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Balancing numbers which are concatenations of three repdigits

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In this study, it is shown that the only balancing numbers which are concatenations of three repdigits are 204 and 1189. The proof depends on lower bounds for linear forms and some tools from Diophantine approximation.

Key words and phrases: balancing number, concatenation, repdigit, Diophantine equation, linear form in logarithms.

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Introduction

Let (B_n) be the sequence of balancing numbers given by $B_0 = 0$, $B_1 = 1$, and

$$B_n = 6B_{n-1} - B_{n-2} \quad \text{for} \quad n \geq 2.$$

The Binet formula for balancing numbers is

$$B_n = \frac{\alpha^n + \beta^n}{4\sqrt{2}},$$

where $\alpha = 3 + \sqrt{8}$ and $\beta = 3 - \sqrt{8}$, which are the roots of the characteristic equation $x^2 - 6x + 1 = 0$. It can be seen that $5 < \alpha < 6$, $0 < \beta < 1$ and $\alpha\beta = 1$. The relation between the nth balancing number B_n and α is given by

$$\alpha^{n-1} \leq B_n < \alpha^n, \quad n \geq 1. \tag{1}$$

The inequality (1) can be proved by induction. A base b-repdigit is a positive integer whose digits are all equal. When $b = 10$, we omit the base and we simply say that N is a repdigit. That is, N is of the form

$$N = \frac{d\,(10^m - 1)}{9} = \overline{\underbrace{d \dots d}_{m \text{ times}}}$$

for some positive integers d, m with $1 \leq d \leq 9$ and $m \geq 1$. Investigation of the repdigits in the second-order linear recurrence sequences has been of interest to mathematicians. Given $k \geq 2$, we say that N is a concatenations of k repdigits if N can be written in the form

$$\overline{\underbrace{d_1 \dots d_1}_{m_1 \text{ times}}\underbrace{d_2 \dots d_2}_{m_2 \text{ times}} \dots \underbrace{d_k \dots d_k}_{m_k \text{ times}}},$$

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where $m_1, m_2, m_3 \geq 1$, $1 \leq d_1 \leq 9$ and $0 \leq d_2, d_3, \ldots, d_k \leq 9$. In [14], the authors showed that the only balancing number which is concatenation of two repdigits is 35. In [1], the authors solved the problem of finding the Fibonacci numbers which are concatenations of two repdigits. In [6,7], M. Ddamulira tackled the problem of finding the Padovan and tribonacci numbers that are concatenations of two repdigits, respectively. In [10,11], we found all Lucas numbers which are concatenations of two and three repdigits, respectively. In [15], P. Trojovský considered the Diophantine equation

$$F_n = \overline{a\underbrace{b\ldots b}_{m \text{ times}}\underbrace{c\ldots c}_{k \text{ times}}},$$

in positive integer numbers m, n, and k with $2 \leq k \leq m$, where $1 \leq a \leq 9$ and $0 \leq b, c \leq 9$. He showed that the largest Fibonacci number satisfying the above equation is 17711. In this paper, we study the equation

$$B_n = \overline{\underbrace{d_1\ldots d_1}_{m_1 \text{ times}}\underbrace{d_2\ldots d_2}_{m_2 \text{ times}}\underbrace{d_3\ldots d_3}_{m_3 \text{ times}}}, \tag{2}$$

where $m_1, m_2, m_3 \geq 1$, $1 \leq d_1 \leq 9$ and $0 \leq d_2, d_3 \leq 9$. Thus, we showed that the only balancing numbers which are concatenations of three repdigits are 204, 1189. Our study can be viewed as a continuation of the former works on this subject. In Section 2, we introduce necessary lemmas. Then we prove our main theorem in Section 3.

1 Auxiliary results

In [10], in order to solve Diophantine equations of the similar form, the authors have used Baker's theory of lower bounds for a nonzero linear form in logarithms of algebraic numbers. Since such bounds are of crucial importance in effectively solving of Diophantine equations of the similar form, we will use the same method. Now we start with recalling some basic notions from algebraic number theory.

Let η be an algebraic number of degree d with minimal polynomial

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0\prod_{i=1}^{d}\left(x - \eta^{(i)}\right) \in \mathbb{Z}[x],$$

where the a_i's are relatively prime integers with $a_0 > 0$ and the $\eta^{(i)}$'s are conjugates of η. Then

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d}\log\left(\max\left\{|\eta^{(i)}|, 1\right\}\right)\right) \tag{3}$$

is called the logarithmic height of η. In particular, if $\eta = a/b$ is a rational number with $\gcd(a,b) = 1$ and $b \geq 1$, then $h(\eta) = \log\left(\max\{|a|, b\}\right)$.

For algebraic numbers η and γ, the function h has the following basic properties (see [5]):

$$\begin{aligned}
h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2,\\
h\left(\eta\gamma^{\pm 1}\right) &\leq h(\eta) + h(\gamma),\\
h\left(\eta^m\right) &= |m|h(\eta).
\end{aligned}$$

Now we give a theorem which is deduced from [13, Corollary 2.3] due to E.M. Matveev and provides a large upper bound for the subscript n in (2) (also see [4, Theorem 9.4]).

Lemma 1. *Assume that $\gamma_1, \gamma_2, \ldots, \gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, b_2, \ldots, b_t$ are rational integers, and*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1$$

is not zero. Then

$$|\Lambda| > \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 (1 + \log D)(1 + \log B) A_1 A_2 \cdots A_t\right),$$

where

$$B \geq \max\left\{ |b_1|, \ldots, |b_t| \right\} \quad \text{and} \quad A_i \geq \max\left\{ Dh(\gamma_i), |\log \gamma_i|, 0.16 \right\}$$

for all $i = 1, \ldots, t$.

The following lemma is given in [3]. This lemma is an immediate variation of the result from [9] due to A. Dujella and A. Pethő, which is a version of a lemma of A. Baker and H. Davenport [2]. This lemma will be used to reduce the upper bound for the subscript n in the equation (2). Let $\|x\|$ denote the distance from x to the nearest integer. That is, $\|x\| = \min\left\{ |x - n| : n \in \mathbb{Z} \right\}$ for any real number x. Then we have the following assertion.

Lemma 2. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational number γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := \|\mu q\| - M\|\gamma q\|$. If $\epsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w},$$

in positive integers u, v, and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

The following lemma is given in [8].

Lemma 3. *Let $a, x \in \mathbb{R}$. If $0 < a < 1$ and $|x| < a$, then*

$$\left|\log(1 + x)\right| < \frac{-\log(1 - a)}{a} |x|$$

and

$$|x| < \frac{a}{1 - e^{-a}} \left|e^x - 1\right|.$$

The following lemma can be deduced from [12].

Lemma 4. *The only repdigits in the balancing sequence are 1 and 6.*

2 Main Theorem

Theorem. *The only balancing numbers which are concatenations of three repdigits are* 204 *and* 1189.

Proof. Assume that the equation (2) holds. We checked the first 39 balancing numbers. We found $B_n \in \{204, 1189\}$, which are all the solutions to the Diophantine equation (2) for $d_1, d_2 \in \{0, 1, \ldots, 9\}$ with $d_1 > 0$. From now on, we assume that $n \geq 40$ and the cases $d_1 = d_2 \neq d_3$ and $d_1 \neq d_2 = d_3$ in equation (2) are impossible since the only balancing number, which is concatenation of two repdigits is 35, which has two digits. Furthermore, the case $d_1 = d_2 = d_3$ in equation (2) is impossible since the largest repdigit in the balancing sequence is $B_2 = 6$, which is stated in Lemma 4. Let

$$B_n = \overline{\underbrace{d_1 \ldots d_1}_{m_1 \text{ times}}\underbrace{d_2 \ldots d_2}_{m_2 \text{ times}}\underbrace{d_3 \ldots d_3}_{m_3 \text{ times}}}.$$

Then

$$B_n = \underbrace{d_1 \ldots d_1}_{m_1 \text{ times}} \cdot 10^{m_2+m_3} + \underbrace{d_2 \ldots d_2}_{m_2 \text{ times}} \cdot 10^{m_3} + \underbrace{d_3 \ldots d_3}_{m_3 \text{ times}}.$$

That is,

$$B_n = \frac{d_1\left(10^{m_1}-1\right)}{9}10^{m_2+m_3} + \frac{d_2\left(10^{m_2}-1\right)}{9}10^{m_3} + \frac{d_3\left(10^{m_3}-1\right)}{9}, \tag{4}$$

or

$$B_n = \frac{1}{9}\Big(d_1 10^{m_1+m_2+m_3} - (d_1-d_2)\,10^{m_2+m_3} - (d_2-d_3)\,10^{m_3} - d_3\Big). \tag{5}$$

Combining the right side of inequality (1) with (4), we obtain

$$10^{m_1+m_2+m_3-1} < B_n < \alpha^n < 10^n.$$

From this, we get $m_1 + m_2 + m_3 < n + 1$. Now, we can manipulate the equation (5) as

$$\frac{9\alpha^n}{4\sqrt{2}} - d_1 10^{m_1+m_2+m_3} = \frac{9\beta^n}{4\sqrt{2}} - (d_1-d_2)\,10^{m_2+m_3} - (d_2-d_3)\,10^{m_3} - d_3. \tag{6}$$

Taking absolute values of both sides of the equation (6), it is seen that

$$\begin{aligned}
\left|\frac{9\alpha^n}{4\sqrt{2}} - d_1 10^{m_1+m_2+m_3}\right| &\leq \frac{9\beta^n}{4\sqrt{2}} + (d_1-d_2)\,10^{m_2+m_3} + (d_2-d_3)\,10^{m_3} + d_3 \\
&\leq \frac{9\alpha^{-n}}{4\sqrt{2}} + 9\cdot 10^{m_2+m_3} + 9\cdot 10^{m_3} + 9 \\
&\leq \frac{9\alpha^{-n}}{4\sqrt{2}} + \left(9\cdot 10^{m_2+m_3} + 9\cdot 10^{m_3} + 0.9\cdot 10^{m_3}\right) \\
&= \frac{9\alpha^{-n}}{4\sqrt{2}} + 10^{m_3}\left(9\cdot 10^{m_2} + 9.9\right) \\
&\leq \frac{9\alpha^{-n}}{4\sqrt{2}} + 10^{m_3}\left(9\cdot 10^{m_2} + 0.99\cdot 10^{m_2}\right) \\
&< \frac{0.09\cdot\alpha^{-n}\cdot 10^{m_2+m_3}}{4\sqrt{2}} + 9.99\cdot 10^{m_2+m_3} \\
&< 9.991\cdot 10^{m_2+m_3},
\end{aligned}$$

where we used the fact that $n \geq 40$. Therefore

$$\left|\frac{9\alpha^n}{4\sqrt{2}} - d_1 10^{m_1+m_2+m_3}\right| < 9.991 \cdot 10^{m_2+m_3}. \tag{7}$$

Dividing both sides of (7) by $d_1 10^{m_1+m_2+m_3}$, we obtain

$$\left|\frac{9}{4\sqrt{2}d_1}\alpha^n 10^{-m_1-m_2-m_3} - 1\right| < \frac{9.991}{10^{m_1}}. \tag{8}$$

Now, let us apply Lemma 1 with $\gamma_1 := 9/\left(4\sqrt{2}d_1\right)$,$\gamma_2 := \alpha$, $\gamma_3 := 10$ and $b_1 := 1$, $b_2 := n$, $b_3 := -m_1 - m_2 - m_3$. Note that the numbers γ_1, γ_2, and γ_3 are positive real numbers and elements of the field $K = \mathbb{Q}(\sqrt{2})$. It is obvious that the degree of the field K is 2. So $D = 2$. Let

$$\Lambda_1 := \frac{9}{4\sqrt{2}d_1}\alpha^n 10^{-m_1-m_2-m_3} - 1.$$

If $\Lambda_1 = 0$, then we get

$$\alpha^n = \frac{4\sqrt{2}d_1}{9} \cdot 10^{m_1+m_2+m_3}.$$

This is impossible as α^{2n} is irrational for $n \geq 1$. Therefore, Λ_1 is nonzero. Moreover, since

$$h(\gamma_1) = h\left(\frac{9}{4\sqrt{2}d_1}\right) \leq h(9/d_1) + h(4\sqrt{2}) \leq \log 9 + \frac{1}{2}\log 32 < 3.94,$$

and

$$h(\gamma_2) = h(\alpha) = \frac{\log \alpha}{2} < 0.9, h(\gamma_3) = h(10) = \log 10 < 2.31,$$

by (3), we can take $A_1 := 7.88$, $A_2 := 1.8$, and $A_3 := 4.62$.

On the other hand, as $m_1 + m_2 + m_3 < n + 1$ and $B \geq \max\left\{|1|, |n|, |-m_1 - m_2 - m_3|\right\}$, we can take $B := n + 1$. Let

$$C := 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2) \cdot 1.8 \cdot 4.62.$$

Thus, taking into account the inequality (8) and using Lemma 1, we obtain

$$9.991 \cdot 10^{-m_1} > |\Lambda_1| > \exp\left(-C \cdot \left(1 + \log(n+1)\right) \cdot 7.88\right).$$

By a simple computation, it follows that

$$m_1 \log 10 < 6.36 \cdot 10^{13} \cdot \left(1 + \log(n+1)\right) + \log 9.991. \tag{9}$$

Rearranging the equation (5) as

$$\frac{9\alpha^n}{4\sqrt{2}} - \left(d_1 10^{m_1} - (d_1 - d_2)\right) 10^{m_2+m_3} = \frac{9\beta^n}{4\sqrt{2}} - (d_2 - d_3)\, 10^{m_3} - d_3 \tag{10}$$

and taking absolute values of both sides of the equation (10), we get

$$\begin{aligned}
\left|\frac{9\alpha^n}{4\sqrt{2}} - \left(d_1 10^{m_1} - (d_1 - d_2)\right) 10^{m_2+m_3}\right| &\leq \frac{9\beta^n}{4\sqrt{2}} + (d_2 - d_3)\, 10^{m_3} + d_3 \\
&\leq \frac{9\alpha^{-n}}{4\sqrt{2}} + 9 \cdot 10^{m_3} + 9 \\
&\leq \frac{9\alpha^{-n}}{4\sqrt{2}} + (9 \cdot 10^{m_3} + 0.9 \cdot 10^{m_3}) \\
&< \frac{(0.9)\alpha^{-n} 10^{m_3}}{4\sqrt{2}} + 9.9 \cdot 10^{m_3} \\
&< 9.91 \cdot 10^{m_3},
\end{aligned}$$

i.e.

$$\left|\frac{9\alpha^n}{4\sqrt{2}} - \big(d_1 10^{m_1} - (d_1 - d_2)\big)10^{m_2+m_3}\right| < 9.91 \cdot 10^{m_3}. \tag{11}$$

Dividing both sides of (11) by $\big(d_1 10^{m_1} - (d_1 - d_2)\big)10^{m_2+m_3}$, we obtain

$$\left|1 - \left(\frac{9}{4\sqrt{2}\big(d_1 10^{m_1} - (d_1 - d_2)\big)}\right)\alpha^n 10^{-m_2-m_3}\right| < \frac{1.11}{10^{m_2}}. \tag{12}$$

Taking

$$\gamma_1 := \frac{9}{4\sqrt{2}\big(d_1 10^{m_1} - (d_1 - d_2)\big)}, \quad \gamma_2 := \alpha, \quad \gamma_3 := 10$$

and $b_1 := 1$, $b_2 := n$, $b_3 := -m_2 - m_3$, we can apply Lemma 1. The numbers γ_1, γ_2, and γ_3 are positive real numbers and elements of the field $K = \mathbb{Q}(\sqrt{2})$ and so $D = 2$. Let

$$\Lambda_2 := 1 - \left(\frac{9}{4\sqrt{2}\big(d_1 10^{m_1} - (d_1 - d_2)\big)}\right)\alpha^n 10^{-m_2-m_3}.$$

By the same arguments used before for Λ_1, we conclude that $\Lambda_2 \neq 0$. By using (3) and the properties of the logarithmic height, we get

$$\begin{aligned}
h(\gamma_1) &= h\left(\frac{9}{4\sqrt{2}(d_1 10^{m_1} - (d_1 - d_2))}\right) \\
&\le h(9) + h(4\sqrt{2}) + h(d_1 10^{m_1}) + h\,(d_1 - d_2) + \log 2 \\
&< 3\log 9 + \frac{1}{2}\log 32 + m_1 \log 10 + \log 2 \\
&< 9.02 + m_1 \log 10, \\
h(\gamma_2) &= h(\alpha) = \frac{\log \alpha}{2} < 0.9, \\
h(\gamma_3) &= h(10) = \log 10 < 2.31.
\end{aligned}$$

So, we can take $A_1 := 18.04 + 2m_1 \log 10$, $A_2 := 1.8$, and $A_3 := 4.62$. As $m_2 + m_3 < n$ and $B \ge \max\{|1|, |-n|, |-m_2 - m_3|\}$, we can take $B := n$. Thus, taking into account the inequality (12) and using Lemma 1, we obtain

$$1.11 \cdot 10^{-m_2} > |\Lambda_2| > \exp\big(-C \cdot (1 + \log n)\,(18.04 + 2m_1 \log 10)\big),$$

or

$$m_2 \log 10 < 8.07 \cdot 10^{12} \cdot (1 + \log n)\,(18.04 + 2m_1 \log 10) + \log 1.11. \tag{13}$$

Rearranging the equation (5) as

$$\frac{9\alpha^n}{4\sqrt{2}} - \big(d_1 10^{m_1+m_2} - (d_1 - d_2)\,10^{m_2} - (d_2 - d_3)\big)10^{m_3} = \frac{9\beta^n}{4\sqrt{2}} - d_3 \tag{14}$$

and taking absolute values of both sides of the equation (14), we get

$$\left|\frac{9\alpha^n}{4\sqrt{2}} - \big(d_1 10^{m_1+m_2} - (d_1 - d_2)\,10^{m_2} - (d_2 - d_3)\big)10^{m_3}\right| \le \frac{9\beta^n}{4\sqrt{2}} + d_3 = \frac{9\alpha^{-n}}{4\sqrt{2}} + 9 < 9.1,$$

i.e.

$$\left|\frac{9\alpha^n}{4\sqrt{2}}-\left(d_1 10^{m_1+m_2}-(d_1-d_2)\,10^{m_2}-(d_2-d_3)\,\right)10^{m_3}\right|<9.1. \tag{15}$$

Dividing both sides of (15) by $9\alpha^n/\left(4\sqrt{2}\right)$, we obtain

$$\left|1-\left(\frac{4\sqrt{2}\left(d_1 10^{m_1+m_2}-(d_1-d_2)\,10^{m_2}-(d_2-d_3)\,\right)}{9}\right)\alpha^{-n}10^{m_3}\right|\leq 5.72\cdot\alpha^{-n}. \tag{16}$$

Taking

$$\gamma_1:=\left(\frac{4\sqrt{2}\left(d_1 10^{m_1+m_2}-(d_1-d_2)\,10^{m_2}-(d_2-d_3)\,\right)}{9}\right),\quad \gamma_2:=\alpha,\quad \gamma_3:=10,$$

and $b_1:=1$, $b_2:=-n$, $b_3:=m_3$, we can apply Lemma 1. The numbers γ_1, γ_2, and γ_3 are positive real numbers and elements of the field $K=\mathbb{Q}(\sqrt{2})$ and so $D=2$. Let

$$\Lambda_3:=1-\left(\frac{4\sqrt{2}(d_1 10^{m_1+m_2}-(d_1-d_2)10^{m_2}-(d_2-d_3))}{9}\right)\alpha^{-n}10^{m_3}.$$

We can ensure that $\Lambda_3\neq 0$, just as for Λ_1. By using (3) and the properties of the logarithmic height, we get

$$\begin{aligned}
h(\gamma_1)&=h\left(\frac{4\sqrt{2}(d_1 10^{m_1+m_2}-(d_1-d_2)10^{m_2}-(d_2-d_3))}{9}\right)\\
&\leq h(9)+h(4\sqrt{2})+h\left(d_1 10^{m_1+m_2}\right)+h((d_1-d_2)\,10^{m_2})+h((d_2-d_3)+2\log 2\\
&\leq 4\log 9+\frac{1}{2}\log 32+(m_1+m_2)\log 10+m_2\log 10+2\log 2\\
&<11.91+(m_1+m_2)\log 10+m_2\log 10,\\
h(\gamma_2)&=h(\alpha)=\frac{\log\alpha}{2}<0.9,\\
h(\gamma_3)&=h(10)=\log 10<2.31.
\end{aligned}$$

So, we can take $A_1:=23.82+2m_1\log 10+4m_2\log 10$, $A_2:=1.8$, and $A_3:=4.62$. As $m_3<n-1$ and $B\geq\max\{|1|,|-n|,|m_3|\}$, we can take $B:=n$. Thus, taking into account the inequality (16) and using Lemma 1, we obtain

$$1.02\cdot\alpha^{-n}>|\Lambda_3|>\exp\left(-C\cdot(1+\log n)(23.82+2m_1\log 10+4m_2\log 10)\right),$$

or

$$n\log\alpha-\log(1.02)<8.07\cdot 10^{12}\cdot(1+\log n)\,(23.82+2m_1\log 10+4m_2\log 10)\,. \tag{17}$$

Using the inequalities (9), (13), and (17), a computer search with Mathematica gives us that $n<2.36\cdot 10^{46}$. Now, let us try to reduce the upper bound on n by applying Lemma 2. Let

$$z_1:=\log(\Lambda_1+1)=(m_1+m_2+m_3)\log 10-n\log\alpha-\log\left(9/(4\sqrt{2}d_1)\right).$$

From (8), we get

$$|\Lambda_1|=\left|e^{-z_1}-1\right|<\frac{9.991}{10^{m_1}}<0.9995\quad\text{for}\ \ m_1\geq 1.$$

Choosing $a := 0.9995$, we get the inequality

$$|z_1| = \left|\log(\Lambda_1 + 1)\right| < \frac{\log 2000}{0.9995} \cdot \frac{9.991}{10^{m_1}} < \frac{75.98}{10^{m_1}},$$

by Lemma 3. Thus, it follows that

$$0 < \left|(m_1 + m_2 + m_3)\log 10 - n\log\alpha - \log\left(\frac{9}{4\sqrt{2}d_1}\right)\right| < \frac{75.98}{10^{m_1}}.$$

Dividing this inequality by $\log\alpha$, we get

$$0 < \left|(m_1 + m_2 + m_3)\frac{\log 10}{\log\alpha} - n - \frac{\log\left(9/(4\sqrt{2}d_1)\right)}{\log\alpha}\right| < 43.2 \cdot 10^{-m_1}. \tag{18}$$

Now, we can apply Lemma 2. Put

$$\gamma := \frac{\log 10}{\log\alpha} \notin \mathbb{Q}, \quad \mu := -\frac{\log\left(9/(4\sqrt{2}d_1)\right)}{\log\alpha}, \quad A := 43.2, \quad B := 10, \quad \text{and} \quad w := m_1.$$

Let $M := 2.36 \cdot 10^{46}$. Then $M > m_1 + m_2 + m_3$ and the denominator of the 97th convergent of γ exceeds $6M$. Furthermore,

$$\epsilon := \|\mu q_{97}\| - M\|\gamma q_{97}\| > 0.03.$$

Thus, the inequality (18) has no solutions for

$$m_1 \geq 50.83 > \frac{\log\left(Aq_{97}/\epsilon\right)}{\log B}.$$

So $m_1 \leq 50$. Using the inequalities (13) and (17) together and substituting this upper bound for m_1 into (17), we obtain $n < 2.23 \cdot 10^{33}$. Now, let

$$z_2 := \log(\Lambda_2 + 1) = (m_2 + m_3)\log 10 - n\log\alpha - \log\left(\frac{9}{4\sqrt{2}\left(d_1 10^{m_1} - (d_1 - d_2)\right)}\right).$$

From (12), it is seen that

$$|\Lambda_2| = \left|e^{-z_2} - 1\right| < (1.11) \cdot 10^{-m_2} < 0.2$$

for $m_2 \geq 1$. Choosing $a := 0.2$, we get the inequality

$$|z_2| = \left|\log(\Lambda_2 + 1)\right| < \frac{\log(5/4)}{0.2} \cdot \frac{1.11}{10^{m_2}} < \frac{1.24}{10^{m_2}},$$

by Lemma 3. This shows that

$$0 < \left|(m_2 + m_3)\log 10 - n\log\alpha - \log\left(\frac{9}{4\sqrt{2}\left(d_1 10^{m_1} - (d_1 - d_2)\right)}\right)\right| < 1.24 \cdot 10^{-m_2}$$

Dividing both sides of the above inequality by $\log\alpha$, we get

$$0 < \left|\frac{(m_2 + m_3)\log 10}{\log\alpha} - n - \frac{\log\left(9/(4\sqrt{2})\left(d_1 10^{m_1} - (d_1 - d_2)\right)\right)}{\log\alpha}\right| < 0.71 \cdot 10^{-m_2}. \tag{19}$$

Putting $\gamma := \dfrac{\log 10}{\log \alpha}$ and taking $m_2 + m_3 < M := 2.23 \cdot 10^{33}$, we found that q_{75}, the denominator of the 75th convergent of γ exceeds $6M$. Taking

$$\mu := -\frac{\log\left(9/(4\sqrt{2})\left(d_1 10^{m_1} - (d_1 - d_2)\right)\right)}{\log \alpha}$$

and considering the fact that $m_1 \leq 50$, $d_1 \neq d_2$, $1 \leq d_1 \leq 9$ and $0 \leq d_2 \leq 9$, a quick computation with Mathematica gives us the inequality

$$\epsilon = \epsilon(\mu) := \|\mu q_{75}\| - M\|\gamma q_{75}\| > 0.$$

Let $A := 0.71$, $B := 10$, and $w := m_2$ in Lemma 2. Then with the help of Mathematica, we can say that the inequality (19) has no solutions for

$$m_2 \geq 39.92 > \frac{\log(Aq_{75}/\epsilon)}{\log B}.$$

Therefore $m_2 \leq 39$. As $m_1 \leq 50$ and $m_2 \leq 39$, substituting this upper bounds for m_1 and m_2 into (17), we obtain $n < 1.14 \cdot 10^{17}$. Now, let

$$z_3 := m_3 \log 10 - n \log \alpha + \log\left(\frac{4\sqrt{2}\left(d_1 10^{m_1+m_2} - (d_1 - d_2)\, 10^{m_2} - (d_2 - d_3)\right)}{9}\right).$$

From (16), we can write

$$|\Lambda_3| = |e^{z_3} - 1| < 1.02 \cdot \alpha^{-n} < 0.01$$

for $n \geq 40$. Choosing $a := 0.01$, it is seen that

$$|z_3| = \left|\log(\Lambda_3 + 1)\right| < \frac{\log(100/99)}{0.01} \cdot \frac{1.02}{\alpha^n} < \frac{1.03}{\alpha^n},$$

by Lemma 3. Thus, it follows that

$$0 < \left| m_3 \log 10 - n \log \alpha + \log\left(\frac{4\sqrt{2}\left(d_1 10^{m_1+m_2} - (d_1 - d_2)\, 10^{m_2} - (d_2 - d_3)\right)}{9}\right)\right| < 1.03 \cdot \alpha^{-n}.$$

Dividing both sides of the above inequality by $\log \alpha$, we get

$$0 < \left| m_3 \frac{\log 10}{\log \alpha} - n + \frac{\log\left(4\sqrt{2}\left(d_1 10^{m_1+m_2} - (d_1 - d_2) 10^{m_2} - (d_2 - d_3)\right)/9\right)}{\log \alpha}\right| < 0.59 \cdot \alpha^{-n}. \tag{20}$$

Putting $\gamma := \dfrac{\log 10}{\log \alpha}$ and taking $m_3 < M := 1.14 \cdot 10^{17}$, we found that q_{41}, the denominator of the 41st convergent of γ exceeds $6M$. Taking

$$\mu := \frac{\log\left(4\sqrt{2}\left(d_1 10^{m_1+m_2} - (d_1 - d_2)\, 10^{m_2} - (d_2 - d_3)\right)/9\right)}{\log \alpha}$$

and considering the fact that $m_1 \leq 50$, $m_2 \leq 39$, $1 \leq d_1 \leq 9$ and $0 \leq d_2, d_3 \leq 9$, a quick computation with Mathematica gives us the inequality

$$\epsilon = \epsilon(\mu) := \|\mu q_{41}\| - M\|\gamma q_{41}\| > 0$$

except for the case $d_1 = d_2 \neq d_3$, $d_1 \neq d_2 = d_3$ and $d_1 = d_2 = d_3$. Let $A := 0.59$, $B := \alpha$, and $w := n$ in Lemma 2. Then with the help of Mathematica, we can say that the inequality (20) has no solution for

$$n \geq 35.74 > \frac{\log(Aq_{41}/\epsilon)}{\log B}.$$

Therefore, $n \leq 35$. This contradicts our assumption that $n \geq 40$. This completes the proof. $\square$

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Ердуван Ф., Кескін Р. *Збалансовані числа, які є конкатенацією трьох репдиджитів* // Карпатські матем. публ. — 2024. — Т.16, №1. — С. 148–157.

У цій статті ми встановили, що єдиними збалансованими числами, які є конкатенацією трьох репдиджитів (натуральних чисел, в записі яких всі цифри однакові) є 204 та 1189. Для доведення ми використовуємо точні нижні межі для лінійних форм і деякі засоби діофантової апроксимації.

Ключові слова і фрази: збалансоване число, конкатенація, репдиджит, діофантове рівняння, лінійна форма від логарифмів.