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Accelerated Krasnoselski-Mann type algorithm for hierarchical fixed point and split monotone variational inclusion problems in Hilbert spaces

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In this paper, a new accelerated extrapolation Krasnoselski-Mann type algorithm for finding common element in the solution set of the hierarchical fixed point and split monotone variational inclusion problems are introduced in the setting of a real Hilbert space. We then prove that a sequence generated by the algorithm converges strongly to such common element which also approximates solution of some fixed point problem of demimetric mapping in the space. Finally, some applications and numerical experiment are given to show effectiveness of the proposed algorithm over the recently known related results in the literature. The established results extend and generalize many recent ones announced in the literature.

Key words and phrases: hierarchical fixed point problem, split monotone inclusion problem, variational inequality problem, demimetric mapping, inverse strongly monotone operator.

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Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. A mapping $T : C \to H$ is called nonexpansive if $\|Tu - Tv\| \le \|u - v\|$ for all $u, v \in C$. Let $F(T)$ denotes the fixed point set of T, i.e.

$$F(T) = \{u \in C : \ u = Tu\}.$$

The theory of variational inequalities which was first studied by G. Grillo and G. Stampacchia [15] is one of the important tool in the study of different classes of problems arising in several branches of pure and applied sciences in a unified and general framework. There are currently many efficient methods for solving variational inequalities in the literature (see, e.g., [1–3, 10, 11, 13, 15, 17, 32] and the references therein).

Let $A : C \to H$ be a nonlinear mapping. The variational inequality problem (VIP for short) associated with C and A is defined as follows:

$$\text{find } \ u^* \in C \ \text{ such that } \ \langle Au^*, u^* - u\rangle \le 0 \ \ \forall\, u \in C. \tag{1}$$

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In particular, if C in VIP (1) is replaced with the set $F(T)$ of fixed points of a nonexpansive self mapping T on C and A is of the form $A := I - S$, where I denotes the identity mapping on C and S is a nonexpansive self map of C, then VIP (1) is of the form

$$\text{find } u^* \in C \text{ such that } \langle u^* - Su^*, u^* - u\rangle \leq 0 \ \forall\, u \in F(T). \tag{2}$$

This problem (2) is called hierarchical fixed point problem (HFPP for short) which was first introduced and studied by A. Moudafi and P.-E. Mainge [22].

Let $\Theta := \{u^* \in C : \ (P_{F(T)} \circ S)u^* = u^*\}$ denotes the solution of HFPP (2). Then finding solution of (2) is equivalent of solving the fixed point problem:

$$\text{find } u^* \in C \text{ such that } u^* = (P_{F(T)} \circ S)u^*. \tag{3}$$

Furthermore, observe that by employing the normal cone $N_{F(T)}$ of $F(T)$ defined by

$$N_{F(T)} = \begin{cases} \{w \in H : \ \langle w, v - u\rangle \leq 0,\ \forall\, v \in F(T)\}, & \text{if } u \in F(T), \\ \varnothing, & \text{otherwise,} \end{cases}$$

it can easily be proved that the HFPP (2) is equivalent to the following variational inclusion problem: find $u^* \in C$ such that $0 \in N_{F(T)}u^* + (I - S)u^*$. We know that HFPP (2) includes as special case a variational inequality problem over a fixed point set, that is, the so called hierarchical variational inequality problem (HVIP for short). In fact, if F is k-strongly monotone and L-Lipschitzian with $\gamma \in (0, 2k/L^2)$, then by setting $S = I - \gamma F$ we get the following HVIP studied by I. Yamada and N. Ogura [32]: find $u^* \in F(T)$ such that $\langle Fu^*, u^* - u\rangle \leq 0$ for all $u \in F(T)$. Based on the relation (3), the HFPP (2) is noted to have an iterative algorithm $x_{n+1} = (P_{F(T)} \circ S)x_n$. If the mapping S is averaged rather than just a nonexpansive and the fixed point of $P_{F(T)} \circ S$ exists, then the algorithm converges. This method is observed to have disadvantage due to some difficulty in computing the operator $P_{F(T)} \circ S$. To overcome the difficulty, A. Moudafi [23] introduced the following Krasnoselski-Mann type algorithm for solving HFPP (2)

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\big(\sigma_n Sx_n + (1 - \sigma_n)Tx_n\big), \quad \forall\, n \geq 0,$$

where $\{\alpha_n\}$ and $\{\sigma_n\}$ are two real sequences in $(0, 1)$. It is worthy to note that some algorithms in signal processing and image reconstruction can be written as the Krasnoselski-Mann iterative algorithm, which provides a unified frame for analysing various concrete algorithms (see, e.g., [8,12,33]). Since then, many researchers developed and analysed iterative algorithms for finding common element of solution sets of HFPP (2) and other problems. For example, K.R. Kazmi et al. [18] developed and analysed the following Krasnoselski-Mann type algorithm

$$\begin{cases} x_0 \in C, \\ u_n = (1 - \alpha_n)x_n + \alpha_n\big(\sigma_n Sx_n + (1 - \sigma_n)Tx_n\big), \\ x_{n+1} = U\big(u_n + \gamma A^*(V - I)Au_n\big) \quad \forall\, n \geq 0, \end{cases} \tag{4}$$

where $U = T^F_{r_n}(I - r_n f)$, $V = T^G_{r_n}(I - r_n g)$, $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ are bifunctions, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are θ_1 and θ_2 inverse strongly monotone mappings respectively, $S, T : C \to C$ are nonexpansive with step size $\lambda \in (0, 1/||A||^2)$ and $\{\alpha_n\}$, $\{\sigma_n\}$ are real sequences in $(0, 1)$ satisfying: $\sum_{n=0}^{\infty} \sigma_n < \infty$, $\lim_{n\to\infty} ||x_n - u_n||/(\alpha_n\sigma_n) = 0$ and $\liminf_{n\to\infty} r_n > 0$.

They proved that the algorithm (4) approximates to a common element in the solution sets of the HFPP (2) and the following split mixed equilbrium problem of finding $x \in C$ such that

$$F(x^*, x) + \langle f x^*, x - x^* \rangle \geq 0, \quad \forall\, x \in C,$$

and such that $y^* = Ax^* \in Q$ solves

$$G(y^*, y) + \langle g y^*, y - y^* \rangle \geq 0, \quad \forall\, y \in Q.$$

In another development, they extended their study to include the following split monotone variational inclusion problem (SMVIP for short) which was first introduced and studied by A. Moudafi [21]:

$$\text{find } \; x^* \in H_1 \; \text{ such that } \; 0 \in f(x^*) + M(x^*) \tag{5}$$

and such that

$$y^* = Ax^* \in H_2 \; \text{ solves } \; 0 \in g(y^*) + N(y^*), \tag{6}$$

where $M : H_1 \to 2^{H_1}$ and $N : H_2 \to 2^{H_2}$ are multi-valued maximal monotone mappings, $A : H_1 \to H_2$ is a bounded linear operator. Let the solution set of the SMVIP (5)–(6) be denoted by Ω, i.e. $\Omega = \{x^* \in H_1 : \; x^* \in \mathrm{Sol}(\mathrm{MVIP}\,(5)) \text{ and } Ax^* \in \mathrm{Sol}(\mathrm{MVIP}\,(6))\}$. They prove that the sequence $\{x_n\}$ iteratively generated by

$$\begin{cases} x_0 \in H_1, \\ u_n = (1-\alpha_n)x_n + \alpha_n(\sigma_n S x_n + (1-\sigma_n) T x_n), \\ x_{n+1} = U(u_n + \gamma A^*(V - I)Au_n) \quad \forall\, n \geq 0, \end{cases} \tag{7}$$

converges weakly to $x \in \Theta \cap \Omega$, where $U = J_\lambda^M (I - \lambda f)$, $V = J_\lambda^N (I - \lambda g)$ and $\lambda \subset (0, \alpha)$ with $\alpha = 2\min\{\theta_1, \theta_2\}$.

J.K. Kim and P. Majee [19] modified algorithm (4) by replacing the nonexpansive self mapping T with averaged of finite family $\{T_i\}_{i=1}^N$ of k_i-strictly pseudocontractive non-self mappings and choosing a step size that does not require prior knowledge of the operator norm. They proved that the algorithm

$$\begin{cases} x_0 \in C, \\ u_n = (1-\alpha_n)x_n + \alpha_n(\tau_n S x_n + (1-\tau_n) T_N^n T_{N-1}^n \ldots T_1^n x_n), \\ x_{n+1} = U(u_n + \gamma A^*(V - I)Au_n) \quad \forall n \; \geq 1, \end{cases} \tag{8}$$

approximates to a common solution of the split mixed equilibrium problem and hierarchical fixed point problem. To speed up the rate of convergence of algorithm (8), P. Chuasuk and A. Kaewcharoen [9] recently proposed and analysed the following inertial Krasnoselki-Mann type algorithm for approximating a common solution of a hierarchical fixed point problems for k-strictly pseudocontractive non-self mappings and the split generalized mixed equilibrium

$$\begin{cases} x_0, x_1 \in C, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = (1-\alpha_n)w_n + \alpha_n(\beta_n S w_n + (1-\beta_n) T_n^N T_n^{N-1} \ldots T_n^1 w_n), \\ x_{n+1} = U(u_n + \delta_n A^*(V - I)Au_n) \quad \forall\, n \geq 1, \end{cases}$$

with step size $\delta_n := \sigma_n \|(T^G_{r_n}(I - r_n g) - I)Au_n\|^2 / \|A^*(T^G_{r_n}(I - r_n g) - I)Au_n\|^2$. Very recently, D.-J. Wen [31] modified algorithm (7) by replacing the nonexpansive self mapping T with $(1 - \mu_n D)$ and prove that the sequence $\{x_n\}$ iteratively generated by

$$\begin{cases} x_0 \in H_1, \\ u_n = (1-\alpha_n)x_n + \alpha_n(\sigma_n S x_n + (1-\sigma_n)(1-\mu_n D)x_n), \\ x_{n+1} = U(u_n + \gamma A^*(V - I)Au_n) \quad \forall\, n \geq 0, \end{cases} \tag{9}$$

converges strongly to an element of $\Theta \cap \Omega$, where D is a strong monotone operator and $\{\mu_n\}$ is a positive real sequence satisfying: $\lim_{n\to\infty} \mu_n = 0$, $\sum_{n=0}^{\infty} \mu_n = \infty$ and $\lim_{n\to\infty} (\mu_n - \mu_{n-1})/\mu_n = 0$.

Motivated and inspired by the results of A. Moudafi [21], K.R. Kazmi et al. [18], P. Chuasuk and A. Kaewcharoen [9], K.P. Kim and K. Majee [19] and D.-J. Wen [31], we introduce a new accelerated extrapolation Krasnoselski-Mann type algorithm (see algorithm 1 below) for finding common element in the solution set of the HFPP and SMVIP which also approximates some solution of fixed point problem of demimetric mapping in the setting of real Hilbert spaces. In respect to this, the following motivations that signify the contributions of our proposed method (algorithm) are highlighted:

(a) the proposed method involves inertia term that speed up the convergence rate;

(b) the implementation of our iterative algorithm does not need any prior knowledge about bounded linear operator norms;

(c) the algorithm involves a class of demimetric mappings which is known to include as special cases, many important classes of nonlinear mappings such as nonexpansive, quasi-nonexpansive and demicontractive etc. (see [28] for more details);

(d) the established result extend and generalize the corresponding ones in A. Moudafi [21] and K.R. Kazmi et. al. [18], from weak convergence to strong convergence;

(e) the result improved the corresponding ones in D.-J. Wen [31] and K.P. Kim and K. Majee [19], in the sense that it solves some fixed point problem of demimetric mapping in addition to HFPP and SMVIP with faster rate of convergence;

(f) as application, we used our proposed algorithm to solve the split variational inequality problem and split convex minimization problem;

(g) finally, we give numerical example to illustrate the convergence behaviour of our proposed algorithm with efficiency over some related results in literature.

1 Preliminaries

Throughout this section, the symbols "$\to$" and "$\rightharpoonup$" represent the strong and weak convergences, respectively. A mapping $T : C \to H$ is called

(1) L-Lipschitz continuous with $L > 0$ if for all $x, y \in C$

$$\|Tx - Ty\| \leq L\|x - y\|;$$

if $L = 1$, then T is called nonexpansive;

(2) quasi-nonexpansive if $\|Tx - y\| \leq \|x - y\|$ for all $x \in C, y \in F(T)$;

(3) generalized hybrid [20] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2, \quad \forall\, x, y \in C;$$

(4) h-demicontractive [16] if $F(T) \neq \varnothing$ and there exists $h \in [0,1)$ such that

$$\|Tx - y\|^2 \leq \|x - y\|^2 + h\|x - Tx\|^2 \quad \text{for all} \quad x \in C, y \in F(T);$$

(6) h-demimetric [27] if $F(T) \neq \varnothing$ and there exists $h \in (-\infty, 1)$ such that for any $x \in C$ and $y \in F(T)$, we have

$$\langle x - y, x - Tx\rangle \geq \frac{1-h}{2}\|x - Tx\|^2.$$

Observe that $(1,0)$-generalized hybrid mapping is nonexpansive and every generalized hybrid mapping with nonempty fixed point set is quasi-nonexpansive. Also, the class of h-demicontractive covers that of nonexpansive and quasi-nonexpasive. The class of h-demimetric mappings includes that of h-demicontractive and generalized hybrid mappings as special cases. In fact, every generalized hybrid mapping with nonempty fixed point is a 0-demimetric mapping.

Definition 1. *Let B be a nonlinear operator from H into H and $x, y \in H$. Then B is said to be*

(a) monotone if $\langle Bx - By, x - y\rangle \geq 0$;

(b) α-inverse strongly monotone, if there exists $\alpha > 0$ such that $\langle Bx - By, x - y\rangle \geq \alpha\|Bx - By\|^2$;

(c) β-strongly monotone if there exists a constant $\beta > 0$ such that $\langle Bx - By, x - y\rangle \geq \beta\|x - y\|^2$.

The variational inequality problem introduced and studied by G. Grillo and G. Stampacchia [15] is to find $u \in C$ such that $\langle Bu, v - u\rangle \geq 0$ for all $v \in C$, where C is a nonempty closed and convex subset of H. We denot the set of solution of variational inequality problem by $VI(C, B)$.

Note that if B is α-inverse strongly monotone mapping, then:

(i) B is monotone and $1/\alpha$-Lipschitz continuous;

(ii) $I - \lambda B : C \to H$ is nonexpansive for any $\lambda \in (0, 2\alpha)$.

See [5,29] for more results of inverse strongly monotone mapping.

If the operator B is multivalued, that is $B : H \to 2^H$ with effective domain denoted by $D(B) := \{x \in H : \ Bx \neq \varnothing\}$, then B is said to be monotone operator on H if $\langle x - y, u - v\rangle \geq 0$ for $x, y \in D(B)$, $u \in Bx$, $v \in By$. A monotone operator B on H is said to maximal if its graph is not contained in the graph of any other monotone operator on H. For any maximal monotone operator B on H and $r > 0$, we define a single-valued operator called resolvent operator of B for r as $J_r := (I + rB)^{-1} : H \to D(B)$. It is well known that the resolvent operator J_r of B for $r > 0$ is firmly nonexpansive that is

$$\|J_r x - J_r y\|^2 \leq \langle x - y, J_r x - J_r y\rangle \quad \forall\, x, y \in H.$$

And the set of null points of B is defined as $B^{-1}0 = \{z \in H : \ 0 \in Bz\}$. It is known that $B^{-1}0 = F(J_r)$ and $B^{-1}0$ is closed and convex (see [26]).

Definition 2 ([7,22]). *A sequence $\{B_n\}$ of maximal monotone mappings defined on H is said to be graph convergent to B if $\{\text{graph}(B_n)\}$ converges to $\{\text{graph}(B)\}$ in the sense of Kuratowski-Painleve's, i.e.* $\limsup_{n\to\infty} \text{graph}(B_n) \subset \text{graph(B)} \subset \liminf_{n\to\infty} \text{graph}(B_n)$.

Lemma 1 ([7]).

(1) Let B be a maximal monotone mapping on H, then $\{t_n B\}$ graph converges to $N_{B^{-1}(0)}$ as $t_n \to 0$ provided that $B^{-1}(0) \neq \varnothing$.

(2) Let $\{B_n\}$ be a sequence of maximal monotone mappings on H which graph converges to B defined on H. If A is a Lipschitz maximal monotone operator on H, then $\{A + B_n\}$ graph converges to $A + B$ and $A + B$ is maximal monotone.

Lemma 2 ([29]). *Let H be a real Hilbert space. Then for all $x, y \in H$ and $\alpha \in \mathbb{R}$, the following hold:*

(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;

(2) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2$.

Lemma 3 ([14]). *Let $T : C \to H$ be a nonexpansive mapping, then T is demiclosed on C in the sense that if $\{x_n\}$ converges weakly to $x \in C$ and $\{x_n - Tx_n\}$ converges strongly to 0 then $x \in F(T)$.*

Lemma 4 ([28,30]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $k \in (-\infty, 0)$ and let T be a k-demimetric mapping of C into H such that $F(T) \neq \varnothing$. Let λ be a real number with $0 < \lambda \leq 1 - k$ and defined $S = (1-\lambda) + \lambda T$. Then:*

(i) $F(T) = F(S)$;

(ii) $F(T)$ *is closed and convex;*

(iii) S *is a quasi-nonexpansive mapping of C into H.*

The following lemmas play key role in the prove of our main results.

Lemma 5 ([4]). *Let H be a real Hilbert space and $F : H \to H$ a β-strongly monotone and L-Lipschitz continuous mapping on H. If $\alpha \in (0,1)$, $\eta \in [0, 1-\alpha]$ and $\mu \in (0, 2\beta/L^2)$, then for all $x, y \in H$, we have*

$$\|[(1-\eta)x - \alpha\mu F(x)] - [(1-\eta)y - \alpha\mu F(y)]\| \leq (1 - \eta - \alpha\delta)\|x - y\|,$$

where $\delta = 1 - \sqrt{1 - \mu(2\beta - \mu L^2)} \in (0, 1]$.

Lemma 6 ([25]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0,1)$ with condition $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n \quad \forall\, n \geq 1.$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0, \quad \text{then} \quad \lim_{n\to\infty} a_n = 0.$$

2 Main results

We begin this with the following assumptions under which the strong convergence results are established.

Assumption 1. *Let H_1 and H_2 be two real Hilbert spaces and C and Q nonempty closed and convex subsets of H_1 and H_2, respectively. Suppose the following conditions are satisfied:*

(C1) $E : H_1 \to 2^{H_1}$ *and* $G : H_2 \to 2^{H_2}$ *are maximal monotone operators with resolvents* $J_\mu^E = (I + \mu E)^{-1}$ *and* $T_\mu^G = (I + \mu G)^{-1}$ *for E and G, respectively;* $f : C \to H_1$ *and* $g : Q \to H_2$ *are* α_1*- and* α_2*- inverse strongly monotone mappings;* $U := J_\mu^E(I - \mu f)$ *and* $V = T_\mu^G(I - \mu g)$*, where* $\mu \in (0, \alpha)$, $\alpha := \min\{2\min\{\alpha_1, \alpha_2\}, 1\}$*, and* $S, T : C \to C$ *are nonexpansive mappings;* $A : H_1 \to H_2$ *is a bounded linear operator such that* $A \neq 0$*;*

(C2) $F : H_1 \to H_1$ *is β-strongly monotone and L-Lipschitz continuous operator on H_1 with* $L > 0$ *such that* $\tau = 1 - \sqrt{1 - \eta(2\beta - \eta L^2)}$*, where* $\eta \in (0, 2\beta/L^2)$*;*

(C3) $W : H_1 \to H_1$ *is an h-demimetric mapping with* $h \in (-\infty, 1)$ *such that W is demiclosed at zero and* $K := (1 - \kappa) + \kappa W$*, where* $\kappa \in (0, 1 - h)$*;*

(C4) $\{c_n\}$ *is a positive sequence with* $c_n = \circ(\gamma_n)$*,* $\{\beta_n\} \subset (\gamma, 1 - \gamma_n)$ *for some* $\gamma > 0$*, a sequence* $\{\gamma_n\} \subset (0,1)$ *satisfies* $\lim\limits_{n\to\infty} \gamma_n = 0$*,* $\sum\limits_{n=1}^{\infty} \gamma_n = \infty$ *and* $\{\zeta_n\}$ *is a sequence in* $(0,1)$ *such that* $\zeta_n \in [a, b] \subset (0,1)$*;*

(C5) *solution set* $\Gamma = \Theta \cap \Omega \cap F(W)$ *is nonempty.*

In this section, using accelerated extrapolation Krasnoselski-Mann type method, a modified iterative algorithm for solving hierarchical fixed point and split monotone variational inclusion problems is constructed. It also approximates some solution of fixed point problem of demimetric mapping.

Algorithm 1.

Initialization. *Choose* $x_0, x_1 \in H_1$ *to be arbitrary.*

Iterative Steps. *Calculate* x_{n+1} *as follows.*

Step 1. *Given the iterates* x_{n-1} *and* x_n *for each* $n \geq 1$*, choose* θ_n *such that* $0 \leq \theta_n \leq \bar{\theta}_n$*, where*

$$\bar{\theta}_n = \begin{cases} \min\{\theta, c_n/\|x_n - x_{n-1}\|\}, & \text{if } x_n \neq x_{n-1}, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Step 2. *Compute*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ v_n = (1 - \alpha_n)w_n + \alpha_n(\sigma_n S w_n + (1 - \sigma_n)T w_n), \\ u_n = U(v_n - \lambda_n A^*(I - V)Av_n), \end{cases} \tag{11}$$

where $\{\alpha_n\}$ *and* $\{\sigma_n\}$ *are real sequences in* $(0,1)$ *and for any fixed value* $\epsilon > 0$*, the step size* λ_n *is chosen as follows*

$$0 < \epsilon \leq \lambda_n \leq \frac{\|(I - V)Av_n\|^2}{\|A^*(I - V)Av_n\|^2} - \epsilon, \tag{12}$$

if $Av_n \neq VAv_n$*, otherwise* $\lambda_n = \lambda$, $\lambda \geq 0$.

Step 3. Compute

$$\begin{cases} z_n = (1-\zeta_n)Ku_n + \zeta_n u_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)z_n - \gamma_n \eta F(z_n), \quad n \in \mathbb{N}. \end{cases} \tag{13}$$

Set $n := n+1$ and return to Step 1.

Remark. *We know that from (C4) of Assumption 1, $c_n = o(\gamma_n)$, i.e. $\lim_{n\to\infty} c_n/\gamma_n = 0$. Also, from (10), $\theta_n \leq \bar{\theta_n} \leq c_n/\|x_n - x_{n-1}\|$ for all $n \geq 1$ and $x_n \neq x_{n-1}$. This implies $\theta_n \leq c_n/\|x_n - x_{n-1}\|$. Thus,*

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \frac{c_n}{\alpha_n} \to 0 \quad as \quad n \to \infty. \tag{14}$$

We begin with the following lemma that the step size of Algorithm 1 is well-defined.

Lemma 7. *The step size in* (12) *is well-defined.*

Proof. By *(C1)* of Assumption 1, is easy to see that $(I - r_n g)$ and $T^G_{r_n}$ are nonexpansive mappings. Hence $V := T^G_{r_n}(I - r_n g)$ is also nonexpansive mapping. Since by *(C5)*, $\Gamma \neq \emptyset$, let $p \in \Gamma$, then $VAp = Ap$, so V is nonexpansive mapping with nonempty fixed point, which means that V is 0-demimetric mapping, thus

$$\begin{aligned} \|v_n - p\|\|A^*(Av_n - VAv_n)\| &\geq \langle v_n - p, A^*(Av_n - VAv_n)\rangle \\ &= \langle Av_n - Ap, Av_n - VAv_n\rangle \geq \frac{1}{2}\|Av_n - VAv_n\|^2. \end{aligned} \tag{15}$$

If $Av_n \neq VAv_n$, then $\|Av_n - VAv_n\| > 0$, so from (15) we have $\|u_n - p\|\|Au_n - VAu_n\| > 0$. Hence, $\|A^*(Av_n - VAv_n)\| \neq 0$ and therefore the step size λ_n in (12) is well-defined. □

Next, we show that the sequence defined by Algorithm 1 is bounded.

Lemma 8. *Let $\{x_n\}$ be the sequence generated by Algorithm 1 such that Assumption 1 holds, then $\{x_n\}$ is bounded.*

Proof. Let $p \in \Gamma$, from (11) of Step 2, we have

$$\begin{aligned} \|v_n - p\| &= \|(1-\alpha_n)w_n + \alpha_n(\sigma_n Sw_n + (1-\sigma_n)Tw_n) - p\| \\ &\leq (1-\alpha_n)\|w_n - p\| + \alpha_n(\sigma_n\|Sw - n\| + (1-\sigma_n)\|Tw_n - p\|) \leq \|w_n - p\|. \end{aligned} \tag{16}$$

But

$$\begin{aligned} \|w_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\ &\leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\| = \|x_n - p\| + \gamma_n . \frac{\theta_n}{\gamma_n}\|x_n - x_{n-1}\|. \end{aligned}$$

We know from (14) that $\{(\theta_n/\gamma_n)\|x_n - x_{n-1}\|\}$ converges to zero as $n \to \infty$. Therefore, it is bounded and so there exists $M > 0$ such that $(\theta_n/\gamma_n)\|x_n - x_{n-1}\| \leq M$ for all $n \neq 1$. Hence,

$$\|w_n - p\| \leq \|x_n - p\| + \gamma_n M. \tag{17}$$

It follows from (16) and (17) that

$$\|v_n - p\| \leq \|x_n - p\| + \gamma_n M. \tag{18}$$

Also, from the step size λ_n in (12), we have

$$\lambda_n \leq \frac{\|Av_n - VAv_n\|^2}{\|A^*(Av_n - VAv_n)\|^2} - \epsilon$$

if and only if

$$\epsilon\|A^*(Av_n - VAv_n)\|^2 \leq \|Av_n - VAv_n\|^2 - \lambda_n\|A^*(1-V)Av_n\|^2. \tag{19}$$

Thus with left side of λ_n in (12) and (19), we get

$$\begin{aligned} \epsilon^2\|A^*(Av_n - VAv_n)\|^2 &< \lambda_n\epsilon\|A^*(Av_n - VAv_n)\|^2 \\ &\leq \lambda_n\left[\|Av_n - VAv_n\|^2 - \lambda_n\|A^*(I-V)Av_n\|^2\right]. \end{aligned} \tag{20}$$

Using the fact that $p \in \Gamma$, we have $Up = p$ and $VAp = Ap$. Now let $y_n = v_n - \lambda_n A^*(I-V)Av_n$ so that with the condition that V is 0-demimetric, (19) and (20), we get

$$\begin{aligned} \|y_n - p\|^2 &= \|v_n - \lambda_n A^*(I-V)Av_n - p\|^2 \\ &= \|v_n - p\|^2 - 2\lambda_n\langle v_n - p, A^*(I-V)Av_n\rangle + \|\lambda_n A^*(I-W)Av_n\|^2 \\ &= \|v_n - p\|^2 - 2\lambda_n\langle Av_n - Ap, (I-V)Av_n\rangle + \lambda_n^2\|A\|^2\|(I-V)Av_n\|^2 \\ &\leq \|v_n - p\|^2 - \lambda_n\|(I-V)Av_n\|^2 + \lambda_n^2\|A^*(I-V)Av_n\|^2 \\ &= \|v_n - p\|^2 - \lambda_n\Big(\|Av_n - VAv_n\|^2 - \lambda_n\|A^*(1-V)Av_n\|^2\Big) \\ &\leq \|v_n - p\|^2 - \epsilon^2\|A^*(Av_n - VAv_n)\|^2 \qquad (21) \\ &\leq \|v_n - p\|^2. \qquad (22) \end{aligned}$$

Using the fact that $U = J_\mu^E(I - \mu f)$, where J_μ^E is nonexpansive and f is α_1-inverse strongly monotone mapping, we get

$$\begin{aligned} \|_n - p\|^2 &= \|Uy_n - Up\|^2 = \|J_\mu^E(I-\mu f)y_n - J_\mu^E(I-\mu f)p\|^2 \\ &\leq \|y_n - p - \mu(f(y_n) - f(p))\|^2 \\ &= \|y_n - p\|^2 + \mu^2\|f(y_n) - f(p)\|^2 - 2\mu\langle y_n - p, f(y_n) - f(p)\rangle \\ &\leq \|y_n - p\|^2 - \mu(2\alpha_1 - \mu)\|f(y_n) - f(p)\|^2 \qquad (23) \\ &\leq \|y_n - p\|^2. \qquad (24) \end{aligned}$$

It follows from (18), (22) and (24) that

$$\|u_n - p\| \leq \|x_n - p\| + \gamma_n M. \tag{25}$$

And combining (21) and (23), we obtain

$$\|u_n - p\|^2 \leq \|v_n - p\|^2 - \epsilon^2\|A^*(Av_n - VAv_n)\|^2 - \mu(2\alpha_1 - \mu)\|f(y_n) - f(p)\|^2. \tag{26}$$

Also, using *(C3)*, Lemma 4 and (25), we obtain

$$\begin{aligned} \|z_n - p\| &= \|(1-\zeta_n)Ku_n + \zeta_n u_n - p\| \\ &\leq (1-\zeta_n)\|Ku_n - p\| + \zeta_n\|x_n - p\| = \|u_n - p\| \leq \|x_n - p\| + \gamma_n M. \end{aligned} \tag{27}$$

Similarly, with the use of Lemma 5, (13) and (27), we have

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\beta_n x_n + (1-\beta_n)z_n - \gamma_n\mu F(z_n) - p\| \\
&= \|(1-\beta_n)z_n - \gamma_n\eta F(z_n) - [(1-\beta_n)p - \gamma_n\eta F(p)] + \beta_n(x_n-p) - \gamma_n\eta F(p)\| \\
&\le \|(1-\beta_n)z_n - \gamma_n\eta F(z_n) - [(1-\beta_n)p - \gamma_n\eta F(p)]\| + \beta_n\|x_n-p\| + \gamma_n\mu\|F(p)\| \\
&\le (1-\beta_n-\gamma_n\tau)\|z_n-p\| + \beta_n\|x_n-p\| + \gamma_n\eta\|F(p)\| \\
&\le (1-\beta_n-\gamma_n\tau)[\|x_n-p\| + \gamma_n M] + \beta_n\|x_n-p\| + \gamma_n\eta\|F(p)\| \\
&\le (1-\gamma_n\tau)\|x_n-p\| + \gamma_n M + \gamma_n\eta\|F(p)\| \\
&= (1-\gamma_n\tau)\|x_n-p\| + \gamma_n\tau\frac{M+\eta\|F(p)\|}{\tau} \\
&\le \max\{\|x_n-p\|, \tau^{-1}(M+\mu\|F(p)\|)\}.
\end{aligned}$$

Thus, by induction for all $n \geq 1$ we obtain $\|x_n - p\| \leq \max\{\|x_1 - p\|, \tau^{-1}(M + \mu\|F(p)\|)\}$. Therefore, $\{x_n\}$ is bounded. Hence, the sequences $\{Sx_n\}$, $\{Tx_n\}$, $\{w_n\}$, $\{v_n\}$, $\{u_n\}$ and $\{z_n\}$ are all bounded. This completes the proof. □

Lemma 9. *Let $\{x_n\}$ be the sequence generated by Algorithm 1 such that Assumption 1 holds, then for any $p \in \Gamma$, the following relations hold:*

$$\begin{aligned}
\|x_{n+1}-p\|^2 \le{}& (1-\gamma_n\tau)\|x_n-p\|^2 + \theta_n\|x_n-x_{n-1}\|^2 + 2\gamma_n\eta\langle F(p), p-x_{n+1}\rangle \\
&- (1-\beta_n-\gamma_n\tau)(1-\zeta_n)[\alpha_n(1-\alpha_n)\|w_n - T_n w_n\|^2 + \epsilon^2\|A^*((I-V)Av_n)\|^2 \\
&+ \zeta_n\|Ku_n - x_n\|^2 - \eta(2\alpha_1-\mu)\|f(y_n)-f(p)\|^2]
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1}-p\|^2 \le{}& (1-\gamma_n\tau)\|x_n-p\|^2 + \theta_n\|x_n-x_{n-1}\|^2 \\
&+ 2\gamma_n\eta\langle F(p), p-x_{n+1}\rangle + 2\alpha_n\sigma_n(1-\beta_n-\gamma_n\tau)\langle p - Sp, T_n w_n - p\rangle,
\end{aligned}$$

where $T_n = \sigma_n S + (1-\sigma_n)T$.

Proof. Let $T_n = \sigma_n S + (1-\sigma_n)T$. So for $p \in \Gamma$ we get

$$\begin{aligned}
\|T_n w_n - p\|^2 &= \|\sigma_n(Sw_n - p) + (1-\sigma_n)(Tw_n - p)\|^2 \\
&= \sigma_n\|Sw_n-p\|^2 + (1-\sigma_n)\|Tw_n-p\|^2 - \sigma_n(1-\sigma_n)\|Sw_n - Tw_n\|^2 \\
&\le \|w_n-p\|^2 - \sigma_n(1-\sigma_n)\|Sw_n-Tw_n\|^2 \qquad (28)\\
&\le \|w_n-p\|\|^2. \qquad (29)
\end{aligned}$$

But

$$\begin{aligned}
\|w_n-p\|^2 &= \|x_n - p + \theta_n(x_n-x_{n-1})\|^2 \\
&= \|x_n-p\|^2 + 2\theta_n\langle x_n-p, x_n-x_{n-1}\rangle + \theta_n^2\|x_n-x_{n-1}\|^2 \\
&\le \|x_n-p\|^2 + 2\theta_n\|x_n-p\|\|x_n-x_{n-1}\| + \theta_n^2\|x_n-x_{n-1}\|^2 le\|x_n-p\|^2 \qquad (30)\\
&\quad + \theta_n\|x_n-x_{n-1}\|[2\|x_n-p\| + \theta_n\|x_n-x_{n-1}\|] \\
&\le \|x_n-p\|^2 + \theta_n\|x_n-x_{n-1}\|M_1
\end{aligned}$$

for some constant $M_1 > 0$. It follows from (29) and (30) that

$$\|T_n w_n - p\|^2 \le \|x_n-p\|^2 + \theta_n\|x_n-x_{n-1}\|M_1.$$

Also, using (29), we obtain

$$\begin{aligned}\|v_n - p\|^2 &= \|(1-\alpha_n)(w_n - p) + \alpha_n(T_n w_n - p)\|^2 \\ &\le \|w_n - p\|^2 - \alpha_n(1-\alpha_n)\|w_n - T_n w_n\|^2 \\ &\le \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|M_1 - \alpha_n(1-\alpha_n)\|w_n - T_n w_n\|^2.\end{aligned} \tag{31}$$

It follows from (26) and (31) that

$$\begin{aligned}\|u_n - p\|^2 \le \|x_n - p\|^2 &+ \theta_n\|x_n - x_{n-1}\|M_1 - \alpha_n(1-\alpha_n)\|w_n - T_n w_n\|^2 \\ &- \epsilon^2\|A^*((I-V)Av_n)\|^2 - \mu(2\alpha_1 - \mu)\|f(y_n) - f(p)\|^2.\end{aligned} \tag{32}$$

And by Lemmas 2 and 4, we obtain

$$\begin{aligned}\|z_n - p\|^2 &= \|(1-\zeta_n)Ku_n + \zeta_n u_n - p\|^2 \\ &= (1-\zeta_n)\|Ku_n - p\|^2 + \zeta_n\|u_n - p\|^2 - \zeta_n(1-\zeta_n)\|Ku_n - u_n\|^2 \\ &\le \|u_n - p\|^2 - \zeta_n(1-\zeta_n)\|Ku_n - u_n\|^2 \qquad (33) \\ &\le \|u_n - p\|^2. \qquad (34)\end{aligned}$$

Combining (32) and (33), we obtain

$$\begin{aligned}\|z_n - p\|^2 \le\ & \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|M_1 - (1-\zeta_n)[\alpha_n(1-\alpha_n)\|w_n - T_n w_n\|^2 \\ &+ \epsilon^2\|A^*((I-V)Av_n)\|^2 + \zeta_n\|Ku_n - u_n\|^2 + \mu(2\alpha_1 - \mu)\|f(y_n) - f(p)\|^2].\end{aligned} \tag{35}$$

On the other hand, by using Lemma 2, we obtain

$$\begin{aligned}\|T_n w_n - p\|^2 &= \|\sigma_n(Sw_n - Sp) + (1-\sigma_n)(Tw - n - p) + \sigma_n(Sp - p)\|^2 \\ &\le \|\sigma_n(Sw_n - Sp) + (1-\sigma_n)(Tw - n - p)\|^2 + 2\sigma_n\langle p - Sp, T_n w_n - p\rangle \\ &\le \sigma_n\|Sw_n - p\|^2 + (1-\sigma_n)\|Tw_n - p\|^2 + 2\sigma_n\langle p - Sp, T_n w_n - p\rangle \\ &\le \|w_n - p\|^2 + 2\sigma_n\langle p - Sp, T_n w_n - p\rangle.\end{aligned} \tag{36}$$

Also, with the use of (30) and (36), we get

$$\begin{aligned}\|v_n - p\|^2 &= \|(1-\alpha_n)(w_n - p) + \alpha_n(T_n w_n - p)\|^2 \\ &\le (1-\alpha_n)\|w_n - p\|^2 + \alpha_n\|T_n w_n - p\|^2 \\ &\le (1-\alpha_n)\|w_n - p\|^2 + \alpha_n[\|w_n - p\|^2 + 2\sigma_n\langle p - Sp, T_n w_n - p\rangle] \\ &= \|w_n - p\|^2 + 2\alpha_n\sigma_n\langle p - Sp, T_n w_n - p\rangle \\ &\le \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|M_1 + 2\alpha_n\sigma_n\langle p - Sp, T_n w_n - p\rangle.\end{aligned} \tag{37}$$

It follows from (24) and (37) that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|M_1 + 2\alpha_n\sigma_n\langle p - Sp, T_n w_n - p\rangle. \tag{38}$$

Combining (34) and (38), we obtain

$$\|z_n - p\|^2 \le \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|M_1 + 2\alpha_n\sigma_n\langle p - Sp, T_n w_n - p\rangle. \tag{39}$$

Similarly, using (13), Lemma 5, *(C1)* and *(C2)*, we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\beta_n x_n + (1-\beta_n)z_n - \gamma_n\eta F(z_n) - p\|^2\\
&= \|[(1-\beta_n)z_n - \gamma_n\eta F(z_n)] - [(1-\beta_n)p - \gamma_n\eta F(p)] + \beta_n(x_n-p) - \gamma_n\eta F(p)\|^2\\
&\le \|[(1-\beta_n)z_n - \gamma_n\eta F(z_n)] - [(1-\beta_n)p - \gamma_n\eta F(p)] + \beta_n(x_n-p)\|^2\\
&\quad + 2\gamma_n\eta\langle F(p), p - x_{n+1}\rangle\\
&\le \{\|[(1-\beta_n)z_n - \gamma_n\eta F(z_n)] - [(1-\beta_n)p - \gamma_n\eta F(p)]\| + \beta_n\|x_n-p\|\}^2\\
&\quad + 2\gamma_n\eta\langle F(p), p - x_{n+1}\rangle\\
&\le \{(1-\beta_n-\gamma_n\tau)\|z_n-p\| + \beta_n\|x_n-p\|\}^2 + 2\gamma_n\eta\langle F(p), p-x_{n+1}\rangle\\
&\le (1-\beta_n-\gamma_n\tau)\|z_n-p\|^2 + \beta_n\|x_n-p\|^2 + 2\gamma_n\eta\langle F(p), p-x_{n+1}\rangle.
\end{aligned} \tag{40}$$

It follows from (35) and (40) that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\beta_n-\gamma_n\tau)[\|x_n-p\|^2 + \theta_n\|x_n-x_{n-1}\|M_1] + \beta_n\|x_n-p\|^2\\
&\quad + 2\gamma_n\eta\langle F(p), p-x_{n+1}\rangle - (1-\zeta_n)(1-\beta_n-\gamma_n\tau)\\
&\quad \times [\alpha_n(1-\alpha_n)\|w_n - T_n w_n\|^2 + \epsilon^2\|A^*((I-V))Av_n\|^2\\
&\quad + \zeta_n\|Ku_n - u_n\|^2 - \mu(2\alpha_1-\mu)\|f(y_n)-f(p)\|^2]\\
&\le (1-\gamma_n\tau)\|x_n-p\|^2 + \theta_n\|x_n-x_{n-1}\|M_1 + 2\gamma_n\eta\langle F(p), p-x_{n+1}\rangle\\
&\quad - (1-\zeta_n)(1-\beta_n-\gamma_n\tau)[\alpha_n(1-\alpha_n)\|w_n-T_nw_n\|^2\\
&\quad + \epsilon^2\|A^*((I-V))Av_n\|^2 + \zeta_n\|Ku_n-u_n\|^2\\
&\quad - \mu(2\alpha_1-\mu)\|f(y_n)-f(p)\|^2].
\end{aligned} \tag{41}$$

It also follows from (39) and (40) that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\beta_n-\gamma_n\tau)[\|x_n-p\|^2 + \theta_n\|x_n-x_{n-1}\|M_1\\
&\quad + 2\alpha_n\sigma_n\langle p-Sp, T_nw_n-p\rangle] + \beta_n\|x_n-p\|^2 + 2\gamma_n\eta\langle F(p), p-x_{n+1}\rangle\\
&\le (1-\gamma_n\tau)\|x_n-p\|^2 + \theta_n\|x_n-x_{n-1}\|M_1\\
&\quad + 2\gamma_n\eta\langle F(p), p-x_{n+1}\rangle + 2\alpha_n\sigma_n(1-\beta_n-\gamma_n\tau)\langle p-Sp, T_nw_n-p\rangle.
\end{aligned} \tag{42}$$

Hence inequalities (41) and (42) yield the result. This completes the proof. ☐

Theorem 1. *Let Assumption 1 holds. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to an element $p \in \Gamma$, where p is a unique solution of the hierarchical fixed point problem* (2).

Proof. From (42) we see that $\|x_{n+1}-p\|^2 \le (1-\gamma_n\tau)\|x_n-p\|^2 + \gamma_n\tau S_n$, where

$$S_n = \Big[\frac{\theta_n}{\tau\gamma_n}\|x_n-x_{n-1}\|M_1 + \frac{2\eta}{\tau}\langle F(p), p-x_{n+1}\rangle + \frac{\alpha_n\sigma_n(1-\beta_n-\gamma_n\tau)}{\gamma_n\tau}\langle p-Sp, T_nw_n-p\rangle\Big].$$

Let $p \in \Gamma$, then following Lemmas 5 and 9, we only need to show that $\limsup_{i\to\infty} S_{n_i} \le 0$ for every subsequence $\{\|x_{n_i}-p\|\}$ of $\{\|x_n-p\|\}$ satisfying $\liminf_{i\to\infty}(\|x_{n_i+1}-p\| - \|x_{n_i}-p\|) \ge 0$. But from (41), we see that

$$\begin{aligned}
\|x_{n_i+1}-p\|^2 &\le (1-\gamma_{n_i}\tau)\|x_{n_i}-p\|^2 + \theta_{n_i}\|x_{n_i}-x_{n_i-1}\|M_1 + 2\gamma_{n_i}\eta\langle F(p), p-x_{n_i+1}\rangle\\
&\quad - \alpha_{n_i}(1-\alpha_{n_i})(1-\beta_{n_i}-\gamma_{n_i}\tau)(1-\zeta_{n_i})||w_{n_i}-T_{n_i}w_{n_i}||^2\\
&\quad - \zeta_{n_i}(1-\zeta_{n_i})||Ku_{n_i}-u_{n_i}||^2
\end{aligned} \tag{43}$$

and

$$\begin{aligned}\|x_{n_i+1}-p\|^2 \le{}& (1-\gamma_{n_i}\tau)\|x_{n_i}-p\|^2+\theta_{n_i}\|x_{n_i}-x_{n_i-1}\|M_1+2\gamma_{n_i}\eta\langle F(p),p-x_{n_i+1}\rangle\\ &-(1-\beta_{n_i}-\gamma_{n_i}\tau)(1-\zeta_{n_i})\big[\epsilon^2\|A^*((I-V)Av_{n_i})\|^2\\ &+\mu(2\alpha_1-\mu)\|f(y_{n_i})-f(p)\|^2\big].\end{aligned} \tag{44}$$

Let $\{\|x_{n_i}-p\|\}$ be a subsequence of $\{\|x_n-p\|\}$ such that $\liminf\limits_{i\to\infty}(\|x_{n_i+1}-p\|-\|x_{n_i}-p\|)\ge 0$ holds, then

$$\begin{aligned}&\liminf_{i\to\infty}(\|x_{n_i+1}-p\|^2-\|x_{n_i}-p\|^2)\\ &\qquad=\liminf_{i\to\infty}\{(\|x_{n_i+1}-p\|+\|x_{n_i}-p\|)(\|x_{n_i+1}-p\|-\|x_{n_i}-p\|)\}\ge 0.\end{aligned} \tag{45}$$

Combining (43) and (45), we obtain

$$\begin{aligned}\limsup_{i\to\infty}\,&\alpha_{n_i}(1-\alpha_{n_i})(1-\beta_{n_i}-\gamma_{n_i}\tau)\|w_{n_i}-T_{n_i}w_{n_i}\|^2+\zeta_{n_i}(1-\zeta_{n_i})\|Ku_{n_i}-u_{n_i}\|^2\\ &\le\limsup_{i\to\infty}[\|x_{n_i+1}-p\|^2-\|x_{n_i}-p\|^2]\\ &\quad+\limsup_{i\to\infty}\Big[\gamma_{n_i}\Big\{\beta_{n_i}\frac{\theta_{n_i}}{\gamma_{n_i}}\|x_{n_i}-x_{n_i+1}\|M_1-\tau\|x_{n_i}-p\|^2+\gamma\|F(p)\|\|p-x_{n_i+1}\|\Big\}\Big]\\ &=-\liminf_{i\to\infty}[\|x_{n_i+1}-p\|^2-\|x_{n_i}-p\|^2]\le 0.\end{aligned}$$

This implies that

$$\lim_{i\to\infty}\|w_{n_i}-T_{n_i}w_{n_i}\|=0=\lim_{i\to\infty}\|Ku_{n_i}-u_{n_i}\|. \tag{46}$$

Similarly, combining (44) and (45), we get

$$\begin{aligned}\limsup_{i\to\infty}\,&(1-\beta_{n_i}-\gamma_{n_i}\tau)(1-\zeta_{n_i})\epsilon^2\|A^*((I-V)Av_{n_i})\|^2+\mu(2\alpha_1-\mu)\|f(y_{n_i})-f(p)\|^2\\ &\le\limsup_{i\to\infty}[\|x_{n_i+1}-p\|^2-\|x_{n_i}-p\|^2]\\ &\quad+\limsup_{i\to\infty}\Big[\gamma_{n_i}\Big\{\frac{\theta_{n_i}}{\gamma_{n_i}}\|x_{n_i}-x_{n_i+1}\|M_1-\tau\|x_{n_i}-p\|^2+\gamma\|F(p)\|\|p-x_{n_i+1}\|\Big\}\Big]\\ &=-\liminf_{i\to\infty}[\|x_{n_i+1}-p\|^2-\|x_{n_i}-p\|^2]\le 0.\end{aligned}$$

This implies that

$$\lim_{i\to\infty}\|A^*((I-V)Av_{n_i})\|=0=\lim_{i\to\infty}\|f(y_{n_i})-f(p)\|. \tag{47}$$

Thus, from (15) and with boundedness of $\{v_n\}$, there exists $M_2>0$ such that $||v_n-p||\le M_2$, then

$$\|Av_n-VAv_n\|^2\le 2M_2\|A^*(Av_n-VAv_n)\|. \tag{48}$$

Combining (47) and (48), we obtain $\lim\limits_{i\to\infty}\|Av_{n_i}-VAv_{n_i}\|^2=0$. So,

$$\lim_{i\to\infty}\|Av_{n_i}-VAv_{n_i}\|=0. \tag{49}$$

From (11) and (46), we get

$$\lim_{i\to\infty}\|w_{n_i}-v_{n_i}\|=\lim_{i\to\infty}\alpha_{n_i}\|w_{n_i}-T_{n_i}w_{n_i}\|=0. \tag{50}$$

It follows from (10) and (11) that

$$\|w_{n_i} - x_{n_i}\| = \theta_{n_i}\|x_{n_i} - x_{n_i-1}\| = \gamma_{n_i} \cdot \frac{\theta_{n_i}}{\gamma_{n_i}}\|x_{n_i} - x_{n_i-1}\| \to 0 \quad \text{as} \quad i \to \infty. \tag{51}$$

Also, from (50) and (51) we obtain

$$\lim_{i\to\infty}\|x_{n_i} - v_{n_i}\| = 0. \tag{52}$$

Since $y_n := v_n - \lambda_n A^*((I-V)Av_n)$, then using (12), (47) and (49), we get

$$\lim_{i\to\infty}\|y_{n_i} - v_{n_i}\| = 0. \tag{53}$$

Thus, with $U := J^E_\mu(I-\mu f)$ and J^E_μ been firmly nonexpansive mappings, we have

$$\begin{aligned}
2\|u_n - p\|^2 &= 2\|Uy_n - Up\|^2 = 2\|J^E_\mu(I-\mu f)(y_n) - J^E_\mu(I-\mu f)(p)\|^2 \\
&\le 2\langle (I-\mu f)y_n - (I-\mu f)p, u_n - p\rangle \\
&= \|(I-\mu f)y_n - (I-\mu f)p\|^2 + \|u_n - p\|^2 - \|u_n - y_n - \mu(f(y_n) - f(p))\|^2 \\
&\le \|y_n - p\|^2 + \|u_n - p\|^2 + 2\mu\langle y_n - u_n, f(y_n - f(p))\rangle - \mu\|f(y_n) - f(p)\|^2 \\
&\le \|y_n - p\|^2 + \|u_n - p\|^2 - (1-\mu)\|y_n - u_n\|^2 + \mu(1-\mu)\|f(y_n) - f(p)\|^2,
\end{aligned}$$

hence

$$\|u_n - p\|^2 \le \|y_n - p\|^2 - (1-\mu)\|y_n - u_n\|^2 + \mu(1-\mu)\|f(y_n) - f(p)\|^2. \tag{54}$$

Combining (22), (31) and (54), we get

$$\begin{aligned}
\|u_n - p\|^2 \le \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|M_1 - \|u_n - y_n\|^2 \\
- (1-\mu)\|y_n - u_n\|^2 + \mu(1-\mu)\|f(y_n) - f(p)\|^2.
\end{aligned} \tag{55}$$

And with (34), (40) and (55), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 \le (1-\gamma_n\tau)\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + 2\gamma_n\mu\langle F(p), p - x_{n+1}\rangle \\
+ \theta_n\|x_n - x_{n-1}\|M_1 - (1-\beta_n-\gamma_n\tau)(1-\eta)\|u_n - y_n\|^2 \\
+ \mu(1-\mu)\|f(y_n) - f(p)\|^2.
\end{aligned} \tag{56}$$

Combining (45), (47) and (56), we obtain

$$\begin{aligned}
&\limsup_{i\to\infty}(1-\beta_{n_i}-\gamma_{n_i}\tau)(1-\mu)\|u_{n_i} - y_{n_i}\|^2 \\
&\quad\le \limsup_{i\to\infty}\left[\|x_{n_i+1} - p\|^2 - \|x_{n_i} - p\|^2 + \mu(1-\mu)\|f(y_{n_i}) - f(p)\|^2\right] \\
&\quad\quad + \limsup_{i\to\infty}\left[\gamma_{n_i}\left\{\beta_{n_i}\frac{\theta_{n_i}}{\gamma_{n_i}}\|x_{n_i} - x_{n_i+1}\|M_1 - \tau\|x_{n_i} - p\|^2 + \gamma\|F(p)\|\|p - x_{n_i+1}\|\right\}\right] \\
&\quad= -\liminf_{i\to\infty}\left[\|x_{n_i+1} - p\|^2 - \|x_{n_i} - p\|^2\right] \le 0.
\end{aligned}$$

Thus

$$\lim_{i\to\infty}\|u_{n_i} - y_{n_i}\| = 0. \tag{57}$$

And combining (52), (53) and (57), we obtain

$$\lim_{i\to\infty}\|u_{n_i} - x_{n_i}\| = 0. \tag{58}$$

With (53) and (57), we have $\lim\limits_{i\to\infty}\|u_{n_i}-v_{n_i}\|=0$. Also, from *(C3)* and (46), we get

$$\lim_{i\to\infty}\|Wu_{n_i}-u_{n_i}\|=0. \tag{59}$$

Since $z_n=(1-\zeta_n)Ku_n+\zeta_n x_n$, then $\|z_n-u_n\|=(1-\zeta_n)\|Ku_n-u_n\|$, it follows from (46) that $\lim\limits_{i\to\infty}\|z_{n_i}-u_{n_i}\|=0$. Combining this with (58), we obtain

$$\lim_{i\to\infty}\|z_{n_i}-x_{n_i}\|=0. \tag{60}$$

We know from (13) that $\|x_{n+1}-x_n\|\leq(1-\beta_n)\|x_n-z_n\|+\gamma_n\tau\|F(z_n)\|$, thus, with (60) we get

$$\lim_{i\to\infty}\|x_{n_i+1}-x_{n_i}\|=0. \tag{61}$$

With the use of (46) and (51), we get $\lim\limits_{i\to\infty}\|x_{n_i}-T_{n_i}w_{n_i}\|=0$. It follows from (61) and (62) that

$$\lim_{i\to\infty}\|x_{n_i+1}-T_{n_i}w_{n_i}\|=0. \tag{62}$$

Using (28) and Cauchy-Schwartz inequality, we get

$$\begin{aligned}\sigma_{n_i}(1-\sigma_{n_i})\|Sw_{n_i}-Tw_{n_i}\|^2 &\leq \|w_{n_i}-p\|^2-\|T_{n_i}w_{n_i}-p\|^2\\ &=\|w_{n_i}-T_{n_i}w_{n_i}\|^2+2\langle w_{n_i}-T_{n_i}w_{n_i},T_{n_i}w_{n_i}-p\rangle\\ &\leq\|w_{n_i}-T_{n_i}w_{n_i}\|^2+2\|w_{n_i}-T_{n_i}w_{n_i}\|\|T_{n_i}w_{n_i}-p\|\to 0 \text{ as } n\to\infty.\end{aligned}$$

Therefore, we obtain $\lim\limits_{i\to\infty}\|Sw_{n_i}-Tw_{n_i}\|=0$. This equality together with (46) implies $\|w_{n_i}-Tw_{n_i}\|\leq\|w_{n_i}-T_{n_i}w_{n_i}\|+\|T_{n_i}w_{n_i}-Tw_{n_i}\|=\|w_{n_i}-T_{n_i}w_{n_i}\|+\sigma_{n_i}\|Sw_{n_i}-Tw_{n_i}\|\to 0$ as $n\to\infty$. Thus,

$$\lim_{i\to\infty}\|w_{n_i}-Tw_{n_i}\|=0. \tag{63}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $\{x_{n_{i_j}}\}$ converges weakly to $q\in H_1$ as $j\to\infty$. It follows from (51) that $\{w_{n_{i_j}}\}$ converges weakly to q. This together with (63), nonexpansivity of T and Lemma 3 implies $q\in F(T)$.

Let us show that q solves HFPP (2). Using (11) we get

$$v_{n_i}-w_{n_i}=\alpha_{n_i}\sigma_{n_i}\Big((I-S)+\frac{1-\sigma_{n_i}}{\sigma_{n_i}}(I-T)\Big)w_{n_i}.$$

Thus,

$$\frac{1}{\alpha_{n_i}\sigma_{n_i}}(v_{n_i}-w_{n_i})=\Big((I-S)+\frac{1-\sigma_{n_i}}{\sigma_{n_i}}(I-T)\Big)w_{n_i}.$$

By *(1)* of Lemma 1, the operator sequence $\{((1-\sigma_{n_i})/\sigma_{n_i})(I-T)\}$ graph converges to $N_{F(T)}$. The operator sequence $\{(I-S)+((1-\sigma_{n_i})/\sigma_{n_i})(I-T)\}$ graph converges to $(I-S)+N_{F(T)}$. Using (50) and the fact that the graph of $(I-S)+N_{F(T)}$ is weakly strongly closed, we get that $0\in(I-S)q+N_{F(T)}q$, thus $q\in\Omega$.

We next show that $q\in\Omega$. Since $x_{n_{i_j}}\rightharpoonup q$, then from (57) and (58), we know that $y_{n_{i_j}}\rightharpoonup q$ and $u_{n_{i_j}}\rightharpoonup q$ as $j\to\infty$. Note that $u_{n_{i_j}}=U(y_{n_{i_j}})$, where $U:=J_\mu^E(I-\mu f)$ and $J_\mu^E:=(I+\mu E)$, can be written as

$$\frac{(y_{n_{i_j}}-u_{n_{i_j}})-\mu f(y_{n_{i_j}})}{\mu}\in E(u_{n_{i_j}}).$$

By taking limit as $j \to \infty$ in the above formula and using the fact that f is $(1/\alpha_1)$-Lipschitz continuous and the graph of a maximal monotone operator is weakly-strongly closed, we get $0 \in f(q) + E(q)$. Since from (52) we have that $v_{n_{i_j}} \rightharpoonup q$, with A been continuous, we have $Av_{n_{i_j}} \rightharpoonup Aq$. By nonexpansivity of $V := J_\mu^G(I - \mu g)$, we get that V is demiclosed at zero together with (49) we obtain $0 \in g(Aq) + G(Aq)$. Also, since $x_{n_{i_j}} \rightharpoonup q$, it follows form (58) that $u_{n_{i_j}} \rightharpoonup q$ and from (59) we know that $\lim_j \|Wu_{n_{i_j}} - u_{n_{i_j}}\| = 0$. Since $T_{n_{i_j}} w_{n_{i_j}} \rightharpoonup q$, then

$$\limsup_{i\to\infty} \langle p - Sp, p - T_{n_i} w_{n_i} \rangle = \lim_{j\to\infty} \langle p - Sp, p - T_{n_{i_j}} w_{n_{i_j}} \rangle = \langle p - Sp, p - q \rangle$$

for any p that solves HFPP (2). Then

$$\limsup_{i\to\infty} \langle p - Sp, p - T_{n_i} w_{n_i} \rangle = \langle p - Sp, p - q \rangle \leq 0. \tag{64}$$

Since F is k-strongly monotone and L-lipschitzian with $\mu \in (0, 2k/L^2)$, we can set $S = I - \mu F$ so that if p solves HVIP (3), i.e. $\langle F(p), p - q \rangle \leq 0$. Thus, with the use of (62) we get

$$\begin{aligned} \limsup_{i\to\infty} \langle F(p), p - x_{n_i+1} \rangle &= \lim_{j\to\infty} \langle F(p), p - x_{n_{i_j}+1} \rangle \\ &= \lim_{j\to\infty} [\langle F(p), p - T_{n_{i_j}} w_{n_{i_j}} \rangle + \langle F(p), T_{n_{i_j}} w_{n_{i_j}} - x_{n_{i_j}+1} \rangle] \leq 0. \end{aligned} \tag{65}$$

It follows from *(C4)*, (64) and (65) that $\limsup_{i\to\infty} S_{n_i} \leq 0$, where

$$S_{n_i} = \left[\frac{\theta_{n_i}}{\tau \gamma_{n_i}} \|x_{n_i} - x_{n_i-1}\| M_1 + \frac{2\mu}{\tau} \langle F(p), p - x_{n_i+1} \rangle + \frac{\alpha_{n_i} \sigma_n (1 - \beta_{n_i} - \gamma_{n_i} \tau)}{\gamma_{n_i} \tau} \langle p - Sp, T_{n_i} w_{n_i} - p \rangle \right],$$

and we know from (42) that

$$\|x_{n_i+1} - p\|^2 \leq (1 - \gamma_{n_i} \tau) \|x_{n_i} - p\|^2 + \gamma_{n_i} \tau S_{n_i}. \tag{66}$$

Therefore, applying Lemma 5 in (66), we have that the sequence $\{x_n\}$ converges strongly to $p \in \Gamma$. This completes the proof. □

3 Application

3.1 Split variational inequality problem

Let C and Q be two nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $N_C(x)$ denotes the normal cone of C at point $x \in C$, that is

$$N_C(x) := \{a \in H_1 : \langle a, y - x \rangle \leq 0\}.$$

From (5) and (6), we let $M = N_C$ and $N = N_Q$, where N_Q is the normal cone of Q. If $A : H_1 \to H_2$ is bounded linear operator, then SMVIP (5)–(6) reduces to a problem of finding $x^* \in H_1$ such that for any $\mu > 0$

$$x^* = (I + \mu N_C)^{-1}(x^* - \mu f(x^*)) \tag{67}$$

and

$$Ax^* = (I + \mu N_Q)^{-1}(Ax^* - \mu g(Ax^*)). \tag{68}$$

But we know that

$$\begin{aligned} y = (I+\mu N_C)^{-1}x \Leftrightarrow x \in (y+\mu N_C y) \Leftrightarrow x \in (y+\mu N_C y) \Leftrightarrow x-y \in \mu N_C y \\ \Leftrightarrow \frac{1}{\mu}\langle x-y, z-y\rangle \leq 0 \Leftrightarrow \langle x-y, z-y\rangle \leq 0 \Leftrightarrow y = P_C x, \quad \forall z \in C. \end{aligned} \tag{69}$$

Thus, using (69) and the projection technique in [29], (67) and (68) reduces to the following split variational inequality problem (SVIP for short):

$$\text{find } \ x^* \in C \ \text{ such that } \ \langle f(x^*), x-x^*\rangle \geq 0 \ \ \forall\, x \in C \tag{70}$$

and

$$y^* := Ax^* \in Q \ \text{ solves } \ \langle g(y^*), y-y^*\rangle \geq 0 \ \ \forall\, y \in Q. \tag{71}$$

Denote the solution set of (70)–(71) by

$$\Omega_2 := \{x^* \in C : x^* \in \mathrm{Sol}(VIP\,(70)) \ \text{ and } \ y^* = Ax^* \in \mathrm{Sol}(VIP\,(71))\}.$$

Hence, using (70)–(71), Algorithm 1 reduces to the following inertial Krasnoselki-Mann type method for solving SVIP and HFPP.

Algorithm 2.

Initialization. *Choose $x_0, x_1 \in H_1$ to be arbitrary.*

Iterative Steps. *Calculate x_{n+1} as follows.*

Step 1. *Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where*

$$\bar{\theta}_n = \begin{cases} \min\{\theta, \mu_n/\|x_n - x_{n-1}\|\}, & \text{if } x_n \neq x_{n-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Step 2. *Compute*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ v_n = (1-\alpha_n)w_n + \alpha_n(\sigma_n S w_n + (1-\sigma_n)T w_n), \\ u_n = P_C(I-\mu f)(v_n - \lambda_n A^*(I - P_Q(I-\mu g))Av_n), \end{cases}$$

where $\{\alpha_n\}$ and $\{\sigma_n\}$ are real sequences in $(0,1)$ and for any fixed value $\epsilon > 0$, the step size λ_n is chosen as follows

$$0 < \epsilon \leq \lambda_n \leq \frac{\|(I-P_Q(I-\mu g))Av_n\|^2}{\|A^*(I-P_Q(I-\mu g))Av_n\|^2} - \epsilon,$$

if $Av_n \neq P_Q(I-\mu g)Av_n$, otherwise $\lambda_n = \lambda$, $\lambda \geq 0$.

Step 3. *Compute*

$$\begin{cases} z_n = (1-\zeta_n)Ku_n + \zeta_n u_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)z_n - \gamma_n \mu F(z_n), \quad n \in \mathbb{N}. \end{cases}$$

Set $n := n+1$ and return to Step 1.

Using Algorithm 2, we obtain the following new result for solving SVIP and HFPP.

Theorem 2. *Let Assumption 1 holds with $E = N_C$, $G = N_Q$ and $\Omega_2 \neq \varnothing$. Then, the sequence generated by Algorithm 2 converges strongly to an element of $\Gamma := \Theta \cap \Omega_2 \cap F(W)$.*

Proof. Setting $U = P_C(I-\mu f)$ and $V = P_Q(I-\mu g)$ in Theorem 1, we obtain the desired result from Theorem 1. □

3.2 Split convex minimization problem

Let $g_1 : H_1 \to \mathbb{R}$ and $g_2 : H_2 \to \mathbb{R}$ be two convex and differentiable functions with L_1, L_2-Lipschitz continuous gradients say ∇g_1 and ∇g_2, respectively. Let $h_1 : H_1 \to \mathbb{R}$ and $h_2 : H_2 \to \mathbb{R}$ be two convex and lower semi-continuous function. Then for any bounded linear operator $A : H_1 \to H_2$ we define the split convex minimization problem (SCMP for short) as follows:

$$\text{find } x^* \in H_1 \text{ that solves } g_1(x^*) + h_1(x^*) = \min_{x \in H_1}[g_1(x) + h_1(x)] \tag{72}$$

such that

$$y^* = Ax^* \in H_2 \text{ solves } g_2(y^*) + h_2(y^*) = \min_{y \in H_2}[g_2(y) + h_2(y)]. \tag{73}$$

Thus by Fermat's rule, the SCMP (72)–(73) is equivalent to the problem:

$$\text{find } x^* \in H_1 \text{ that solves } 0 \in [\nabla g_1(x^*) + \partial h_1(x^*)] \tag{74}$$

such that

$$y^* = Ax^* \in H_1 \text{ solves } 0 \in [\nabla g_2(y^*) + \partial h_2(y^*)], \tag{75}$$

where ∇g_1 and ∇g_2 are gradients of g_1 and g_2, respectively; ∂h_1 and ∂h_2 are subdifferential of h_1 and h_2, respectively. Let the solution set of SCMP (74)–(75) be denoted by

$$\Omega_3 := \{x^* \in H_1 : x^* \in \text{Sol(SCMP (74)) and } y^* = Ax^* \in \text{Sol(SCMP (75))}\}.$$

It is known [24] that the subdifferentials ∂h_1 of h_1 and ∂h_2 of h_2 are maximal monotone, and since g_1 and g_2 are convex and differentiable functions with L_1, L_2-Lipschits continuous gradients $\nabla g_1, \nabla g_2$, then ∇g_1 and ∇g_2 are $(1/L_1), (1/L_2)$-inverse strongly monotone (see [6]). Hence, letting $E = \partial h_1, G = \partial h_2, f = \nabla g_1$ and $g = \nabla g_1$ in Assumption 1, from Algorithm 1 we obtain the following new algorithm.

Algorithm 3.

Initialization. *Choose $x_0, x_1 \in H_1$ to be arbitrary.*

Iterative Steps. *Calculate x_{n+1} as follows.*

Step 1. *Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta_n}$, where*

$$\bar{\theta}_n = \begin{cases} \min\{\theta, \mu_n / \|x_n - x_{n-1}\|\}, & \text{if } x_n \neq x_{n-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Step 2. *Compute*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ v_n = (1-\alpha_n)w_n + \alpha_n(\sigma_n S w_n + (1-\sigma_n)T w_n), \\ u_n = J_\mu^{\partial h_1}(I - \mu\nabla g_1)(v_n - \lambda_n A^*(I - J_\mu^{\partial h_2}(I - \mu \nabla g_2))Av_n), \end{cases}$$

where $J_\mu^{\partial h_1} := (I + \mu\partial h_1)^{-1}$, $J_\mu^{\partial h_2} := (I + \mu\partial h_2)^{-1}$, $\{\alpha_n\}$ and $\{\sigma_n\}$ are real sequences in $(0,1)$ and for any fixed value $\epsilon > 0$, the step size λ_n is chosen as follows

$$0 < \epsilon \leq \lambda_n \leq \frac{\|(I - J_\mu^{\partial h_2}(I - \mu\nabla g_2))Av_n\|^2}{\|A^*(I - J_\mu^{\partial h_2}(I - \mu\nabla g_2))Av_n\|^2} - \epsilon,$$

if $Av_n \neq J_\mu^{\partial h_2}(I - \mu\nabla g_2)Av_n$; otherwise $\lambda_n = \lambda$, $\lambda \geq 0$.

Step 3. Compute

$$\begin{cases} z_n = (1-\zeta_n)Ku_n + \zeta_n u_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)z_n - \gamma_n \mu F(z_n), \quad n \in \mathbb{N}. \end{cases}$$

Set $n := n+1$ and return to Step 1.

Using Algorithm 2, we obtain the following new result for solving SCMP and HFPP.

Theorem 3. *Let Assumption 1 holds with $E = \partial h_1, G = \partial h_2, f = \nabla g_1$, $g = \nabla g_1$ and $\Omega_3 \neq \varnothing$. Then the sequence generated by Algorithm 3 converges strongly to an element of $\Gamma := \Theta \cap \Omega_3 \cap F(W)$.*

Proof. Setting $U = J_\mu^{\partial h_1}(I - \mu \nabla g_1)$ and $V = J_\mu^{\partial h_2}(I - \mu\nabla g_2)$ in Theorem 1 for all $\mu \in (0, L)$, where $L := \min\{2\min\{(1/L_1), (1/L_2)\}, \}$, we obtain the desired result from Theorem 1. □

4 Numerical illustration

Next, we give numerical experiment to substantiate the efficiency of our Algorithm 1 in comparison with algorithm of K.R. Kazmi et al. [18] (see (7)) and algorithm of D.-J. Wen [31] (see (9)) in a infinite dimensional Hilbert space.

Example 1. *Let $H_1 = H_2 = L_2([0,1])$ with norm*

$$\|x\| := \Big(\int_0^1 |x(t)|^2 dt\Big)^{1/2} \quad \textit{for all} \quad x \in L_2([0,1])$$

and inner product

$$\langle x, y\rangle := \int_0^1 x(t)y(t)dt \quad \textit{for all} \quad x, y \in L_2([0,1]).$$

Let $A, E, G : L_2([0,1]) \to L_2([0,1])$ be operators defined as follows:

$$Ax(t) = \int_0^1 x(t)dt, \quad Ex(t) = 5x(t), \quad \textit{and} \quad Gx(t) = 4x(t)$$

for all $x \in L_2([0,1])$ and $t \in [0,1]$. Then A is bounded and linear, E and G are maximal monotone operators with resolvents $J_\mu^E x(t) = x(t)/(1+5\mu)$ and $T_\mu^G x(t) = x(t)/(1+4\mu)$, $\mu > 0$, respectively. Also, let maps $f, g : L_2([0,1]) \to L_2([0,1])$ be defined by $f(x(t)) = 2x(t)$ and $g(x(t)) = 3x(t)$, then f and g are $1/2, 1/3$ inverse strongly monotone with order 2 and 3, respectively. Thus, for $\alpha = \min\{1/2, 2/3\}$ and $\mu \in (0, \alpha)$, we get

$$U(x(t)) := J_\mu^E (I - \mu g)x(t) = (1-3\mu)/(1+5\mu)x(t)$$

and

$$V(x(t)) := T_\mu^G(I - \mu f)x(t) = (1-2\mu)/(1+4\mu)x(t).$$

Furthermore, we define the mappings $F, S, T, W : L_2([0,1]) \to L_2([0,1])$ by $F(x(t)) = 2x(t)$, $S(x(t)) = \int_0^1 \frac{x(t)}{2}dt$, $T(x(t)) = x(t)$ and $W(x(t)) = -4x(t)$. Then S is strongly monotone and Lipschitz continuous, S and T are nonexpansive and W is 9-demimetric. Thus, we can choose $\kappa = 1/3$, so that $K(x(t)) = -2/3x(t)$. We assume also that $\alpha_n = (n+5)/(100n)$,

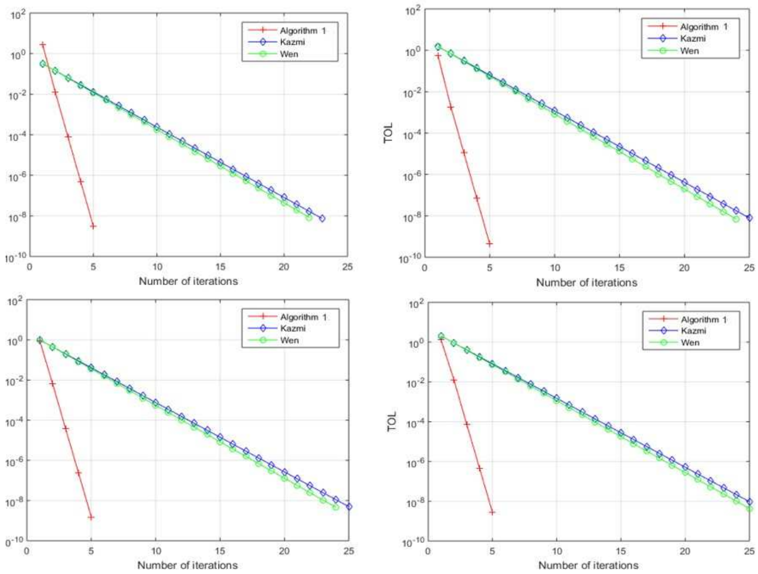

Figure 1. The error plotting of comparison of Algorithm 1, algorithm of K.R. Kazmi and algorithm of D.-J. Wen

$\sigma_n = 24n/(25n+1)$, $\gamma_n = 1/(n+1)^{(0.3)}$, $\beta_n = 0.8 - \gamma_n$, $\zeta_n = 1/3 - 1/4n$, $c_n = \gamma_n/n^{(0.03)}$, $\theta = 1/(2n+1)$, $\epsilon = 0.01$, $\eta = 0.65$ and in addition for K.R. Kazmi [18] (algorithm (7)) and D.-J. Wen [31] (algorithm (9)), we take $\lambda = 0.1$, $\mu_n = 1/(5n)$ and $D(x(t)) = 3x(t)$. Then, we let the iteration terminate if $\|x_{n+1} - x_n\| \leq \epsilon$, where $\epsilon = 10^{-8}$. The numerical experiments are listed on Table 1. Also, we illustrate the efficiency of strong convergence of the proposed Algorithm 1 in comparison with convergence of algorithms (7) and (9) in Figure 1.

		Algorithm 1	Algorithm (7)	Algorithm (9)
$x_0 = t^3 - 3$	no. of iter.	5	25	24
$x_1 = t$	cpu (time)	4.1740	5.6569	5.3922
$x_0 = t$	no. of iter.	5	23	22
$x_1 = t^3 - 3$	cpu (time)	4.6335	6.2346	6.32286
$x_0 = e^t$	no. of iter.	5	25	24
$x_1 = 2t^2$	cpu (time)	4.1514	11.8047	11.3466
$x_0 = e^{2t}$	no. of iter.	5	25	25
$x_1 = t^3 + 1$	cpu (time)	7.3563	12.5365	12.5672

Table 1. Comparison of Algorithm 1, algorithm (7) and algorithm (9)

Conclusion

A new accelerated Krasnoselki-Mann type algorithm for solving hierarchical fixed point and split monotone variational inclusion problems is introduced in the setting of a real Hilbert space. It is then proved that the algorithm approximates to a common solution of the said problems which is also solution to some fixed point problem of demimetric mapping in the space. A numerical example is given to ascertain the implementation of the proposed algorithm.

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Уґуннаді Ґ.К., Харуна Л.Й., Харбау М.Х. *Прискорений алгоритм типу Красносельського-Манна для ієрархічної задачі про нерухому точку та задачі монотонного варіаційного включення у гільбертових просторах* // Карпатські матем. публ. — 2023. — Т.15, №1. — С. 158–179.

У цій статті представлено новий прискорений алгоритм екстраполяції типу Красносельського-Манна для знаходження спільного елемента в множині розв'язків ієрархічної задачі про нерухому точку та розщепленої проблеми монотонного варіаційного включення у дійсному гільбертовому просторі. Доведено, що послідовність, згенерована алгоритмом, сильно збігається до такого спільного елемента, який також наближає розв'язок деякої задачі про нерухому точку деміметричного відображення в просторі. Наприкінці наведено деякі застосування та чисельні експерименти, щоб показати ефективність запропонованого алгоритму порівняно з нещодавно відомими відповідними результатами у літературі. Встановлений результат поширює та узагальнює багато останніх, описаних в літературі.

Ключові слова і фрази: ієрархічна задача про нерухому точку, розщеплена проблема монотонного включення, проблема варіаційної нерівності, деміметричне відображення, обернений сильно монотонний оператор.