ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2025, **17** (1), 200–210
doi:10.15330/cmp.17.1.200-210

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2025, Т.17, №1, С.200–210

Chen inequalities for immersions in trans-Sasakian space forms with slant factor

Mohd. Danish Siddiqi[1], Aliya Naaz Siddiqui[2,✉]

In this research article, our focus is directed towards the exploration of trans-Sasakian manifolds that incorporate a distinctive type of non-metric connection referred to as a quarter-symmetric non-metric ($QSNM$) connection. We delve into the derivation of the mathematical expressions governing the curvature tensor $\widetilde{R}$ of trans-Sasakian space forms, utilizing the aforementioned $QSNM$-connection. Our primary efforts are centered around the establishment of Chen inequalities. These inequalities find application in the characterization of slant submanifolds in the trans-Sasakian space forms and connected by a $QSNM$-connection. Furthermore, our investigation encompasses the classification of Chen invariants. This classification is extended to α-Sasakian, β-Kenmotsu and cosymplectic manifolds, all of which are endowed with the distinctive $QSNM$-connection.

Key words and phrases: Chen inequality, slant submanifold, trans-Sasakian manifold, quarter-symmetric non-metric connection.

[1] Department of Mathematics, College of Science, Jazan University, P.O. Box 277, Jazan, 4512, Saudi Arabia
[2] Division of Mathematics, School of Basic Sciences, Galgotias University, Greater Noida, Uttar Pradesh 203201, India
✉ Corresponding author
E-mail: msiddiqi@jazanu.edu.sa (Mohd. Danish Siddiqi),
aliyanaazsiddiqui9@gmail.com (Aliya Naaz Siddiqui)

1 Introduction

The concept of submanifolds in an ambient space holds a captivating allure as it intricately weaves together the extrinsic properties and intrinsic. Specifically, the Riemannian invariants serve as inherent features of manifolds with Riemannian geometry. In 1993, B.-Y. Chen [5] formulated an inequality that relates the sectional curvature $\mathcal{K}$, the scalar curvature σ (an intrinsic invariant), and the mean curvature function $|\mathcal{H}|$ (an extrinsic invariant) of a submanifold $\mathcal{M}$ in a real space form characterized by a constant curvature c.

Moreover, B.-Y. Chen [6] extended his contributions by delving into the realm of Riemannian invariants applicable to a Riemannian manifold. This expansion further enriched the understanding of these manifold properties. Recently, A.N. Siddiqui et. al. also studied Chen inequality in latest statistical ambient space (for more details see [21, 22]).

For any point $p \in \mathcal{M}$, let $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m\}$ be an orthonormal basis of the tangent space $T_p\mathcal{M}$. Let $\mathcal{M}$ be a Riemannian manifold of size m. Next, we obtain σ at p as follows

$$\sigma = \sum_{1 \le i < j \le m} \mathcal{K}\left(\varepsilon_i \wedge \varepsilon_j\right).$$

УДК 514.7
2020 *Mathematics Subject Classification:* 53C40, 53B25, 53D15.

We express our sincere thanks to the anonymous referee for many helpful suggestions.

For any point $p \in \mathcal{M}$, we indicate

$$(Inf\mathcal{K})(p) = Inf\left\{\mathcal{K}(\pi) : \pi \subset T_p\mathcal{M},\ \dim(\pi) = 2\right\},$$

where the sectional curvature of $\mathcal{M}$ associated with plane section $\pi \subset T_p\mathcal{M}$ at $p \in M$ is denoted by $\mathcal{K}(\pi)$.

At any point $p \in \mathcal{M}$, the Chen invariant $\delta_{\mathcal{M}}$ is specified as

$$\delta_{\mathcal{M}}(p) = \sigma(p) - (Inf\mathcal{K})(p). \tag{1}$$

For instances involving C-totally real submanifolds and slant submanifolds in a contact manifold (Sasakian space form) characterized by a constant φ-sectional curvature, F. Defever et. al. [8] and D. Cioroboiu et. al. [7] derived separate inequalities that bear resemblance to the inequality originally presented by Chen for submanifolds in a real space form. The squared mean curvature of the curve and the intrinsic invariant $\delta_{\mathcal{M}}$ are used to express the following inequality

$$\delta_M \leq \frac{(m-2)m^2}{2(m-1)}\|\mathcal{H}\|^2 + \frac{1}{2}(m-2)(m+1)c.$$

In the scenario that $\mathcal{M}$ is an invariant submanifold of complex space form $\mathcal{M}(c)$, the aforementioned inequality is also valid.

On a different note, back in 1985, J.A. Oubina [18] contributed to the propagation of knowledge regarding a fresh category of almost contact Riemannian manifolds recognized as trans-Sasakian manifolds. Geometries such as cosymplectic, α-Sasakian, Sasakian, β-Kenmotsu, and Kenmotsu structures are all included in the category of trans-Sasakian structures.

H.A. Hayden established a metric connection on a Riemannian manifold with non-zero torsion [13]. Numerous scholars (see [3, 11]) have scrutinized the characteristics of Riemannian manifolds endowed with semi-symmetric (or symmetric) and non-metric connections. S. Golab [12] investigated the possibility of quarter-symmetric linear connections inside a differential manifold. If the torsion tensor $\overline{T}$ of a linear connection has the following form

$$\overline{T}(u,v) = \gamma(v)\varphi u - \gamma(u)\varphi v,$$

it is considered quarter-symmetric for all vector fields u, v on a manifold, where φ is a tensor of type $(1,1)$, and γ is a 1-form.

The quarter-symmetric connection reduces to a semi-symmetric connection when $\varphi = I$. Consequently, the quarter-symmetric connection presents itself as an extension or broader framework encompassing semi-symmetric connections. The denotation of a metric connection is assigned to $\overline{\nabla}$ when a Riemannian metric g exists on the manifold $\mathcal{M}$ in a manner that satisfies $\overline{\nabla}g = 0$. Conversely, if this condition is not met, the connection is classified as non-metric.

A semi-symmetric metric connection was introduced on an almost contact manifold in the work of A. Sharfuddin and S.I. Husain [20]. The detailed description is as follows

$$\overline{T}(u,v) = \gamma(v)u - \gamma(u)v.$$

This manuscript focuses on the exploration of Chen inequalities concerning slant submanifolds in a trans-Sasakian space form. The analysis is carried out in the context of a quarter-symmetric non-metric connection denoted as $(QSNM)$.

2 Preliminaries

Let $\mathcal{M}$ be an almost contact metric manifold of dimension n with an almost contact metric structure $(\varphi, \zeta, \gamma, g)$, that is, φ is a $(1,1)$ tensor field, ζ is a vector field, γ is a 1-form, and g is a compatible Riemannian metric such that

$$\varphi^2 u = -u + \gamma(u)\zeta, \quad \gamma(\zeta) = 1, \quad \varphi(\zeta) = 0, \quad \gamma o \varphi = 0,$$

$$g(\varphi u, \varphi v) = g(u, v) - \gamma(u)\gamma(v), \quad g(u, \varphi v) = -g(\varphi u, v), \quad g(u, \zeta) = \gamma(u)$$

for all $u, v \in T\mathcal{M}$.

Definition 1. *An almost contact metric structure $(\varphi, \zeta, \gamma, g)$ on $\mathcal{M}$ is called a trans-Sasakian structure if*

$$(\nabla_u \varphi)(v) = \alpha\big(g(u, v)\zeta - \gamma(v)u\big) + \beta\big(g(\varphi u, v)\zeta - \gamma(v)\varphi u\big).$$

If α and β are two smooth functions on $\mathcal{M}$, then we say that the trans-Sasakian structure is of type (α, β).

It is important to observe that trans-Sasakian structures with characteristics $(0, 0)$ align with the cosymplectic structures, while those with characteristics $(\alpha, 0)$ correspond to α-Sasakian structures. Similarly, trans-Sasakian structures with features $(0, \beta)$ can be identified as β-Kenmotsu structures. The investigation into trans-Sasakian manifolds (TS-manifolds) has been explored by many researchers [1, 2, 9, 10, 14, 16], leading to the acquisition of the subsequent outcomes:

$$\nabla_U \zeta = -\alpha\varphi u + \beta\big(u - \gamma(u)\zeta\big), \tag{2}$$

$$(\nabla_u \gamma)(v) = -\alpha g(\varphi u, v) + \beta g(\varphi u, \varphi v),$$

$$\begin{aligned} \mathrm{Re}ie(u, v)\zeta =& \big(\alpha^2 - \beta^2\big)\left[\gamma(v)u - \gamma(u)v\right] - (u\alpha)\varphi v - (u\beta)\varphi^2 v \\ &+ 2\alpha\beta\big[\gamma(v)\varphi u - \eta(u)\varphi v\big] + (v\alpha)\varphi u + (v\beta)\varphi^2 u, \end{aligned}$$

$$\mathrm{Re}ie(\zeta, u)\zeta = \big(\alpha^2 - \beta^2 - \zeta\beta\big)\left[\gamma(u)\zeta - u\right],$$

$$S(u, \zeta) = \big[(n-1)\big(\alpha^2 - \beta^2\big) - \zeta\beta\big]\gamma(u) - (\varphi u)\alpha - (n-2)(u\beta). \tag{3}$$

When $\varphi(\text{grad } \alpha) = (n-2)\text{grad } \beta$, then from equation (3) the following is provided:

$$S(u, \zeta) = (n-1)\big(\alpha^2 - \beta^2\big)\gamma(u),$$

$$S(\zeta, \zeta) = (n-1)\big(\alpha^2 - \beta^2\big),$$

$$\sigma = n(n-1)\big(\alpha^2 - \beta^2\big).$$

3 Curvature analysis on TS-manifolds for $QSNM$-connection

Let $\widetilde{\mathrm{Re}ie}$ and $\mathrm{Re}ie$ represent the curvature tensors in relation to the quarter-symmetric non-metric connection denoted as $(QSNM)$ $\widetilde{\nabla}$ and the Levi-Civita connection ∇ applied to a TS-manifold $\mathcal{M}$, respectively.

In the subsequent part of this section, we will establish the connection between $\widetilde{R}$ and R concerning the $QSNM$-connection $\widetilde{\nabla}$ operating on $\mathcal{M}$, as well as in the context of the Levi-Civita connection ∇ on the same manifold.

In the previous work by M.M. Tripathi [23], the following connection was introduced

$$\widetilde{\nabla}_u v = \nabla_u v - \eta(u)\varphi v - \gamma(u)v - \gamma(v)u + g(u,v)\zeta,$$

where $\widetilde{\nabla}$ is $(QSNM)$-connection. Using Definition 1 and equation (2), we obtain

$$(\widetilde{\nabla}_u \varphi)(v) = \alpha\big(g(u,v)\zeta - \gamma(v)u\big) + (\beta-1)\big(g(\varphi u,v)\zeta - \gamma(v)\varphi u\big),$$

$$\widetilde{\nabla}_u \zeta = -\alpha\varphi u + (\beta-1)u - \beta\gamma(u)\zeta.$$

In [19], C. Patra and A. Bhattacharyya studied trans-Sasakian manifold admitting quarter-symmetric non-metric connection. Now, we obtain curvature, Ricci and scalar curvature tensors by using $QSNM$-connection.

Theorem 1. *Let $\mathcal{M}$ be a TS-manifold with the $QSNM$-connection $\widetilde{\nabla}$. Then the following equality is provided*

$$\begin{aligned}\widetilde{Rie}\,(u,v)w &= \mathrm{Re}ie(u,v)w + \alpha\Big[\big(g(v,w)\gamma(u) - g(u,w)\gamma(v)\big)\zeta + \big(\gamma(v)\gamma(w) - g(\varphi v,w)\big)u \\ &+ \big[-\gamma(u)\gamma(w) + g(\varphi u,w)\big]v - 2g(u,\varphi v)w - g(v,w)\varphi u + g(u,w)\varphi v - 2g(u,\varphi v)\varphi w\Big] \\ &+ (\beta-1)\Big\{\big[g(\varphi v,w)\gamma(u) - g(v,w)\gamma(u) - g(\varphi u,w)\gamma(v) + g(u,w)\gamma(v)\big]\zeta \\ &- \big(\gamma(v)\gamma(w) - g(v,w)\big)u + \big[\gamma(u)\gamma(w) - g(u,w)\big]v + \gamma(v)\gamma(w)\varphi u - \gamma(u)\gamma(w)\varphi v\Big\} \\ &+ \beta\big[g(v,w)u - g(u,w)v\big]\end{aligned} \tag{4}$$

for any vector fields u, v and w on $\mathcal{M}$.

Proof. The curvature tensor $\widetilde{\mathrm{Re}ie}$ is as follows

$$\widetilde{\mathrm{Re}ie}(u,v)w = \widetilde{\nabla}_u\widetilde{\nabla}_v w - \widetilde{\nabla}_v\widetilde{\nabla}_u w - \widetilde{\nabla}_{[u,v]}w.$$

From the above equality and (3) we have the proof. □

4 Chen inequality for Θ-slant submanifolds in $QSNM$-manifolds with $QSNM$-connection

We employ the Gauss equation for a submanifold $\mathcal{M}$ (see [24])

$$\mathrm{Re}ie(u,v,w,z) = \widetilde{\mathrm{Re}ie}(u,v,w,z) - g\big(\hbar(u,z),\hbar(v,w)\big) + g\big(\hbar(u,w),\hbar(v,z)\big) \tag{5}$$

for all $u,v,w,z \in T\mathcal{M}$. Here $\hbar$ denotes the second fundamental form of $\mathcal{M}$. We use

$$\hbar^s_{i,j} = g\big(\hbar(\varepsilon_i,\varepsilon_j),\varepsilon_s\big), \quad \text{and} \quad \|\hbar\|^2 = \sum_{i,j}^{m} g\big(\hbar(\varepsilon_i,\varepsilon_j),\hbar(\varepsilon_i,\varepsilon_j)\big) \tag{6}$$

for any $\varepsilon_i, \varepsilon_j \in T\mathcal{M}$ and $\varepsilon_s \in T^\perp\mathcal{M}$.

Next, let $\mathcal{M}$ is a Θ-slant submanifold of M, $\dim(\mathcal{M}) = n = 2k$. Consider a point $p \in \mathcal{M}$, and let $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m\}$ form an orthonormal frame for $T_p\mathcal{M}$ and $\{\varepsilon_{m+1}, \ldots, \varepsilon_n\}$ constitute an orthonormal frame for $T_p^\perp\mathcal{M}$.

When we evaluate the equations (4) and (5) for the cases where $u = w = \varepsilon_i$ and $V = Z = \varepsilon_j$, it becomes evident that

$$\begin{aligned}
\sum_{i,j=1}^{m} \widetilde{\mathrm{Re}\,ie}(\varepsilon_i, \varepsilon_j, \varepsilon_j, \varepsilon_i) &= \sum_{i=1}^{m} \mathrm{Re}\,ie(\varepsilon_i, \varepsilon_j, \varepsilon_j, \varepsilon_i) + \alpha \Big[\sum_{i,j=1}^{m} \big(g(\varepsilon_j, \varepsilon_j)\gamma(\varepsilon_i) - g(\varepsilon_i, \varepsilon_j)\gamma(\varepsilon_j)\big) g(\varepsilon_i, \varepsilon_i) \\
&+ \sum_{i,j=1}^{m} \big(\gamma(\varepsilon_j)\gamma(\varepsilon_j) - g(\varphi\varepsilon_j, \varepsilon_j)\big) g(\varepsilon_i, \varepsilon_i) + \sum_{i,j=1}^{m} \big[-\gamma(\varepsilon_i)\gamma(\varepsilon_j) + g(\varphi\varepsilon_i, \varepsilon_j)\big] g(\varepsilon_i, \varepsilon_i) \\
&- \sum_{i,j=1}^{m} 2g(\varepsilon_i, \varphi\varepsilon_j) g(\varepsilon_j, \varepsilon_i) - \sum_{i,j=1}^{m} g(\varepsilon_j, \varepsilon_j) g(\varphi\varepsilon_i, \varepsilon_i) + \sum_{i,j=1}^{m} g(\varepsilon_i, \varepsilon_j) g(\varphi\varepsilon_j, \varepsilon_i) \\
&- \sum_{i,j=1}^{m} 2g(\varepsilon_i, \varphi\varepsilon_j) g(\varphi\varepsilon_j, \varepsilon_i) \Big] + (\beta - 1) \Big[\sum_{i,j=1}^{m} \big(g(\varepsilon_j, \varepsilon_j)\gamma(\varepsilon_i) - g(\varepsilon_j, \varepsilon_j)\gamma(\varepsilon_i) - g(\varphi\varepsilon_i, \varepsilon_j)\gamma\varepsilon_j\big) \\
&+ g(\varepsilon_i, \varepsilon_j)\gamma(\varepsilon_j) g(\zeta, \varepsilon_i) - \sum_{i,j=1}^{m} \big(\gamma(\varepsilon_j)\gamma(\varepsilon_j) - g(\varepsilon_j, \varepsilon_j)\big) g(\varepsilon_i, \varepsilon_i) \\
&+ \sum_{i,j=1}^{m} \big(\gamma(\varepsilon_i)\gamma(\varepsilon_j) - g(\varepsilon_i, \varepsilon_j)\big) g(\varepsilon_j, \varepsilon_i) + \sum_{i,j=1}^{m} \gamma(\varepsilon_j)\gamma(\varepsilon_j) g(\varphi\varepsilon_i, \varepsilon_i) - \sum_{i,j=1}^{m} \gamma(\varepsilon_i)\gamma(\varepsilon_j) g(\varphi\varepsilon_j, \varepsilon_i) \Big] \\
&+ \beta \Big[\sum_{i,j=1}^{m} g(\varepsilon_j, \varepsilon_j) g(\varepsilon_i, \varepsilon_i) - \sum_{i,j=1}^{m} g(\varepsilon_i, \varepsilon_j) g(\varepsilon_j, \varepsilon_i) \Big].
\end{aligned}$$

Thus, we arrive at

$$\begin{aligned}
\sum_{i,j=1}^{m} \widetilde{\mathrm{Re}\,ie}(\varepsilon_i, \varepsilon_j, \varepsilon_j, \varepsilon_i) = S(\varepsilon_j, \varepsilon_j) &+ \alpha \big[-g(\varepsilon_j, \varepsilon_j) + m\gamma(\varepsilon_j)\gamma(\varepsilon_j) - mg(\varphi\varepsilon_j, \varepsilon_j) \big] \\
&+ (\beta - 1)\big[g(\varphi\varepsilon_j, \varepsilon_j) + (m-2) g(\varphi\varepsilon_j, \varphi\varepsilon_j) \big] + \beta(m-1) g(\varepsilon_j, \varepsilon_j).
\end{aligned} \tag{7}$$

For $u \in T\mathcal{M}$ we have $\varphi u = Pu + Qu$, $Pu \in T\mathcal{M}$, $Qu \in T^{\perp}\mathcal{M}$. Given an orthonormal frame $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m\}$ for a differential distribution D, we proceed to define the squared norms of P and Q as follows

$$\sum_{i,j=1}^{m} g^2(\varepsilon_i, P\varepsilon_j) = \|P\|^2 \quad \text{and} \quad \sum_{i=1}^{m} \|Q\varepsilon_i\|^2 = \|Q\|^2.$$

Also, for any $i = 1, 2, \dots, m$, where $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m, \zeta\}$ is a local orthonormal frame, we have

$$\sum_{j=1}^{m} g^2(\varepsilon_i, \varphi\varepsilon_j) = \cos^2\Theta.$$

Furthermore, the scalar curvature σ at p can be expressed in the following manner

$$2\sigma = \sum_{i \neq j}^{m} \mathcal{K}(\varepsilon_i \wedge \varepsilon_j) + 2\sum_{i=1}^{m} \mathcal{K}(\varepsilon_i \wedge \zeta).$$

Consider an orthonormal frame $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m\}$ of $T_p\mathcal{M}$ with

$$\varepsilon_2 = \frac{1}{\cos\Theta} P\varepsilon_1, \dots, \varepsilon_{2k} = \frac{1}{\cos\Theta} P\varepsilon_{2k-1}.$$

And we have

$$g\,(\varphi\varepsilon_1, \varphi\varepsilon_2) = g\Big(P\varepsilon_1, \frac{1}{\cos\Theta} P\varepsilon_1\Big) = \cos\Theta,$$

also in a similar manner,

$$g\,(\varphi\varepsilon_i, \varphi\varepsilon_{i+1}) = \cos\Theta$$

for $i = 3, 5, \dots, 2k-1$.

Now, recall the following results.

Lemma 1 ([5]). *Let $v_1, v_2, \dots, v_k$, $\kappa \geq 1$, and v be $\kappa + 1$ real numbers such that*

$$\left(\sum_{i=1}^{\kappa} v_i\right)^2 = (\kappa - 1)\left(\sum_{i=1}^{\kappa} v_i^2 + v\right).$$

Then $2v_1v_2 \geq v$ and the equality holds if and only if $v_1 = v_2 = v_3 = \cdots = v_k$.

Theorem 2 ([4, 15]). *Let $\mathcal{M}'$ be any submanifold of an almost contact metric manifold $(\mathcal{M}, \varphi, \gamma, \zeta, g)$ such that $\zeta \in T\mathcal{M}'$. Then*

1) *$\mathcal{M}'$ is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that $P^2 = -\lambda(I - \gamma \otimes \zeta)$; furthermore, if θ is the slant angle of $\mathcal{M}$, then $\lambda = \cos^2\theta$;*

2) *$g(Pu, Pv) = \cos^2\theta\big[g(u, v) - \gamma(u)\gamma(v)\big]$ for any $u, v \in T\mathcal{M}$.*

Theorem 3. *Let an n-dimensional TS-space form conceding a $QSNM$-connection and a Θ-slant submanifold $\mathcal{M}'$, $\dim(\mathcal{M}') = m$. Then we have*

$$\delta_{\mathcal{M}} \leq \frac{m^2(m-2)}{2(m-1)}\|\mathcal{H}\|^2 - \frac{(\alpha^2-\beta^2)}{2} - m(m-2)\cos^2\Theta - (m-1)\left[\alpha + \beta\left\{\frac{(m-2)}{2}\cos^2\Theta + \frac{5}{4}\right\}\right].$$

Proof. Utilizing equation (7), we arrive at the following expression

$$\widetilde{\mathrm{Re}\,ie}(\varepsilon_i, \varepsilon_j, \varepsilon_j, \varepsilon_i) = m(m-1)(\alpha^2-\beta^2) + \beta(m-1)\big[(m-2)\cos^2\Theta + m\big] - \cos^2\Theta(m-2)(m-1). \tag{8}$$

By incorporating equations (6), (8), and then (5), we deduce the subsequent relationship

$$m(m-1)(\alpha^2-\beta^2) + \beta(m-1)\big[(m-2)\cos^2\Theta + m\big] - \cos^2\Theta(m-2)(m-1) = 2\sigma + \|\hbar\|^2 - m^2\|\mathcal{H}\|^2. \tag{9}$$

Equivalently, equation (9) can be written as

$$2\sigma = m^2\|\mathcal{H}\|^2 - \|\hbar\|^2 + m(m-1)(\alpha^2-\beta^2) + \beta(m-1)\big[(m-2)\cos^2\Theta + m\big] - \cos^2\Theta(m-2)(m-1).$$

Introducing the notation

$$\omega = 2\sigma - \frac{m^2}{m-1}\|\mathcal{H}\|^2 - m(m-1)(\alpha^2-\beta^2) + \beta(m-1)\big[(m-2)\cos^2\Theta + m\big] + \cos^2\Theta(m-2)(m-1), \tag{10}$$

we turn up

$$\omega = m^2\|\mathcal{H}\|^2\left(1 - \frac{m-2}{m-1}\right) - \|\hbar\|^2,$$

$$m^2\|\mathcal{H}\|^2 = (m-1)\big(\omega + \|\hbar\|^2\big). \tag{11}$$

Consider a point p belonging to $T\mathcal{M}$, and let π be a subspace of $T_p\mathcal{M}$ with dimension 2, such that $\pi = span\,\{\varepsilon_1, \varepsilon_2\}$. Defining $\varepsilon_{m+1} = \frac{\mathcal{H}}{\|\mathcal{H}\|}$, we can deduce from equation (11) that

$$\left(\sum_{i=1}^{m} \hbar_{ii}^{m+1}\right)^2 = (m-1)\left(\sum_{i,j=1}^{m}\sum_{r=m+1}^{2m} \left(\hbar_{ij}^{r}\right)^2 + \omega\right),$$

or equivalently,

$$\left(\sum_{i=1}^{m} \hbar_{ii}^{m+1}\right)^2 = (m-1)\left\{\sum_{i=1}^{m}\left(\hbar_{ii}^{m+1}\right)^2 + \sum_{i\neq j}\left(\hbar_{ij}^{m+1}\right)^2 + \sum_{i,j=1}^{m}\sum_{r=m+2}^{2m}\left(\hbar_{ij}^{r}\right)^2 + \omega\right\}. \tag{12}$$

By applying Lemma 1, we deduce from (12) that

$$2\hbar_{11}^{m+1}\hbar_{22}^{m+1} \geq \sum_{i=1}^{m}\left(\hbar_{ii}^{m+1}\right)^2 + \sum_{i\neq j}\left(\hbar_{ij}^{m+1}\right)^2 + \sum_{i,j=1}^{m}\sum_{r=m+2}^{2m}\left(\hbar_{ij}^{r}\right)^2 + \omega.$$

Using the Gauss equation for $U = W = \varepsilon_1$ and $V = Z = \varepsilon_2$, we obtain

$$\begin{aligned}
\mathcal{K}(\pi) &= \left(\alpha^2-\beta^2\right) + (m-1)(\alpha+\beta) + (m-2)(\beta-1)\cos^2\Theta + \sum_{r=m+1}^{2m}\left[\hbar_{11}^{r}\hbar_{22}^{r} - \left(\hbar_{12}^{r}\right)^2\right] \\
&\geq \left(\alpha^2-\beta^2\right) + (m-1)(\alpha+\beta) + (m-2)(\beta-1)\cos^2\Theta \\
&\quad + \frac{1}{2}\left[\sum_{i\neq j}\left(\hbar_{ij}^{m+1}\right)^2 + \sum_{i,j=1}^{m}\sum_{r=m+2}^{2m}\left(\hbar_{ij}^{r}\right)^2 + \omega\right] + \sum_{r=m+2}^{2m}\hbar_{11}^{r}\hbar_{22}^{r} - \sum_{r=m+1}^{2m}\left(\hbar_{12}^{r}\right)^2 \\
&= \left(\alpha^2-\beta^2\right) + (m-1)(\alpha+\beta) + (m-2)(\beta-1)\cos^2\Theta \\
&\quad + \frac{1}{2}\sum_{i\neq j}\left(\hbar_{ij}^{m+1}\right)^2 + \frac{1}{2}\sum_{i,j=1}^{m}\sum_{r=m+2}^{2m}\left(\hbar_{ij}^{r}\right)^2 + \frac{1}{2}\omega + \sum_{r=m+2}^{2m}\hbar_{11}^{r}\hbar_{22}^{r} - \sum_{r=m+1}^{2m}\left(\hbar_{12}^{r}\right)^2 \\
&= \left(\alpha^2-\beta^2\right) + (m-1)(\alpha+\beta) + (m-2)(\beta-1)\cos^2\Theta + \frac{1}{2}\sum_{i\neq j}\left(\hbar_{ij}^{m+1}\right)^2 \\
&\quad + \frac{1}{2}\sum_{r=m=2}^{2m}\sum_{i,j>2}\left(\hbar_{ij}^{r}\right)^2 + \frac{1}{2}\sum_{r=m+2}^{2m}\left(\hbar_{11}^{r}+\hbar_{22}^{r}\right)^2 + \sum_{j>2}\left[\hbar_{11}^{r}\hbar_{22}^{r} - \left(\hbar_{12}^{r}\right)^2\right] + \frac{1}{2}\omega \\
&\geq \left(\alpha^2-\beta^2\right) + (m-1)(\alpha+\beta) + (m-2)(\beta-1)\cos^2\Theta + \frac{\omega}{2},
\end{aligned}$$

or equivalently,

$$\mathcal{K}(\pi) \geq \left(\alpha^2-\beta^2\right) + (m-1)(\alpha+\beta) + (m-2)(\beta-1)\cos^2\Theta + \frac{\omega}{2}. \tag{13}$$

Furthermore, substituting relation (10) into (13), we find that

$$\begin{aligned}
\sigma - Inf\mathcal{K}(\pi) \leq \frac{m^2(m-2)}{2(m-1)}\|\mathcal{H}\|^2 - \frac{\left(\alpha^2-\beta^2\right)}{2} - m(m-2)\cos^2\Theta \\
- (m-1)\left[\alpha + \beta\left\{\frac{(m-2)}{2}\cos^2\Theta + \frac{5}{4}\right\}\right].
\end{aligned} \tag{14}$$

From the implication of relation (14), we can conclude that

$$\begin{aligned}
\delta_{\mathcal{M}} \leq \frac{m^2(m-2)}{2(m-1)}\|\mathcal{H}\|^2 - \frac{\left(\alpha^2-\beta^2\right)}{2} - m(m-2)\cos^2\Theta \\
- (m-1)\left[\alpha + \beta\left\{\frac{(m-2)}{2}\cos^2\Theta + \frac{5}{4}\right\}\right],
\end{aligned}$$

where $\delta_{\mathcal{M}}$ is defined according to the formula (1). This inequality is that to be established. □

Theorem 4. *Let $\mathcal{M}$ be an m-dimensional submanifold of a TS- manifold M with a $QSNM$-connection. Then the equality dominates uniformly if and only if the shape operators A of $\mathcal{M}$ in M take the following forms with the suitable orthonormal frames $\{\varepsilon_1, \ldots, \varepsilon_m\}$ and $\{\varepsilon_{m+1}, \ldots, \varepsilon_n\}$ such as*

$$A_{m+1} = \begin{pmatrix} \tau & 0 & 0 & \ldots & 0 \\ 0 & \lambda & 0 & \ldots & 0 \\ 0 & 0 & \mu & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \mu \end{pmatrix}, \quad \tau + \lambda = \mu,$$

$$A_r = \begin{pmatrix} \hbar^r_{11} & \hbar^r_{22} & 0 & 0 & \ldots & 0 & 0 \\ \hbar^r_{12} & -\hbar^r_{11} & 0 & \ldots & & 0 \\ 0 & 0 & 0 & \ldots & & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & & 0 \end{pmatrix},$$

where we indicate by

$$A_r = A_{\varepsilon_r}, \quad r = m+1, \ldots, n,$$

$$\hbar^r_{ij} = g\left(\hbar\left(\varepsilon_i, \varepsilon_j\right), \varepsilon_r\right), \quad r = m+2, \ldots, n.$$

Proof. By similar arguments adopted in [17], one can arrive at the desired result. □

5 Chen inequality for invariant and anti-invariant submanifolds of TS-manifolds with $QSNM$-connection

Within this section, our focus shifts towards the classification of the Chen inequality for both invariant and anti-invariant submanifolds in a TS-manifold. This classification is conducted concerning the $QSNM$-connection and is articulated with respect to the slant angle Θ, taking into consideration the insights from Theorem 3.

When the submanifold $\mathcal{M}$ is invariant, the slant angle Θ assumes a value of 0, whereas for anti-invariant submanifolds, the slant angle Θ takes on the value of $\pi/2$. These considerations lead us to the subsequent outcomes.

Corollary 1. *Let $\mathcal{M}$ be an m-dimensional invariant submanifold of a TS-manifold M with a $QSNM$-connection. Then*

$$\delta_{\mathcal{M}} \leq \frac{m^2(m-2)}{2(m-1)} \|\mathcal{H}\|^2 - \frac{(\alpha^2 - \beta^2)}{2} - m(m-2) - (m-1)\left[\alpha + \beta\left\{(m-2) + \frac{5}{4}\right\}\right].$$

Corollary 2. *Let $\mathcal{M}$ be an m-dimensional anti-invariant submanifold of a TS-manifold M with a $QSNM$-connection. Then*

$$\delta_{\mathcal{M}} \leq \frac{m^2(m-2)}{2(m-1)} \|\mathcal{H}\|^2 - \frac{(\alpha^2 - \beta^2)}{2} - (m-1)\left[\alpha + \frac{5}{4}\right].$$

Remark 1. *By considering specific values of α and β, we can derive the Chen inequality applicable to θ-slant submanifolds in α-Sasakian, β-Kenmotsu, and cosymplectic manifolds, respectively.*

By utilizing the aforementioned Theorem 3, we can deduce the subsequent outcomes.

Theorem 5. *Let an n-dimensional α-Sasakian space form conceding a $QSNM$-connection and a Θ-slant submanifold $\mathcal{M}$, $\dim(\mathcal{M}) = m$. Then we have*

$$\delta_{\mathcal{M}} \leq \frac{m^2(m-2)}{2(m-1)} \|\mathcal{H}\|^2 - \frac{\alpha^2}{2} - m(m-2)\cos^2\Theta - (m-1)\alpha.$$

Theorem 6. *Let an n-dimensional β-Kenmotsu space form conceding a $QSNM$-connection and a Θ-slant submanifold $\mathcal{M}$, $\dim(\mathcal{M}) = m$. Then we have*

$$\delta_{\mathcal{M}} \leq \frac{m^2(m-2)}{2(m-1)} \|\mathcal{H}\|^2 + \frac{\beta^2}{2} - m(m-2)\cos^2\Theta - (m-1)\left[\beta\left\{\frac{(m-2)}{2}\cos^2\Theta + \frac{5}{4}\right\}\right].$$

Theorem 7. *Let an n-dimensional cosymplectic space form conceding a $QSNM$-connection and a Θ-slant submanifold $\mathcal{M}$, $\dim(\mathcal{M}) = m$. Then we have*

$$\delta_{\mathcal{M}} \leq \frac{m^2(m-2)}{2(m-1)} \|\mathcal{H}\|^2 - m(m-2)\cos^2\Theta - (m-1).$$

We can further categorize the Chen inequality for both invariant and anti-invariant submanifolds in α-Sasakian, β-Kenmotsu, and cosymplectic manifolds under the conditions $\beta = 0$, $\alpha = 0$, and $\alpha = \beta = 0$, respectively. These classifications are carried out while considering the $QSNM$-connection and in relation to the slant angle Θ taking values of 0 and $\pi/2$, respectively. Consequently, the ensuing corollaries come to light.

Corollary 3. *Let $\mathcal{M}$ be an m-dimensional invariant submanifold of an α-Sasakian manifold M with a $QSNM$-connection. Then*

$$\delta_{\mathcal{M}} \leq \frac{m^2(m-2)}{2(m-1)} \|\mathcal{H}\|^2 - \frac{\alpha^2}{2} - m(m-2) - (m-1)\alpha.$$

Corollary 4. *Let $\mathcal{M}$ be an m-dimensional invariant submanifold of a β-Kenmotsu manifold M with a $QSNM$-connection. Then*

$$\delta_{\mathcal{M}} \leq \frac{m^2(m-2)}{2(m-1)} \|\mathcal{H}\|^2 + \frac{\beta^2}{2} - m(m-2) - (m-1)\left[\beta\left\{\frac{(m-2)}{2} + \frac{5}{4}\right\}\right].$$

Corollary 5. *Let $\mathcal{M}$ be an m-dimensional invariant submanifold of a cosymplectic manifold M with a $QSNM$-connection. Then*

$$\delta_{\mathcal{M}} \leq \frac{m^2(m-2)}{2(m-1)} \|\mathcal{H}\|^2 - m(m-2).$$

Corollary 6. *Let $\mathcal{M}$ be an m-dimensional anti-invariant submanifold of an α-Sasakian manifold M with a $QSNM$-connection. Then*

$$\delta_{\mathcal{M}} \leq \frac{m^2(m-2)}{2(m-1)} \|\mathcal{H}\|^2 - \frac{\alpha^2}{2} - (m-1)\alpha.$$

Corollary 7. *Let $\mathcal{M}$ be an m-dimensional anti-invariant submanifold of a β-Kenmotsu manifold M with a $QSNM$-connection. Then*

$$\delta_{\mathcal{M}} \leq \frac{m^2(m-2)}{2(m-1)} \|\mathcal{H}\|^2 + \frac{\beta^2}{2}.$$

Corollary 8. *Let $\mathcal{M}$ be an m-dimensional anti-invariant submanifold of a cosymplectic manifold M with a $QSNM$-connection. Then*

$$\delta_{\mathcal{M}} \leq \frac{m^2(m-2)}{2(m-1)} \|\mathcal{H}\|^2.$$

Remark 2. *Similarly, the equality condition can be derived in the same way as in Theorem 4 for Θ-slant submanifolds in α-Sasakian, β-Kenmotsu, and cosymplectic manifolds. This derivation holds for specific instances where $\beta = 0$, $\alpha = 0$, and both α and β are zero, respectively, while utilizing the $QSNM$-connection.*

References

[1] Ahmad M., Jun J.B., Siddiqi M.D. *On some properties of semi-invariant submanifolds of a nearly trans-Sasakian manifolds admitting a quarter-symmetric non-metric connection*. J. Chungcheong Math. Soc. 2012, **25** (1), 73–80.

[2] Ahmad M., Rahman S., Siddiqi M.D. *Semi-invariant submanifolds of a nearly Sasakian manifold endowed with a semi-symmetric metric connection*. Bull. Allahabad Math. Soc. 2010, **25** (1), 23–33.

[3] Agashe N.S., Chafle M.R. *A semi symmetric non-metric connection in a Riemannian manifold*. Indian J. Pure Appl. Math. 1992, **23**, 399–409.

[4] Cabrerizo J.L., Carriazo A., Fernández L.M., Fernández M. *Structure on a slant submanifold of a contact manifold*. Indian J. Pure Appl. Math. 2000, **31**, 857–874.

[5] Chen B.-Y. *Some pinching and classification theorems for mimimal submanifold*. Arch. Math. 1993, **60** (6), 568–578. doi:10.1007/BF01236084

[6] Chen B.-Y. *A Riemannian invariant and its applications to submanifold theory*. Results Math. 1996, **27**, 17–26. doi:10.1007/BF03322265

[7] Cioroboiu D., Oiaga A. *B.Y. Chen inequalities for slant submanifolds in Sasakian space forms*. Rend. Circ. Mat. Palermo (2) 2003, **52** (2), 367–381. doi:10.1007/BF02872761

[8] Defever F., Mihai I., Verstraelen L. *B.-Y. Chen's inequality for C-totally real submanifolds of Sasakian space forms*. Boll. Unione Mat. Ital. 1997, **7** (11), 365–374.

[9] De U.C., De K. *On a class of three-dimensional trans-Sasakian manifolds*. Commun. Korean Math. Soc. 2012, **27** (4), 795–808.

[10] De U.C., Tripathi M.M. *Ricci tensor in 3-dimensional trans-Sasakian manifolds*. Kyungpook Math. J. 2003, **43**, 247–255.

[11] De U.C., Kamilya D. *Hypersurfaces of Rieamnnian manifold with semi-symmetric non-metric connection*. J. Indian Inst. Sci. 1995, **75**, 707–710.

[12] Golab S. *On semi-symmetric and quarter-symmetric linear connections*. Tensor (N.S.) 1975, **29**, 249–254.

[13] Hayden H.A. *Subspaces of a space with torsion*. Proc. Lond. Math. Soc. (3) 1932, **34**, 27–50. doi:10.1112/plms/s2-34.1.27

[14] Kim J.S., Prasad R., Tripathi M.M. *On generalized Ricci-recurrent trans-Sasakian manifolds*. J. Korean Math. Soc. 2002, **39** (6), 953–961. doi:10.4134/JKMS.2002.39.6.953

[15] Lotta A. *Slant submanifolds in contact geometry*. Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 1996, **39** (87), 183–198.

[16] Marrero J.C. *The local structure of trans-Sasakian manifolds*. Ann. Mat. Pura Appl. (4) 1992, **162** (4), 77–86. doi:10.1007/BF01760000

[17] Mihai A., Ozgur C. *Chen inequalities for submanifolds of real space forms with a semi-symmetric metric connection*. Taiwanese J. Math. 2010, **14** (4), 1465–1477. doi:10.11650/twjm/1500405961

[18] Oubina J.A. *New class of almost contact metric structures*. Publ. Math. Debrecen 1985, **32**, 187–193.

[19] Patra C., Bhattacharyya A. *Trans-Sasakian manifold admitting quarter-symmetric non-metric connection*. Acta Univ. Apulensis 2013, **36**, 39–49.

[20] Sharfuddin A., Husain S.I. *Semi-symmetric metric connections in almost contact manifolds*. Tensor (N.S.) 1976, **30**, 133–139.

[21] Siddiqui A.N., Murathan C., Siddiqi M.D. *The Chen's first inequality for submanifolds of statistical warped product manifolds*. J. Geom. Phys. 2021, **169** (3), 104344. doi:10.1016/j.geomphys.2021.104344

[22] Siddiqui A.N., Chen B.-Y., Siddiqi M.D. *Chen inequalities for statistical submersions between statistical manifolds*. Int. J. Geom. Methods Mod. Phys. 2021, **18** (04), 2150049. doi:10.1142/S0219887821500493

[23] Tripathi M.M. *A new connection in a Riemannian manifold*. Int. Electron. J. Geom. 2008, **1** (1), 15–24.

[24] Yano K., Kon M. Structures on manifolds. In: Series in Pure Mathematics, 3. World Scientific Publ. Co., Singapore, 1984.

Received 15.11.2021
Revised 02.03.2024

Сіддікі М.Д., Сіддікі А.Н. *Нерівності Чена для занурень у транс-Сасакянові просторові форми з косим чинником* // Карпатські матем. публ. — 2025. — Т.17, №1. — С. 200–210.

У цій статті ми зосереджуємо увагу на дослідженні транс-Сасакянових многовидів, які обладнані особливим типом неметричного зв'язку, відомим як чверть-симетричний неметричний ($QSNM$) зв'язок. Ми виводимо математичні формули, що описують тензор кривини $\widetilde{R}$ транс-Сасакянових просторових форм, використовуючи згаданий $QSNM$-зв'язок. Основну увагу приділено встановленню нерівностей Чена. Ці нерівності використовуються для характеристики косих підмноговидів у транс-Сасакянових просторових формах, які пов'язані $QSNM$-зв'язком. Крім того, ми розглядаємо класифікацію інваріантів Чена. Ця класифікація поширюється на α-Сасакянові, β-Кенмоцу та косимплектичні многовиди, усі з яких наділені особливим $QSNM$-зв'язком.

Ключові слова і фрази: нерівність Чена, косий підмноговид, транс-Сасакяновий многовид, чверть-симетричний неметричний зв'язок.