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Fixed point theorems on an orthogonal metric space using Matkowski type contraction

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The purpose of this paper is to prove Boyd-Wong and Matkowski type fixed point theorems in orthogonal metric space which was defined by M.E. Gordji in 2017 and is an extension of the metric space. Some examples are established in support of our main results. Finally, we apply our results to establish the existence of a unique solution of a periodic boundary value problem.

Key words and phrases: fixed point, orthogonal set, orthogonal metric space, metric space.

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Introduction

The Banach contraction principle has become one of the most well-known and important discoveries in mathematics during the last century because of its simple structure and utility. Numerous researchers have expanded and generalized Banach's fixed point theorem from many viewpoints. One common approach to reinforce the Banach contraction principle is to replace the metric space with other generalized metric spaces.

The contraction condition in metric spaces was improved by D.W. Boyd, J.S.W. Wong [2] by using a control function. Y.I. Alber and S. Guerre-Delabriere pioneered the ϕ-weak contraction condition in Hilbert spaces [1]. In metric spaces, every ϕ-weak contractive map has a unique fixed point, as B.E. Rhoades has shown in [9]. J. Matkowski developed the idea to generalize the Banach contraction principle [8].

Recently, for the first time, M.E. Gordji et al. [3] expanded the literature on metric space by introducing the concept of orthogonality and establishing the fixed point result. There are several uses for this novel idea of an orthogonal set as well as numerous forms of orthogonality. M.E. Gordji and H. Habibi [4, 5] proved the fixed point and related results in (generalized) orthogonal metric spaces. For more information, we refer the reader to [6, 10–13].

This article is organized as follows. Section 2 contains some basic definition of orthogonal set from the literature. In Section 3, we established fixed point theorems in the settings of orthogonal metric spaces. Section 4 containes an application of our main result from Section 3 for the existence and uniqueness of solution to differential equation of first order with periodic boundary conditions.

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1 Preliminaries

Definition 1 ([3]). *Let E be a non-empty set and $\perp$ be a binary relation defined on E. If binary relation $\perp$ fulfils the criteria*

$$\exists\, \varsigma_0\, [(\forall\, \vartheta \in E, \vartheta \perp \varsigma_0) \text{ or } (\forall\, \vartheta \in E, \varsigma_0 \perp \vartheta)],$$

then pair $(E, \perp)$ known as an orthogonal set. The element ς_0 is called an orthogonal element.

Definition 2. *Let $(E, \perp)$ be an orthogonal set (O-set). Any two elements $\varsigma, \vartheta \in E$ such that $\varsigma \perp \vartheta$ are said to be orthogonally related.*

An orthogonal set is illustrated in the following non-trivial examples.

Example 1. *Let $E = 2\mathbb{Z}$ and set a binary relation $\perp$ on $2\mathbb{Z}$ as $m \perp n$ if $m.n = 0$. Then $(2\mathbb{Z}, \perp)$ is an orthogonal set with 0 as an orthogonal element.*

Example 2. *Let E be set of all matrices of order n over $\mathbb{R}$, i.e. $E = \mathcal{M}_n(\mathbb{R})$. We define $\perp$ on $\mathcal{M}_n(\mathbb{R})$ as $\mathcal{A} \perp \mathcal{B}$ if $\mathcal{AB} = \mathcal{BA}$. Then $(\mathcal{M}_n(\mathbb{R}), \perp)$ is an orthogonal set since $\mathcal{SA} = \mathcal{AS}$ for a scalar matrix $\mathcal{S} \in \mathcal{M}_n(\mathbb{R})$.*

Remark. *A orthogonal set may have unique, more than one or infinite many orthogonal elements.*

Consider a non-empty set $E(\neq \varnothing)$ and define a binary relation $\perp$ on set E with usual metric d defined on set E, then triplet $(X, \perp, d)$ is called orthogonal metric space (or O-metric space).

Some basic definition and properties of an orthogonal set and orthogonal metric space are given below. For more information and examples the reader is suggested to see [3,7].

Definition 3 ([3]). *Consider a non-empty set E. Let $\perp$ be a binary relation defined on E. A sequence $\{\varsigma_n\}$ is called an orthogonal sequence (briefly O-sequence) if*

$$(\forall\, n \in \mathbb{N}, \varsigma_n \perp \varsigma_{n+1}) \text{ or } (\forall\, n \in \mathbb{N}, \varsigma_{n+1} \perp \varsigma_n).$$

Definition 4 ([3]). *Let $(E, \perp, d)$ be an orthogonal metric space. Then E is said to be an O-complete if every Cauchy O-sequence converges in X.*

Remark 1 ([3]). *Every complete metric space is O-complete and the converse is not true.*

Definition 5 ([3]). *Let $(E, \perp, d)$ be an orthogonal metric space. A function $f : E \to E$ is said to be $\perp$-continuous in $\varsigma \in E$ if for each O-sequence $\{\varsigma_n\}_{n\in\mathbb{N}}$ converging to ς we have $f(\varsigma_n) \to f(\varsigma)$ as $n \to \infty$. Also, f is said to be $\perp$-continuous on E if f is $\perp$-continuous in each $\varsigma \in E$.*

Remark 2. *The authors of [3] found, that O-continuity in conventional metric spaces is weaker than classical continuity.*

Definition 6 ([3]). *Let a pair $(E, \perp)$ be an O-set, where $E(\neq \varnothing)$ is a non-empty set and $\perp$ is a binary relation on E. A mapping $f : E \to E$ is said to be $\perp$-preserving if $f(\varsigma) \perp f(\vartheta)$ whenever $\varsigma \perp \vartheta$ and weakly $\perp$-preserving if $f(\varsigma) \perp f(\vartheta)$ or $f(\vartheta) \perp f(\varsigma)$ whenever $\varsigma \perp \vartheta$.*

2 Main Results

Theorem 1. *Let $(E, d, \perp)$ be an O-complete metric space and suppose that $f : E \to E$ be $\perp$-continuous and $\perp$-preserving, satisfying*

$$d(f(\varsigma), f(\vartheta)) \le \phi(d(\varsigma, \vartheta)), \quad \forall\, \varsigma, \vartheta \in E \quad \textit{with} \quad \varsigma \perp \vartheta,$$

where $\phi : \mathbb{R}^+ \to [0, \infty)$ is upper semi-continuous from right, i.e. for any sequence

$$t_n \to t \ge 0 \;\Rightarrow\; \lim_{n\to\infty} \sup \phi(t_n) \le \phi(t),$$

and satisfies $0 \le \phi(t) < t$ for $t > 0$. Then f has unique fixed point ς^*. Also f is a Picard operator, that is $\lim_{n\to} f^n(\varsigma) = \varsigma^*$ for all $\varsigma \in E$.

Proof. Let $\varsigma_0 \in E$ be an orthogonal element in E, then by definition

$$(\forall\, \vartheta \in E, \varsigma_0 \perp \vartheta) \text{ or } (\forall\, \vartheta \in E, \vartheta \perp \varsigma_0).$$

It follows that $(\varsigma_0 \perp f(\varsigma_0))$ or $(f(\varsigma_0) \perp \varsigma_0)$. Let

$$\varsigma_1 = f(\varsigma_0), \quad \varsigma_2 = f(\varsigma_1) = f^2(\varsigma_0), \quad \varsigma_{n+1} = f(\varsigma_n) = f^{n+1}(\varsigma_0), \quad \forall\, n \in \mathbb{N}.$$

Since f is $\perp$-preserving, $\{\varsigma_n\}$ is an O-sequence.

Set $a_n = d(\varsigma_{n-1}, \varsigma_n)$. Observe that $\{a_n\}$ is a bounded below monotonically decreasing sequence, then $\{a_n\}$ is convergent to a, i.e. $\lim_{n\to\infty} a_n = a$. If $a > 0$, we have $a_{n+1} \le \phi(a_n)$, so that

$$a \le \lim_{t\to a^+} \sup \phi(t) \le \phi(a),$$

which is a contradiction. Contrary, assume that O-sequence $\{\varsigma_n\}$ is not Cauchy O-sequence, then $\exists\, \epsilon > 0$ and sequences $\{m_k\}$, $\{n_k\}$ of integers with $m_k \ge n_k \ge k$ such that

$$d_k = d(\varsigma_{m_k}, \varsigma_{n_k}) \ge \epsilon, \quad d(\varsigma_{m_{k-1}}, \varsigma_{n_k}) < \epsilon, \quad k = 1, 2, 3, \ldots .$$

Now, $\epsilon \le d_k = d(\varsigma_{m_k}, \varsigma_{n_k}) \le d(\varsigma_{m_k}, \varsigma_{m_{k-1}}) + d(\varsigma_{m_{k-1}}, \varsigma_{n_k}) < a_{m_k} + \epsilon$, which implies that $d_k \to \epsilon^+$ as $k \to \infty$. But, now

$$d_k = d(\varsigma_{m_k}, \varsigma_{n_k}) \le d(\varsigma_{m_k}, \varsigma_{m_{k+1}}) + d(\varsigma_{m_{k+1}}, \varsigma_{n_{k+1}}) + d(\varsigma_{n_{k+1}}, \varsigma_{n_k}) \le 2a_k + \phi(d_k).$$

Thus, we have $\epsilon \le \phi(\epsilon)$ as $k \to \infty$, which is a contradiction. Hence our assumption is wrong, so O-sequence $\{\varsigma_n\}$ is a Cauchy O-sequence. Since E is O-complete, then there exists $\varsigma^* \in E$ such that $\{\varsigma_n\} \to \varsigma^*$. Since orthogonal continuity of f implies that $f(\varsigma_n) \to f(\varsigma^*)$, then

$$f(\varsigma^*) = f(\lim_{n\to\infty} \varsigma_n) = \lim_{n\to\infty} f(\varsigma_n) = \lim_{n\to\infty} \varsigma_{n+1} = \varsigma^*.$$

For uniqueness, assume that $\vartheta^* \in E$ such that $f(\vartheta^*) = \vartheta^*$. Also $f^n(\vartheta^*) = \vartheta^*$, now for choice of $\varsigma_0 \in E$, we have $[\varsigma_0 \perp \varsigma^*$ or $\varsigma^* \perp \varsigma_0]$ and $[\varsigma_0 \perp \vartheta^*$ or $\vartheta^* \perp \varsigma_0]$, since f is $\perp$-preserving, we have $[f^n(\varsigma_0) \perp f^n(\varsigma^*)$ or $f^n(\varsigma^*) \perp f^n(\varsigma_0)]$ and $[f^n(\varsigma_0) \perp f^n(\vartheta^*)$ or $f^n(\vartheta^*) \perp f^n(\varsigma_0)]$ for all $n \in \mathbb{N}$. Therefore, by $\phi(t) < t$, $t > 0$,

$$\begin{aligned} d(\varsigma^*, \vartheta^*) = d(f^n(\varsigma^*), f(\vartheta^*)) &\le \phi(d(f^{n-1}(\varsigma^*), f^{n-1}(\vartheta^*))) \\ &< d(f^{n-1}(\varsigma^*), f^{n-1}(\vartheta^*)) < \ldots < d(\varsigma^*, \vartheta^*). \end{aligned}$$

It is a contraction, thus it follows that $\varsigma^* = \vartheta^*$. Finally, let $\varsigma \in E$ be arbitrary. Similarly, we have $[\varsigma_0 \perp \varsigma^*$ and $\varsigma_0 \perp \varsigma]$ or $[\varsigma^* \perp \varsigma_0$ and $\varsigma \perp \varsigma_0]$, and $[f^n(\varsigma_0) \perp f^n(\varsigma^*)$ and $f^n(\varsigma_0) \perp f^n(\varsigma)]$ or $[f^n(\varsigma^*) \perp f^n(\varsigma_0)$ and $f^n(\varsigma) \perp f^n(\varsigma_0)]$ for all $n \in \mathbb{N}$. Hence, for any $n \in \mathbb{N}$ we get

$$\begin{aligned} d(\varsigma^*, f^n(\varsigma)) = d(f^n(\varsigma^*), f^n(\varsigma)) &\leq \phi(d(f^{n-1}(\varsigma^*), f^{n-1}(\varsigma))) \\ &\leq \phi^2(d(f^{n-2}(\varsigma^*), f^{n-2}(\varsigma))) \leq \ldots \leq \phi^n(d(\varsigma^*, \varsigma)) \to 0 \end{aligned}$$

as $n \to \infty$. Hence, proof is complete. □

Example 3. *Let $([0,1], \perp, d)$ be an O-complete metric space, where $\varsigma \perp \vartheta$ if $\varsigma - \vartheta > 0$ and with metric*

$$d(\varsigma, \vartheta) = \begin{cases} |\varsigma - \vartheta|, & \varsigma, \vartheta \in [0,1), \\ \varsigma + \vartheta, & \varsigma = 1 \text{ or } \vartheta = 1. \end{cases}$$

Let a self map f on E be defined as

$$f(\varsigma) = \begin{cases} \varsigma^2/4, & \varsigma, \vartheta \in [0,1), \\ \varsigma - 1, & \varsigma = 1 \text{ or } \vartheta = 1. \end{cases}$$

Now, if we define

$$\phi(t) = \begin{cases} t^2/2, & 0 \leq t < 1, \\ t - 1, & 1 \leq t < \infty, \end{cases}$$

then the hypothesis of Boyd and Wong's theorem is violated, since f is not continuous. As a result, Theorem 1 is useful extension of Boyd and Wong's fixed point theorem.

Theorem 2. *Let $(E, d, \perp)$ be an O-complete metric space and suppose that $f : E \to E$ be $\perp$-continuous and $\perp$-preserving, satisfying*

$$d(f(\varsigma), f(\vartheta)) \leq \phi(d(\varsigma, \vartheta)), \quad \forall\, \varsigma, \vartheta \in E \quad \text{with} \quad \varsigma \perp \vartheta,$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is monotonic non-decreasing function, that satisfies $\lim_{n\to\infty} \phi^n(t) = 0$, $\forall\, t > 0$. Then f has unique fixed point ς^*. Also f is a Picard operator, that is $\lim_{n\to\infty} f^n(\varsigma) = \varsigma^*$ for all $\varsigma \in E$.

Proof. By the definition of orthogonality, there exists an orthogonal element $\varsigma_0 \in E$ such that $(\forall\, \vartheta \in E, \varsigma_0 \perp \vartheta)$ or $(\forall\, \vartheta \in E, \vartheta \perp \varsigma_0)$. It follows that $(\varsigma_0 \perp f(\varsigma_0))$ or $(f(\varsigma_0) \perp \varsigma_0)$. Let $\varsigma_1 = f(\varsigma_0)$, $\varsigma_2 = f(\varsigma_1) = f^2(\varsigma_0)$, $\varsigma_{n+1} = f(\varsigma_n) = f^{n+1}(\varsigma_0)$, $\forall\, n \in \mathbb{N}$. Since f is $\perp$-preserving, $\{\varsigma_n\}$ is an O-sequence. Let $a_n = d(\varsigma_{n-1}, \varsigma_n)$, then

$$\begin{aligned} d(\varsigma_{n+1}, \varsigma_n) = d(f(\varsigma_n), f(\varsigma_{n-1})) &\leq \phi(d(\varsigma_n, \varsigma_{n-1})) = \phi(f(\varsigma_{n-1}), f(\varsigma_{n-2})) \\ &\leq \phi^2(\varsigma_{n-1}, \varsigma_{n-2}) \leq \phi^n(\varsigma_1, \varsigma_0) \end{aligned}$$

as $n \to \infty$, then by the definition of ϕ we get $\lim_{n\to\infty} d(\varsigma_{n+1}, \varsigma_n) = 0$. Now, we show that O-sequence $\{\varsigma_n\}$ is a Cauchy sequence. Also we note that for any $\epsilon > 0$, $\phi(\epsilon) < \epsilon$. Since $\lim_{n\to\infty} a_n = 0$, so for $\epsilon > 0$, we can choose $n \in \mathbb{N}$ such that $a_n \leq \epsilon - \phi(\epsilon)$. Now define $M = \{\varsigma \in E : d(\varsigma, \varsigma_n) < \epsilon\}$ with $\varsigma \perp \vartheta$, then for any $\vartheta \in M$ we have

$$d(f(\vartheta), \varsigma_n) \leq d(f(\vartheta), f(\varsigma_n)) + d(f(\varsigma_n), \varsigma_n) \leq \phi(d(\vartheta, \varsigma_n)) + d(\varsigma_{n-1}, \varsigma_n) \leq \phi(\epsilon) + \epsilon - \phi(\epsilon) \leq \epsilon,$$

this implies that $f(\vartheta) \in M$, i.e. $f(M) \subset M$. It follows that $d(\varsigma_m, \varsigma_n) \leq \epsilon, \forall n \geq m$. By the completeness of X, there exists $\varsigma^* \in E$ such that $\lim_{n\to\infty} \varsigma_n = \varsigma^*$. Since f is $\perp$-continuous, hence $f(\varsigma_n) \to f(\varsigma^*)$, then

$$f(\varsigma^*) = f(\lim_{n\to\infty} \varsigma_n) = \lim_{n\to\infty} f(\varsigma_n) = \lim_{n\to\infty} \varsigma_{n+1} = \varsigma^*.$$

Therefore, ς^* is a fixed point. To prove uniqueness of the fixed point, let $\vartheta^* \in E$ be another fixed point of f different from ς^* such that $f^n(\vartheta^*) = \vartheta^*$, then $d(\varsigma^*, \vartheta^*) > 0$. Now, for choice of $\varsigma_0 \in E$, we have $[\varsigma_0 \perp \vartheta^*]$ or $[\vartheta^* \perp \varsigma_0]$, since f is $\perp$-preserving, we have $[f^n(\varsigma_0) \perp f^n(\vartheta^*)]$ or $[f^n(\vartheta^*) \perp f^n(\varsigma_0)]$ for all $n \in \mathbb{N}$. Then

$$d(\varsigma^*, \vartheta^*) = d(f^n(\varsigma^*), f^n(\vartheta^*)) \leq \phi(d(f^{n-1}(\varsigma^*), f^{n-1}(\vartheta^*))) \leq \ldots \leq \phi^n(d(\varsigma^*, \vartheta^*)),$$

$\lim_{n\to\infty} d(f(\varsigma^*), \vartheta^*) = 0$ this implies that $\varsigma^* = \vartheta^*$.

Finally, let $\varsigma \in E$ be arbitrary. Similarly, we have

$$[\varsigma_0 \perp \varsigma^* \text{ and } \varsigma_0 \perp \varsigma] \text{ or } [\varsigma^* \perp \varsigma_0 \text{ and } \varsigma \perp \varsigma_0],$$

and

$$[f^n(\varsigma_0) \perp f^n(\varsigma^*) \text{ and } f^n(\varsigma_0) \perp f^n(\varsigma)] \text{ or } [f^n(\varsigma^*) \perp f^n(\varsigma_0) \text{ and } f^n(\varsigma) \perp f^n(\varsigma_0)]$$

for all $n \in \mathbb{N}$. Hence, for all $n \in \mathbb{N}$, we get

$$\begin{aligned} d(\varsigma^*, f^n(\varsigma)) = d(f^n(\varsigma^*), f^n(\varsigma)) &\leq \phi(d(f^{n-1}(\varsigma^*), f^{n-1}(\varsigma))) \\ &\leq \phi^2(d(f^{n-2}(\varsigma^*), f^{n-2}(\varsigma))) \leq \ldots \leq \phi^n(d(\varsigma^*, \varsigma)) \to 0 \end{aligned}$$

as $n \to \infty$. This completes the proof. □

Example 4. *Given a function*

$$f(\varsigma) = \begin{cases} \varsigma/5, & \varsigma \in [0, 1/2), \\ \varsigma/6, & \varsigma \in [1/2, 1]. \end{cases}$$

It satisfies the condition of Theorem 2, where $\varsigma \perp \vartheta$ if $\varsigma \geq \vartheta \geq 0$ with usual metric d and ϕ is given by $\phi(t) = t/5$. Hence, the hypothesis of Matkowski's theorem is violated, since f is not continuous. As a result, Theorem 2 is a useful extension of Matkowski's fixed point theorem.

Remark 3. *The main result of M.E. Gordji et. al. [3] is the extension of Banach contraction principle. In this case, if we use $\phi(t) = \alpha t$, the number $\alpha \in [0, 1)$ is such that $d(f(\varsigma), f(\vartheta)) \leq \alpha d(\varsigma, \vartheta)$ with $x \perp y$.*

3 Application

Here, in this part, we discuss usefulness of our main result discussed in previous section of the article by investigating the existence and uniqueness of solution of differential equation of first order with periodic boundary condition.

Consider

$$\begin{aligned} &\mu'(t) = g(t, \mu(t)), \quad t \in [0, \lambda], \\ &\mu(0) = \mu(\lambda), \end{aligned} \tag{1}$$

where $\lambda > 0$ and $g : [0, \lambda] \times \mathbb{R} \to \mathbb{R}$ is continuous function. Let $E = C[0, \lambda]$ be the set of all continuous functions and the metric is $d(\mu, \nu) = \sup_{t\in[0,\lambda]} |\mu(t) - \nu(t)|$ with $\mu \perp \nu$ if $\mu(t) \leq \nu(t), \forall\, t \in [0, \lambda]$.

Theorem 3. *Consider a first order differential equation with boundary condition mentioned in equation* (1) *and suppose that there exists some* $K > 0$ *such that for* $s_1, s_2 \in \mathbb{R}$ *with* $s_1 \geq s_2$

$$0 \leq g(t, s_1) + Ks_1 - [g(t, s_2) + Ks_2] \leq K\phi(s_1 - s_2),$$

where ϕ *is a function given in Theorem 2. If there exists a lower solution for differential equation* (1)*, then this implies that differential equation* (1) *has unique solution.*

Proof. It is possible to rewrite problem (1) as

$$\begin{aligned} \mu'(t) + K\mu(t) &= g(t, \mu(t)) + K\mu(t), \quad t \in [0, \lambda], \\ \mu(0) &= \mu(\lambda), \end{aligned}$$

then it is identical to the integral equation of the form

$$\mu(t) = \int_0^\lambda G(t, s)[g(s, \mu(s)) + K\mu(s)]ds,$$

where

$$G(t, s) = \begin{cases} e^{K(\lambda+s-t)}/e^{K\lambda-1}, & 0 \leq s < t \leq \lambda, \\ e^{K(s-t)}/e^{K\lambda-1}, & 0 \leq t < s \leq \lambda. \end{cases}$$

Let a mapping $T : E \to E$ be defined by

$$(T\mu)(t) = \int_0^\lambda G(t, s)[g(s, \mu(s)) + K\mu(s)]ds.$$

It is evident that a fixed point of T is a solution to the preceding problem (1). Now we will demonstrate that the hypothesis in Theorem 3 is satisfied.

As $\mu \perp \nu$ if $\mu(t) \leq \nu(t), \forall\, t \in [0, \lambda]$, from the hypothesis we obtain

$$g(t, \mu(t)) + K\mu(t) \leq g(t, \nu(t)) + K\nu(t), \quad \forall\, t \in [0, \lambda].$$

As $G(t, s) > 0, \forall\, t, s \in [0, \lambda]$, we have

$$(T\mu)(t) = \int_0^\lambda G(t, s)[g(s, \mu(s)) + K\mu(s)]ds \leq \int_0^\lambda G(t, s)[g(s, \nu(s)) + K\nu(s)]ds = (T\nu)(t).$$

Hence, T is $\perp$-preserving.

Let $\{\mu_n\}$ be an O-Cauchy sequence converging to $\mu \in E$. Then

$$\mu_0(t) \leq \mu_1(t) \leq \mu_2(t) \leq \mu_3(t) \leq \ldots \leq \mu_n(t) \leq \ldots \leq \mu(t), \quad \forall\, t \in [0, \lambda],$$

this implies that $\mu_n \perp \mu, \forall t \in [0, \lambda]$. As T is $\perp$-preserving, then $f(\mu_n) \to f(\mu)$. Therefore, T is O-continuous.

Now, assume that there exists a lower solution, say $\mu_0 \in X$, such that $\mu_0'(t) \leq g(t, \mu_0(t))$, which may be rewritten in the following way

$$\mu_0'(t) + K\mu_0(t) \leq g(t, \mu_0(t)) + K\mu_0(t) \quad \text{for} \quad t \in [0, \lambda].$$

Multiplying e^{Kt} to above inequality, gives

$$(\mu_0(t)e^{Kt})' \leq [g(t, \mu_0(t)) + K\mu_0(t)]e^{Kt} \quad \text{for} \quad t \in [0, \lambda],$$

and thus, we have

$$\mu_0(t)e^{Kt} \leq \mu_0(0) + \int_0^t [g(s,\mu_0(s)) + K\mu_0(s)]e^{Ks}ds \quad \text{for} \quad t \in [0,\lambda], \tag{2}$$

which implies that

$$\mu_0(0)e^{K\lambda} \leq \mu_0(\lambda)e^{K\lambda} \leq \mu_0(0) + \int_0^\lambda [g(s,\mu_0(s)) + K\mu_0(s)]e^{Ks}ds,$$

thereby yielding

$$\mu_0(0) \leq \int_0^\lambda \frac{e^{Ks}}{e^{K\lambda-1}}[g(s,\mu_0(s)) + \mu_0(s)]ds.$$

Using the inequality (2), we get that

$$\begin{aligned}
\mu_0(t)e^{Kt} &\leq \int_0^t [g(s,\mu_0(s)) + \mu_0(s)]e^{Ks}ds + \int_0^\lambda \frac{e^{Ks}}{e^{K\lambda-1}}[g(s,\mu_0(s)) + \mu_0(s)]ds \\
&= \int_0^t \frac{e^{K(s+\lambda)}}{e^{K\lambda-1}}[g(s,\mu_0(s)) + \mu_0(s)]ds + \int_t^0 \frac{e^{Ks}}{e^{K\lambda-1}}[g(s,\mu_0(s)) + \mu_0(s)]ds \\
&\quad + \int_0^\lambda \frac{e^{Ks}}{e^{K\lambda-1}}[g(s,\mu_0(s)) + \mu_0(s)]ds \\
&= \int_0^t \frac{e^{K(s+\lambda)}}{e^{K\lambda-1}}[g(s,\mu_0(s)) + \mu_0(s)]ds + \int_t^\lambda \frac{e^{Ks}}{e^{K\lambda-1}}[g(s,\mu_0(s)) + \mu_0(s)]ds.
\end{aligned}$$

Hence,

$$x_0(t) \leq \int_0^t \frac{e^{K(s+\lambda-t)}}{e^{K\lambda-1}}[g(s,\mu_0(s)) + \mu_0(s)]ds + \int_t^\lambda \frac{e^{K(s-t)}}{e^{K\lambda-1}}[g(s,\mu_0(s)) + \mu_0(s)]ds,$$

i.e.

$$x_0(t) \leq \int_0^t G(t,s)[g(s,\mu_0(s)) + \mu_0(s)]ds = (T\mu_0(t)).$$

Hence, T possesses a fixed point in E. □

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Сінгх Б., Сінгх В., Уддін І., Акар О. *Теореми про фіксовану точку на ортогональному метричному просторі з використанням стиску типу Матковського* // Карпатські матем. публ. — 2022. — Т.14, №1. — С. 127–134.

У цій роботі доведено теореми Бойда-Вонга та Матковського про нерухомі точки в ортогональному метричному просторі, який був означений М.Е. Горджі у 2017 році і є розширенням метричного простору. Наведено кілька прикладів на підтвердження основних результатів. Застосовано отримані результати для встановлення існування єдиного розв'язку періодичної крайової задачі.

Ключові слова і фрази: нерухома точка, ортогональна множина, ортогональний метричний простір, метричний простір.