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Bernstein-Nikol'skii-type inequalities for trigonometric polynomials

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We obtain order estimates for Bernstein-Nikol'skii-type inequalities for trigonometric polynomials with an arbitrary choice of harmonics. It is established that in the case $q = \infty$, $1 < p \leq 2$ these inequalities for trigonometric polynomials with arbitrary choice of harmonics and for ordinary trigonometric polynomials has different order of estimates.

Key words and phrases: (ψ, β)-derivative, Bernstein-Nikol'skii inequality.

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Introduction

In numerous problems of approximation theory of one variable periodic functions, an important role is played by inequalities connecting the norms of polynomials and their derivatives in different metrics (see, e.g., [9, Chap I, Sec. 2]), namely,

for any trigonometric polynomial $t \in T(m) = \left\{t : t(x) = \sum\limits_{k=-m}^{m} c_k e^{ikx}\right\}$ *and any* $r > 0$, $\beta \in \mathbb{R}$ *the following inequality is true:*

$$\|t^r_\beta\|_q \ll m^{r+1/p-1/q}\|t\|_p, \quad 1 \leq p \leq q \leq \infty.$$

Relations of this form are called Bernstein-Nikol'skii inequalities because they combine Bernstein inequalities for $p = q$ and Nikol'skii "inequalities of different metrics" for $r = 0$.

In connection with Bernstein-Nikol'skii inequalities, V.E. Maiorov [3,4] considered a more general statement of the problem for trigonometric polynomials from the set $T(m)$, namely, he studied the quantity

$$\mathcal{T}_m(r,q,p) = \inf_{K_m} \sup_{t \in L(K_m)} \frac{\|t^{(r)}\|_q}{\|t\|_p}, \quad 1 \leq p, q \leq \infty,$$

where the derivative of the order $r \geq 0$ is treated in Weyl's sense, i.e. $\beta = r$, $K_m = \{j_1, \ldots, j_m\}$ is an arbitrary collection of m different integers and

$$L(K_m) = \left\{t \colon t(x) = \sum_{j \in K_m} c_j e^{ijx}\right\}.$$

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1 Definition of classes of functions and approximative characteristics

Let L_q be a space of 2π-periodic functions, that are summable to a power q, $1 \le q < \infty$ (resp., essentially bounded for $q = \infty$), on the segment $[-\pi, \pi]$. The norm in this space is defined as follows

$$\|f\|_{L_q} = \|f\|_q = \left(\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} |f(x)|^q dx\right)^{\frac{1}{q}}, \quad 1 \le q < \infty,$$

$$\|f\|_{L_\infty} = \|f\|_\infty = \operatorname*{ess\,sup}_{x\in[-\pi,\pi]} |f(x)|.$$

For a function $f \in L_1$, we consider its Fourier series

$$\sum_{k\in\mathbb{Z}} \hat{f}(k)e^{ikx},$$

where $\hat{f}(k) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)e^{-ikx}dx$ are the Fourier coefficients of the function f. In what follows, we always assume that the function $f \in L_1$ satisfies the condition

$$\int\limits_{-\pi}^{\pi} f(x)dx = 0.$$

Further, let $\psi \neq 0$ be an arbitrary function of a natural argument and let β be an arbitrary fixed real number. If a series

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\hat{f}(k)}{\psi(|k|)} e^{i(kx+\beta\frac{\pi}{2}\operatorname{sign} k)}$$

is the Fourier series of a summable function, then, following A.I. Stepanets [6, Vol. 1, p. 132], we introduce the (ψ, β)-derivative of the function f and denote it by f^{ψ}_{β}. If $\psi(|k|) = |k|^{-r}$, $r > 0$, $k \in \mathbb{Z} \setminus \{0\}$, then the (ψ, β)-derivative of the function f coincides with its (r, β)-derivative (denoted by f^{r}_{β}) in the Weyl-Nady sense.

By Ψ we denote the set of functions $\psi(\tau)$, $\tau \in \mathbb{N}$, satisfying the following conditions:

1) ψ are positive and nonincreasing;

2) there exists a constant $C > 0$ such that

$$\frac{\psi(\tau)}{\psi(2\tau)} \le C, \quad \tau \in \mathbb{N}.$$

It is easy to check that the functions $\frac{1}{\tau^r}$, $r > 0$, $\tau \in \mathbb{N}$; $\frac{\ln^{\gamma}(\tau+1)}{\tau^r}$, $\gamma \in \mathbb{R}$, $r > 0$, $\tau \in \mathbb{N}$, and some other functions belong to the set Ψ.

In what follows, we formulate the obtained results in terms of order relations. For two nonnegative sequences $\{a(n)\}_{n=1}^{\infty}$ and $\{b(n)\}_{n=1}^{\infty}$ the relation (order inequality) $a(n) \ll b(n)$ means that there exists a constant $C_1 > 0$ such that $a(n) \le C_1 b(n)$. The relation $a(n) \asymp b(n)$ is equivalent to $a(n) \ll b(n)$ and $b(n) \ll a(n)$. Note that the constants C_i, $i = 1, 2, 3, \ldots$, in the order relations may depend on the some parameters. These parameters will sometimes be indicated, in other cases they will be clear from the context.

The main aim of the present paper is to establish order estimates for quantities of the form

$$\mathcal{T}_m(\psi, \beta, \infty, p) = \inf_{K_m} \sup_{t \in L(K_m)} \frac{\|t_\beta^\psi\|_\infty}{\|t\|_p} \tag{1}$$

under certain conditions imposed on ψ, $\beta \in \mathbb{R}$.

Note that the value of $\mathcal{T}_m(r, \infty, p)$ (respectively $\mathcal{T}_m(\psi, \beta, \infty, p)$) is a constant in inequalities of Bernstein-Nykol'skii-type for the polynomials of the set $L(K_m)$, and the harmonics are chosen in this way to make this value the smallest.

2 Auxiliary Statements

Consider a trigonometric polynomial

$$\bar{t}(x) = \sum_{k \le m} \frac{a(k, x, \beta)}{k}, \tag{2}$$

where

$$a(k, x, \beta) = \cos\left(kx - \frac{\beta\pi}{2}\right) + \cos\left((k+1)x - \frac{\beta\pi}{2}\right) \\ - \cos\left(2kx - \frac{\beta\pi}{2}\right) - \cos\left((2k+1)x - \frac{\beta\pi}{2}\right).$$

The trigonometric polynomial $a(k, x, \beta)$ has an order is not higher than $2k+1$, respectively $\bar{t}(x)$ has an order not higher than $2m+1$, therefore $\bar{t}(x) \in T(m)$.

Lemma A ([7]). *For the trigonometric polynomial $\bar{t}(x)$ of the form (2) the estimate $\|\bar{t}\|_\infty \ll 1$ is true.*

Theorem A (see, e.g., [5, p. 159]). *Let $t \in T(m)$, $m > 0$. Then, for $1 \le q \le p \le \infty$, the inequality $\|t\|_p \ll m^{1/q - 1/p} \|t\|_q$ is true.*

Proposition A (see, e.g., [6, Vol. 2, p. 115]). *Suppose that $1 < q < \infty$ and ψ is an arbitrary nonincreasing sequence of nonnegative numbers. Then, for any polynomial $t \in T(m)$, the estimate $\|t_\beta^\psi\|_q \ll \psi^{-1}(m) \|t\|_q$ is true.*

Theorem B (Marcinkiewicz theorem, see, e.g., [11, Vol. 2, p. 346]). *Suppose that $\{\lambda_n\}_{n \in \mathbb{Z}}$ is a given sequence satistying the following conditions:*

1) $|\lambda_n| \le C_2$, $n \in \mathbb{Z}$;

2) $\sum\limits_{\mu = \pm 2^{\nu-1}}^{\pm 2^\nu - 1} |\lambda_{\mu+1} - \lambda_\mu| \le C_2$, $\nu \in \mathbb{N}$.

If

$$f(x) = \sum_{k=-\infty}^{+\infty} \hat{f}(k) e^{ikx} \in L_q, \quad 1 < q < \infty,$$

then

$$F(x) = \sum_{k=-\infty}^{+\infty} \lambda_k \hat{f}(k) e^{ikx} \in L_q$$

and there exists a constant $C_3(q)$ such that

$$\|F\|_q \le C_3(q) C_2 \|f\|_q.$$

Theorem C (Hausdorff-Young theorem, see, e.g., [11, Vol. 2, p. 154]). *Suppose that* $1 < q \leq 2$ *and* $\frac{1}{q} + \frac{1}{q'} = 1$. *Then for any function* $f \in L_q$ *we have*

$$\|f\|_q \geq \left(\sum_{k\in\mathbb{Z}} |\hat{f}(k)|^{q'} \right)^{1/q'}.$$

If a sequence $\{c_k\}$ *is such that* $\sum_{k\in\mathbb{Z}} |c_k|^q < \infty$, *then there exists a function* $f \in L_{q'}$ *for which* $\hat{f}(k) = c_k$ *and*

$$\|f\|_{q'} \leq \left(\sum_{k\in\mathbb{Z}} |\hat{f}(k)|^{q} \right)^{1/q}.$$

Denote

$$\mathcal{E}(m) = \{\varepsilon = (\varepsilon_1, \dots, \varepsilon_m), \varepsilon_j = \pm 1, j = \overline{1, m}\}.$$

Lemma B ([2]). *Suppose that* $0 < q < \infty$ *and* $p_i \in \mathbb{R}^n$, $i = \overline{1, m}$. *Then there exist positive constants* $C_4(q)$ *and* $C_5(q)$, *such that the inequality*

$$C_4(q) \left(\sum_{i=1}^{m} |p_i|^2 \right)^{1/2} \leq \left(2^{-m} \sum_{\varepsilon\in\mathcal{E}(m)} \left| \sum_{i=1}^{m} \varepsilon_i p_i \right|^q \right)^{1/2} \leq C_5(q) \left(\sum_{i=1}^{m} |p_i|^2 \right)^{1/2},$$

is true.

3 Main results

We first prove the auxiliary proposition.

Lemma 1. *Suppose* $\psi \in \Psi$ *and* $\beta \in \mathbb{R}$. *Then the estimate*

$$\sup_{t\in T(m)} \frac{\|t_\beta^\psi\|_\infty}{\|t\|_\infty} \gg \psi^{-1}(m) \tag{3}$$

is true.

Proof. We consider the trigonometric polynomial (2). By using Lemma A for the estimate (3), we can write

$$\sup_{t\in T(m)} \frac{\|t_\beta^\psi\|_\infty}{\|t\|_\infty} \gg \frac{\|\bar{t}_\beta^\psi\|_\infty}{\|\bar{t}\|_\infty} \gg \|\bar{t}_\beta^\psi\|_\infty = \left\| \sum_{k\leq m} \Big(a(k,x,\beta) \Big)_\beta^\psi \cdot k^{-1} \right\|_\infty \geq \left| \sum_{k\leq m} \Big(a(k,0,\beta) \Big)_\beta^\psi \cdot k^{-1} \right|,$$

where

$$\bar{t}(x)_\beta^\psi = \sum_{k\leq m} \Big(a(k,x,\beta) \Big)_\beta^\psi \cdot k^{-1}$$

and

$$\begin{aligned} \Big(a(k,x,\beta) \Big)_\beta^\psi = {} & \psi^{-1}(k) \cos\left(kx - \frac{\beta\pi}{2} \right) + \psi^{-1}(k+1) \cos\left((k+1)x - \frac{\beta\pi}{2} \right) \\ & - \psi^{-1}(2k) \cos\left(2kx - \frac{\beta\pi}{2} \right) - \psi^{-1}(2k+1) \cos\left((2k+1)x - \frac{\beta\pi}{2} \right). \end{aligned}$$

Since the sign of $\Big(a(k,0,\beta)\Big)_{\beta}^{\psi}$ for all k is the same, then

$$\sup_{t\in T(m)} \frac{\|t_{\beta}^{\psi}\|_{\infty}}{\|t\|_{\infty}} \gg \sum_{k\le m} \left| \Big(a(k,0,\beta)\Big)_{\beta}^{\psi} \right| \cdot k^{-1}.$$

Further, if $\psi \in \Psi$, then for $\Big(a(k,0,\beta)\Big)_{\beta}^{\psi}$ we can write

$$\begin{aligned}\Big(a(k,0,\beta)\Big)_{\beta}^{\psi} &\ge \psi^{-1}(k) + \psi^{-1}(k+1) - \psi^{-1}(2k) - \psi^{-1}(2k+1)\\ &= \psi^{-1}(2k)\left(\frac{\psi^{-1}(k)}{\psi^{-1}(2k)} - 1\right) + \psi^{-1}(2k+1)\left(\frac{\psi^{-1}(k+1)}{\psi^{-1}(2k+1)} - 1\right)\\ &\ge \psi^{-1}(2k) + \psi^{-1}(2k+1) = \psi^{-1}(2k+1)\left(\frac{\psi^{-1}(2k)}{\psi^{-1}(2k+1)} + 1\right) \ge \psi^{-1}(k)\end{aligned}$$

and respectively then

$$\sup_{t\in T(m)} \frac{\|t_{\beta}^{\psi}\|_{\infty}}{\|t\|_{\infty}} \gg \sum_{k\le m} \frac{\psi^{-1}(k)}{k} \ge \sum_{k=\frac{m}{2}}^{m} \frac{\psi^{-1}(k)}{k} \ge \psi^{-1}\left(\frac{m}{2}\right) \sum_{k=\frac{m}{2}}^{m} \frac{1}{k} \ge \psi^{-1}\left(\frac{m}{2}\right) \ln 2 \asymp \psi^{-1}(m).$$

Lemma 1 is proved. □

Now we consider a special case of the quantity (1), namely, we establish order estimates for the quantity

$$\sup_{t\in T(m)} \frac{\|t_{\beta}^{\psi}\|_{\infty}}{\|t\|_{p}}, \quad 1 < p < \infty.$$

Theorem 1. *Suppose that ψ is positive and nonincreasing sequence and $\beta \in \mathbb{R}$. Then the relation*

$$\sup_{t\in T(m)} \frac{\|t_{\beta}^{\psi}\|_{\infty}}{\|t\|_{p}} \ll \psi^{-1}(m) m^{1/p}, \quad 1 < p < \infty, \tag{4}$$

is true.

In addition, if $\psi \in \Psi$, then

$$\sup_{t\in T(m)} \frac{\|t_{\beta}^{\psi}\|_{\infty}}{\|t\|_{p}} \asymp \psi^{-1}(m) m^{1/p}, \quad 1 < p < \infty. \tag{5}$$

Proof. We now establish the upper bound in (4). By using Theorem A and Proposition A, we can write

$$\sup_{t\in T(m)} \frac{\|t_{\beta}^{\psi}\|_{\infty}}{\|t\|_{p}} \ll \sup_{t\in T(m)} \frac{m^{1/p}\|t_{\beta}^{\psi}\|_{p}}{\|t\|_{p}} \ll \sup_{t\in T(m)} \frac{\psi^{-1}(m) m^{1/p}\|t\|_{p}}{\|t\|_{p}} = \psi^{-1}(m) m^{1/p}.$$

We now derive the corresponding lower bound in (5). To this end, we present an example of a polynomial for which the lower bounds are realized.

For given m we choose $\tilde{s} \in \mathbb{N}$ such that $2^{\tilde{s}-1} \le m < 2^{\tilde{s}}$. Consider a polynomial

$$\tilde{t}(x) = \frac{1}{2} \sum_{k \in \rho(\tilde{s})} e^{ikx},$$

where $\rho(\tilde{s}) = \{k : 2^{\tilde{s}-1} \le |k| < 2^{\tilde{s}}\}$. Then, according to the definition of the (ψ, β)-derivative for the polynomial $\tilde{t}$, we can write

$$\tilde{t}^{\psi}_{\beta}(x) = \frac{1}{2} \sum_{k \in \rho(\tilde{s})} \psi^{-1}(|k|) e^{ikx} = \sum_{k \in \rho^{+}(\tilde{s})} \psi^{-1}(k) \cos kx,$$

where $\rho^{+}(\tilde{s}) = \{k : 2^{\tilde{s}-1} \le k < 2^{\tilde{s}}\}$.

Further, by using the relation (see, e.g., [6, Vol. 2, p. 42])

$$\Big\| \sum_{k=m}^{l} \cos kt \Big\|_p \asymp (l-m)^{1-1/p}, \quad m, l \in \mathbb{N},\ l > m \ \text{ and } \ 1 < p < \infty,$$

we find

$$\Big\| \sum_{k=2^{\tilde{s}-1}}^{2^{\tilde{s}}-1} \cos kx \Big\|_p \asymp 2^{(\tilde{s}-1)(1-1/p)} \asymp 2^{\tilde{s}(1-1/p)}. \tag{6}$$

Then, by using Lemma 1 and the relation (6) and stating that $\psi(2^{\tilde{s}}) \asymp \psi(m)$ (because $\psi \in \Psi$ and $2^{\tilde{s}-1} \le m < 2^{\tilde{s}}$), we get

$$\begin{aligned} \|(\tilde{t})^{\psi}_{\beta}\|_\infty \ge \psi^{-1}(2^{\tilde{s}}) \|\tilde{t}\|_\infty &\asymp \psi^{-1}(2^{\tilde{s}}) 2^{\tilde{s}} 2^{-\tilde{s}(1-1/p)} \Big\| \sum_{k=2^{\tilde{s}-1}}^{2^{\tilde{s}}-1} \cos kx \Big\|_p \\ &\asymp \psi^{-1}(m) m^{1/p} \Big\| \sum_{k=2^{\tilde{s}-1}}^{2^{\tilde{s}}-1} \cos kx \Big\|_p = \psi^{-1}(m) m^{1/p} \|\tilde{t}\|_p. \end{aligned}$$

Theorem 1 proved. □

In what follows, we formulate and prove the assertions concerning order estimates for the quantity $\mathcal{T}_m(\psi, \beta, \infty, p)$ with $1 < p < \infty$.

Theorem 2. *Suppose that $2 \le p < \infty$, ψ is a positive and nonincreasing sequence and $\beta \in \mathbb{R}$. Then the following estimate is true*

$$\mathcal{T}_m(\psi, \beta, \infty, p) \ll \psi^{-1}(m) m^{1/p}.$$

If $\psi \in \Psi$ and, in addition, there exists $\varepsilon > 0$ such that the sequence $\psi(\tau)\tau^{\varepsilon}$, $\tau \in \mathbb{N}$, does not increase, then

$$\mathcal{T}_m(\psi, \beta, \infty, p) \asymp \psi^{-1}(m) m^{1/p}.$$

Proof. The upper bound follows from Theorem 1, namely

$$\mathcal{T}_m(\psi, \beta, \infty, p) \le \sup_{f \in T(m)} \frac{\|f^{\psi}_{\beta}\|_\infty}{\|f\|_p} \ll \psi^{-1}(m) m^{1/p}.$$

We prove the lower bound. Let $K_m = \{j_1, \dots, j_m\}$, $0 < j_1 < \dots < j_m$, be an arbitrary collection of m different integers and let $m_s = |K_m \cap \rho(s)|$ be the number of elements of this collection from the set $\rho(s)$, $s \in \mathbb{Z}_+$. In what follows, we consider only $s \in \mathbb{Z}_+$ for which $K_m \cap \rho(s) \neq \varnothing$. Hence, the number of these s is finite.

Denote

$$\overline{K}_m = K_m \cap ((\mathbb{Z}\backslash\{0\})\backslash K),$$

where $K = \{\rho(s) : s \leq \mu\}$, the quantity μ is chosen from the condition

$$|K| = \sum_{s \leq \mu} 2^s \leq \frac{m}{2}, \quad m \asymp 2^\mu,$$

where $|K|$ is the number of elements of the set K. Then $|K| \asymp m$.

Since

$$\sum_{s>\mu} m_s \geq C_6 \frac{m}{2}, \tag{7}$$

we have $|\overline{K}_m| \asymp m$.

For each $s \in \mathbb{Z}_+$ such that $\overline{K}_m \neq \varnothing$, we consider a polynomial $\sum_{k\in\overline{K}_m} e^{ikx}$ and show that the relation

$$\Big\| \sum_{k\in\overline{K}_m} \psi(k) e^{ikx} \Big\|_p \gg \psi(2^s) \Big\| \sum_{k\in\overline{K}_m} e^{ikx} \Big\|_p$$

is true. To this end, for $s \in \mathbb{Z}_+$ we consider a sequence $\{\lambda_{k,s}\}$ given by the relation

$$\{\lambda_{k,s}\} = \left\{ \frac{\psi(2^s)}{\psi(k)} \right\}, \quad k \in \overline{K}_m.$$

We now show that the sequence $\{\lambda_{k,s}\}$ satisfies conditions 1) and 2) of Theorem B. For this purpose, it suffices to check the validity of these conditions for positive k such that $k \in K_m \cap \rho(s)$.

Since $\psi \in \Psi$, we get

$$|\lambda_{k,s}| = \left| \frac{\psi(2^s)}{\psi(k)} \right| \leq \frac{\psi(2^s)}{\psi(2^s)} = 1,$$

and

$$\begin{aligned} \sum_{k\in\overline{K}_m} |\lambda_{k+1,s} - \lambda_{k,s}| &= \sum_{k\in\overline{K}_m} \left| \frac{\psi(2^s)}{\psi(k+1)} - \frac{\psi(2^s)}{\psi(k)} \right| \leq \psi(2^s) \sum_{k\in\overline{K}_m} \left(\frac{1}{\psi(k+1)} - \frac{1}{\psi(k)} \right) \\ &\leq \psi(2^s) \left(\frac{1}{\psi(2^s)} - \frac{1}{\psi(2^{s-1})} \right) \leq \frac{\psi(2^s)}{\psi(2^s)} \left(1 - \frac{\psi(2^s)}{\psi(2^{s-1})} \right) = 1 - \frac{\psi(2^s)}{\psi(2^{s-1})}. \end{aligned} \tag{8}$$

Last difference is equal to zero, or less than 1. Since

$$\sum_{k\in\overline{K}_m} |\lambda_{k+1,s} - \lambda_{k,s}| \leq \sum_{k\in\overline{K}_m} |\lambda_{k+1,s}| + \sum_{k\in\overline{K}_m} |\lambda_{k,s}| \leq 2, \tag{9}$$

comparing (8) and (9), we arrive at the estimate

$$\sum_{k\in\overline{K}_m} |\lambda_{k+1,s} - \lambda_{k,s}| \leq 3.$$

Accordingly, if $C_7 = 3$, then for the sequence $\{\lambda_{k,s}\}$ conditions of Theorem B will be fulfilled.

We now act by a multiplicator $\Lambda_{k,s}$ specified by the sequence $\{\lambda_{k,s}\}$ upon the polynomial $\sum\limits_{k\in\overline{K}_m} \psi(k)e^{ikx}$. As a result, we obtain

$$\Lambda_{k,s} \sum_{k\in\overline{K}_m} \psi(k)e^{ikx} = \sum_{k\in\overline{K}_m} \frac{\psi(2^s)}{\psi(k)}\psi(k)e^{ikx} = \psi(2^s) \sum_{k\in\overline{K}_m} e^{ikx}.$$

This yields

$$\Big\|\Lambda_{k,s} \sum_{k\in\overline{K}_m} e^{ikx}\Big\|_p = \psi(2^s)\Big\| \sum_{k\in\overline{K}_m} e^{ikx}\Big\|_p.$$

On the other hand, by Theorem B, we get

$$\Big\|\Lambda_{k,s} \sum_{k\in\overline{K}_m} e^{ikx}\Big\|_p = \psi(2^s)\Big\| \sum_{k\in\overline{K}_m} e^{ikx}\Big\|_p \le C_8\Big\| \sum_{k\in\overline{K}_m} \psi(k)e^{ikx}\Big\|_p.$$

We arrive at the required relation

$$\Big\| \sum_{k\in\overline{K}_m} \psi(k)e^{ikx}\Big\|_p \gg \psi(2^s)\Big\| \sum_{k\in\overline{K}_m} e^{ikx}\Big\|_p. \tag{10}$$

Since $p \ge 2$, by using Theorem C for (10), we can write

$$\begin{aligned}\Big\| \sum_{k\in\overline{K}_m} \psi(k)e^{ikx}\Big\|_p &\le \Big(\sum_{k\in\overline{K}_m} |\psi(k)|^{p'}\Big)^{1/p'} \le \Big(\psi^{p'}(2^{s-1})m_s\Big)^{1/p'} \\ &\asymp \Big(\psi^{p'}(2^s)m_s\Big)^{1/p'} = \psi(2^s)m_s^{1/p'}.\end{aligned} \tag{11}$$

Further, denote

$$L(\overline{K}_m) = \Big\{t : t(x) = \sum_{k\in\overline{K}_m} e^{ikx}\Big\}.$$

Thus, according to (10) and (11), we get

$$\begin{aligned}\sup_{t\in L(K_m)} \frac{\|t_\beta^\psi\|_\infty}{\|t\|_p} &\ge \sup_{t\in L(\overline{K}_m)} \frac{\|t_\beta^\psi\|_\infty}{\|t\|_p} \ge \sup_{t\in L(\overline{K}_m)} \frac{\psi^{-1}(2^s)\|t\|_\infty}{\|t\|_p} \\ &\ge \sup_{t\in L(\overline{K}_m)} \frac{\|t\|_\infty}{\psi(2^s)\|t\|_p} \ge \sup_{s\in\mathbb{Z}_+} \frac{m_s}{\psi(2^s)m_s^{1/p'}} = \sup_{s\in\mathbb{Z}_+} \psi^{-1}(2^s)m_s^{1/p}.\end{aligned} \tag{12}$$

Denote

$$I = \sup_{s\in\mathbb{Z}_+} \psi^{-1}(2^s)m_s^{1/p}, \tag{13}$$

where $m_s \le 2^s$ and $\sum\limits_{s\in\mathbb{Z}_+} m_s = m$.

We choose $\mu > 0$ such that $2^{\mu-1} \le m < 2^\mu$ and

$$\sum_{s\le\mu} m_s \le \sum s \le \mu 2^s \le C_9 2^\mu \le \frac{m}{2}.$$

In this case, the following relation

$$\sum_{s>\mu} m_s \geq C_{10}\frac{m}{2}$$

is true. Moreover, the relation (13) immediately implies that

$$m_s \leq I^p \psi^p(2^s) \tag{14}$$

for any $s \in \mathbb{Z}_+, \overline{K}_m \neq \varnothing$.

Since the sequence $\psi(\tau)\tau^{\varepsilon}$, $\tau \in \mathbb{N}$, is nonincrreasing, in view of (7) and (14), we obtain

$$\begin{aligned}\frac{m}{2} \ll \sum_{s>\mu} m_s \leq \sum_{s>\mu} I^p \psi^p(2^s) = I^p \sum_{s>\mu} \psi^p(2^s) = I^p \sum_{s>\mu} \psi^p(2^s) 2^{sp\varepsilon} 2^{-sp\varepsilon} \\ \ll I^p \psi^p(2^\mu) 2^{\mu p\varepsilon} \sum_{s>\mu} 2^{-sp\varepsilon} \ll I^p \psi^p(2^\mu) 2^{\mu p\varepsilon} 2^{-\mu p\varepsilon} = I^p \psi^p(2^\mu).\end{aligned} \tag{15}$$

By using (15), we get $I \gg \psi^{-1}(2^\mu) m^{1/p}$ and, in view of the inequality $2^{\mu-1} \leq m \leq 2^\mu$, we obtain

$$I \gg \psi^{-1}(m) m^{1/p}. \tag{16}$$

Comparing (12), (13) and (16), we arrive at the required lower bound

$$\mathcal{T}_m(\psi, \beta, \infty, p) \gg \psi^{-1}(m) m^{1/p}.$$

Theorem 2 is proved. $\square$

Theorem 3. *Suppose that $1 < p \leq 2$, ψ is a positive and nonincreasing sequence and $\beta \in \mathbb{R}$. Then the estimate*

$$\mathcal{T}_m(\psi, \beta, \infty, p) \ll \psi^{-1}(m) m^{1/2}$$

is true.

If $\psi \in \Psi$ and, in addition, there exists $\varepsilon > 0$, such that the sequence $\psi(\tau)\tau^{\varepsilon}$, $\tau \in \mathbb{N}$, does not increase, then

$$\mathcal{T}_m(\psi, \beta, \infty, p) \asymp \psi^{-1}(m) m^{1/2}.$$

Proof. We prove the upper bound. We consider a set $K = \{\rho(s) : s \leq \mu\}$, where the quantity μ is chosen from the condition

$$|K| = \sum_{s \leq \mu} 2^s \leq \frac{m}{2}, \quad m \asymp 2^\mu.$$

Then $|K| \asymp m$. By using Lemma B we get

$$\begin{aligned}2^{-|K|} \sum_{\mathcal{E}(|K|)} \Big\| \sum_{k\in K} \varepsilon_k e^{ikx} \Big\|_{p'}^{p'} \asymp 2^{-|K|} \sum_{\mathcal{E}(|K|)} \int_{-\pi}^{\pi} \Big| \sum_{k\in K} \varepsilon_k e^{ikx} \Big|^{p'} dx \\ = \int_{-\pi}^{\pi} 2^{-|K|} \sum_{\mathcal{E}(|K|)} \Big| \sum_{k\in K} \varepsilon_k e^{ikx} \Big|^{p'} dx \asymp \int_{-\pi}^{\pi} \Big(\sum_{k\in K} |e^{ikx}|^2 \Big)^{p'/2} dx \asymp m^{p'/2}.\end{aligned}$$

Then in the set $\mathcal{E}(|K|)$ there is a set $\{\varepsilon_k = \pm 1, k \in K\}$ for which

$$\Big\| \sum_{k\in K} \varepsilon_k e^{ikx} \Big\|_{p'} \ll m^{1/2}. \tag{17}$$

Let us divide the set of indexes K into two subsets:

$$K_+ = \{k \in K : \varepsilon_k = 1\} \quad \text{and} \quad K_- = \{k \in K : \varepsilon_k = -1\}.$$

If $|K| = |K_+| + |K_-| \asymp m$, then from here we conclude that $|K_+| \asymp m$ or $|K_-| \asymp m$.

Let $|K_+| \asymp m$. We consider a polynomial

$$t^*(x) = \sum_{k\in K_+} e^{ikx},$$

for which according to (17)

$$\|t^*\|_{p'} \ll m^{1/2}. \tag{18}$$

Then, according to the definition of the quantity $\mathcal{T}_m(\psi, \beta, \infty, p)$ we can write

$$\mathcal{T}_m(\psi, \beta, \infty, p) \ll \sup_{f\in L(K_+)} \frac{\|f^\psi_\beta\|_\infty}{\|f\|_p}, \tag{19}$$

where $L(K_+) = \left\{ t : t(x) = \sum\limits_{k\in K_+} e^{ikx} \right\}$.

Further, since $f^\psi_\beta = f^\psi_\beta * t^*$, by using the Holder's inequality and the relation (18), we obtain

$$\|f^\psi_\beta\|_\infty \le \|f^\psi_\beta\|_p \|t^*\|_{p'} \ll \|f^\psi_\beta\|_p m^{1/2}. \tag{20}$$

We now successively apply Proposition A for the estimate $\|f^\psi_\beta\|_p$ by using (20), we get

$$\|f^\psi_\beta\|_\infty \ll \|f^\psi_\beta\|_p m^{1/2} \ll \psi^{-1}(m)\|f\|_p m^{1/2}. \tag{21}$$

Finally, substituting (21) into (19), we obtain

$$\mathcal{T}_m(\psi, \beta, \infty, p) \ll \frac{\psi^{-1}(m) m^{1/2} \|f\|_p}{\|f\|_p} = \psi^{-1}(m) m^{1/2}.$$

We prove lower bound. Since at $2 \le p'$, by virtue of the inequality $\|\cdot\|_2 \le \|\cdot\|_{p'}$, we have

$$\mathcal{T}_m(\psi, \beta, \infty, 2) \ge \mathcal{T}_m(\psi, \beta, \infty, p'). \tag{22}$$

Then, using results of Theorem 2 and substituting them in (22), we obtain

$$\mathcal{T}_m(\psi, \beta, \infty, p) \ge \mathcal{T}_m(\psi, \beta, \infty, 2) \gg \psi^{-1}(m) m^{1/2}.$$

Theorem 3 is proved. $\square$

4 Conclusions

In the article, we studied Bernstein-Nikol'skii-type inequalities for trigonometric polynomials with an arbitrary choice of harmonics and, in particular, for ordinary trigonometric polynomials. Here are some comments on the result.

Remark 1. *For $\psi(|k|) = |k|^{-r}$, $k \in \mathbb{Z} \setminus \{0\}$, the corresponding result for Theorem 1 was established by Temlyakov (see, e.g., [8, p. 23]).*

Remark 2. *For $\psi(|k|) = |k|^{-r}$, $k \in \mathbb{Z} \setminus \{0\}$, the corresponding results for Theorems 2 and 3 were obtained in [1].*

Note results of established theorems supplement the exact order estimates for quantities $\mathcal{T}_m(\psi, \beta, q, p)$ for $2 \leq p \leq q < \infty$ and $1 < q \leq p < \infty$ obtained in [10] under different conditions imposed on the sequence $\psi(\tau)$, $\tau \in \mathbb{N}$.

Comparing Theorems 1 and 3, we conclude that under the conditions of Theorem 3 imposed on the sequence $\psi(\tau)$, $\tau \in \mathbb{N}$, the quantity $\mathcal{T}_m(\psi, \beta, \infty, p)$ and the quantity $\sup\limits_{t \in T(m)} \frac{\|t^{\psi}_{\beta}\|_{\infty}}{\|t\|_p}$ for $1 < p \leq 2$ have different orders.

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Власик Г.М., Собчук В.В., Шкапа В.В., Замрій І.В. *Нерівності типу Бернштейна-Нікольського для тригонометричних поліномів* // Карпатські матем. публ. — 2022. — Т.14, №1. — С. 147–157.

Отримано порядкові оцінки для нерівностей типу Бернштейна-Нікольського для тригонометричних поліномів з довільним вибором гармонік. Встановлено, що у випадку $q = \infty$, $1 < p \leq 2$ ці нерівності для тригонометричних поліномів з довільним вибором гармонік і для звичайних тригонометричних поліномів мають різний порядок оцінок.

Ключові слова і фрази: (ψ, β)-похідна, нерівність Бернштейна-Нікольського.