Канторовича стверджує існування та єдиність лінійного продовження додатного адитивного відображення, визначеного на додатному конусі E^+ векторної ґратки E зі значеннями у архімедовій векторній ґратці F на всю векторну ґратку E. Ми доводимо, що якщо E має головну проективну властивість та F порядково σ-повна, то для довільного $e \in E^+$ кожна додатна скінченно-адитивна F-значна міра, що визначена на булевій алгебрі $\mathfrak{F}_e$ фрагментів елемента e має єдине додатне лінійне продовження на ідеал E_e векторної ґратки E, породжений елементом e. Якщо, крім того, міра є τ-неперервною, то лінійне продовження порядково неперервне.

Ключові слова і фрази: додатний оператор, лінійне продовження, простір Рісса, векторна ґратка.