ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2022, **14** (1), 194–212
doi:10.15330/cmp.14.1.194-212

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2022, Т.14, №1, С.194–212

On Wick calculus and its relationship with stochastic integration on spaces of regular test functions in the Lévy white noise analysis

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We deal with spaces of regular test functions in the Lévy white noise analysis, which are constructed using Lytvynov's generalization of a chaotic representation property. Our aim is to study properties of Wick multiplication and of Wick versions of holomorphic functions, and to describe a relationship between Wick multiplication and integration, on these spaces. More exactly, we establish that a Wick product of regular test functions is a regular test function; under some conditions a Wick version of a holomorphic function with an argument from the space of regular test functions is a regular test function; show that when employing the Wick multiplication, it is possible to take a time-independent multiplier out of the sign of an extended stochastic integral with respect to a Lévy process; establish an analog of this result for a Pettis integral (a weak integral); obtain a representation of the extended stochastic integral via formal Pettis integral from the Wick product of the original integrand by a Lévy white noise. As an example of an application of our results, we consider an integral stochastic equation with Wick multiplication.

Key words and phrases: Lévy process, extended stochastic integral, Pettis integral, Wick product.

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Introduction

A theory of test and generalized functions with arguments belonging to infinite-dimensional spaces has many applications in different areas of modern mathematics and physics. There are various approaches to building of such a theory. One of the most successful of them consists in introduction of spaces of the above-mentioned functions in a way that the pairing between test and generalized functions is generated by integration with respect to some probability measure on a dual nuclear space. First it was the Gaussian measure, the corresponding theory is called the *Gaussian white noise analysis* (e.g., [2, 17, 30, 31]), then it were realized numerous generalizations. In particular, important results were obtained when the above-mentioned probability measure is the generalized Meixner measure ([35]), and the Lévy white noise measure (e.g., [7,8,32]), the corresponding theories are called the *Miexner-* and *Lévy white noise analysis*, respectively.

A very important role in the Gaussian analysis belongs to a so-called *chaotic representation property* (CRP): roughly speaking, any square integrable (with respect to the Gaussian measure) random variable can be decomposed in a series of repeated Itô's stochastic integrals with nonrandom integrands (see, e.g., [33] for details). Using CRP, one can construct diverse spaces

УДК 517.98
2010 *Mathematics Subject Classification:* 46F05, 46F25, 60G51, 60H05.

of test and generalized functions, introduce and study stochastic integrals and derivatives on these spaces, etc. Unfortunately, in the Meixner and Lévy white noise analysis there is no CRP, generally speaking [39]; nevertheless, there are various generalizations of this property. For example, in the Meixner analysis one can decompose square integrable random variables in series of generalized Meixner polynomials [35]; in the Lévy analysis there are decompositions connected with a Lévy-Khintchine representation of a Lévy process (Itô's approach [19], see also [6]), decompositions by repeated stochastic integrals from nonrandom integrands with respect to so-called orthogonalized centered power jump processes (Nualart-Schoutens' approach [34], see also [36]), decompositions by special orthogonal functions (Lytvynov's approach [32], see also [5]), special orthogonal decompositions with numeric coefficients (Øksendal's approach [8], see also [7]), etc. The relationships between these generalizations of CRP are described in, e.g., [1,7,8,24,32,38,40].

In the present paper we deal with one of the most useful and challenging generalizations of CRP in the Lévy white noise analysis, which is proposed by E. W. Lytvynov [32]. The idea of this generalization is to decompose random variables, square integrable with respect to the Lévy white noise measure, in series of special orthogonal functions with nonrandom kernels, by analogy with decompositions of random variables, square integrable with respect to the Gaussian measure, by Hermite polynomials (remind that the last decompositions are equivalent to the decompositions by repeated Itô's stochastic integrals). Like using CRP in the Gaussian analysis, one can use Lytvynov's generalization of CRP, in particular, in order to construct and study spaces of regular and nonregular test and generalized functions [20], introduce and investigate various operators and operations on these spaces, etc. Note that the extended stochastic integral and the Hida stochastic derivative on the spaces of *regular* test and generalized functions are introduced and studied in [11,20], operators of stochastic differentiation — in [9,10,14], some elements of a Wick calculus and its relationship with operators of stochastic differentiation and integration on the spaces of *regular* generalized functions — in [12,13]. As for the spaces of *nonregular* test and generalized functions — the corresponding results are presented in [20,26–29]. The paper [25] is a survey of some author's results related to the development of the Lévy white noise analysis in terms of Lytvynov's generalization of CRP.

As is known, in various versions of a white noise analysis a natural multiplication on spaces of generalized functions is a so-called Wick multiplication. In particular, in many cases, using the Wick multiplication, one can take a time-independent multiplier out of the sign of an extended stochastic integral. Moreover, such a result holds true for a Pettis integral (a weak integral). Also, the extended stochastic integral can be presented as a Pettis integral (or a *formal* Pettis integral — depending on the concrete situation) from the Wick product of the original integrand by the corresponding white noise. On the above-mentioned spaces of *nonregular* generalized functions in the Lévy analysis such results were obtained in [29], on the spaces of *regular* generalized functions — in [13].

The aim of the present paper is to introduce by analogy with [22] elements of the Wick calculus on the spaces of *regular test* functions of the Lévy analysis; to transfer the results of [13] to these spaces; and to consider some related topics (in particular, a Pettis integral over a set of infinite Lebesgue measure and a Wick product under the sign of this integral).

The paper is organized in the following manner. In the first section we introduce a Lévy process L and construct a probability triplet connected with L, convenient for our considerations; then we describe Lytvynov's generalization of CRP; construct a regular rigging of the space of square integrable random variables (the positive and negative spaces of this rigging are the spaces of regular test and generalized functions respectively); describe the extended stochastic integral with respect to L on the spaces of regular test and generalized functions; and recall necessary notions of the Wick calculus on the spaces of regular generalized functions. In the second section we introduce elements of the Wick calculus on the spaces of regular test functions by analogy with the Meixner analysis [22]; show that when employing the Wick multiplication, it is possible to take a time-independent multiplier out of the sign of the extended stochastic integral and of the Pettis integral; prove a theorem about a representation of the extended stochastic integral via the formal Pettis integral; and consider an example of an integral stochastic equation with Wick multiplication.

It is important to note that the spaces of regular test and generalized functions in the Lévy analysis and in the Meixner analysis have similar structure, see Subsection 2.1 for details. This allows us to reformulate some results of the Meixner analysis for the Lévy analysis and vice versa. In the present paper we use this opportunity, when possible.

1 Preliminaries

In this paper we denote by $\|\cdot\|_H$ or $|\cdot|_H$ the norm in a space H; by $(\cdot,\cdot)_H$ the real (i.e. bilinear) scalar product in a space H; by $\langle\!\langle\cdot,\cdot\rangle\!\rangle_H$ the dual pairing generated by the scalar product in a space H; by $\mathcal{B}$ the Borel σ-algebra; and by 1_Δ the indicator of a set Δ. Further, we use a designation pr lim (resp., ind lim) for a projective (resp., inductive) limit of a family of spaces, this designation implies that the limit space is endowed with the projective (resp., inductive) limit topology (see, e.g., [3] for a detailed description).

1.1 A Lévy process and its probability space

Denote $\mathbb{R}_+ := [0,+\infty)$. Let $L=(L_u)_{u\in\mathbb{R}_+}$ be a real-valued locally square integrable Lévy process (i.e. a continuous in probability random process on $\mathbb{R}_+$ with stationary independent increments and such that $L_0=0$, see, e.g., [4] for details) without Gaussian part and drift. As is well known (e.g., [8]), the characteristic function of L is

$$\mathbb{E}[e^{i\theta L_u}] = \exp\left[u\int_{\mathbb{R}}(e^{i\theta x}-1-i\theta x)\nu(dx)\right], \tag{1}$$

where ν is the Lévy measure of L, which is a measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, here $\mathbb{E}$ denotes the expectation. We assume that *ν is a Radon measure whose support contains an infinite number of points, $\nu(\{0\})=0$, there exists $\varepsilon>0$ such that $\int_{\mathbb{R}} x^2 e^{\varepsilon|x|}\nu(dx)<\infty$, and $\int_{\mathbb{R}} x^2\nu(dx)=1$.*

Define a measure of the white noise of L. Let $\mathcal{D}$ denote the set of all real-valued infinite-differentiable functions on $\mathbb{R}_+$ with compact supports. As is well known, $\mathcal{D}$ can be endowed by the projective limit topology generated by a family of Sobolev spaces (e.g., [3]). Let $\mathcal{D}'$ be the set of linear continuous functionals on $\mathcal{D}$. It is worth noting that $\mathcal{D}$ and $\mathcal{D}'$ are the positive and negative spaces of a chain

$$\mathcal{D}'\supset L^2(\mathbb{R}_+)\supset\mathcal{D}, \tag{2}$$

where $L^2(\mathbb{R}_+)$ is the space of (classes of) real-valued functions on $\mathbb{R}_+$, square integrable with respect to the Lebesgue measure (e.g., [3]). Denote by $\langle\cdot,\cdot\rangle$ the dual pairing generated by the scalar product in $L^2(\mathbb{R}_+)$, this notation will be preserved for dual pairings in tensor powers of the complexification of chain (2).

Definition 1. *A probability measure μ on $(\mathcal{D}',\mathcal{C}(\mathcal{D}'))$, where $\mathcal{C}$ denotes the cylindrical σ-algebra, with the Fourier transform*

$$\int_{\mathcal{D}'} e^{i\langle\omega,\varphi\rangle}\mu(d\omega) = \exp\Big[\int_{\mathbb{R}_+\times\mathbb{R}}(e^{i\varphi(u)x}-1-i\varphi(u)x)du\nu(dx)\Big],\quad \varphi\in\mathcal{D}, \tag{3}$$

is called the measure of a Lévy white noise.

The existence of μ follows from the Bochner-Minlos theorem (e.g., [18]), see [32]. Below we assume that the *σ-algebra $\mathcal{C}(\mathcal{D}')$ is completed with respect to μ.*

Denote by $(L^2) := L^2(\mathcal{D}',\mathcal{C}(\mathcal{D}'),\mu)$ the space of (classes of) complex-valued functions on $\mathcal{D}'$, square integrable with respect to μ (in what follows, this notation will be used very often). Let $f\in L^2(\mathbb{R}_+)$ and a sequence $(\varphi_k\in\mathcal{D})_{k\in\mathbb{N}}$ converge to f in $L^2(\mathbb{R}_+)$ as $k\to\infty$ (remind that $\mathcal{D}$ is a dense set in $L^2(\mathbb{R}_+)$). One can show [7,8,24,32] that $\langle\circ,f\rangle := (L^2)\text{-}\lim_{k\to\infty}\langle\circ,\varphi_k\rangle$ is a well-defined element of (L^2).

Put $1_{[0,0)}\equiv 0$. It follows from (1) and (3) that $\big(\langle\circ,1_{[0,u)}\rangle\big)_{u\in\mathbb{R}_+}$ can be identified with a Lévy process on the probability space (probability triplet) $(\mathcal{D}',\mathcal{C}(\mathcal{D}'),\mu)$, see, e.g., [7,8]. So, for each $u\in\mathbb{R}_+$ we have $L_u=\langle\circ,1_{[0,u)}\rangle\in(L^2)$.

Note that the derivative in the sense of generalized functions of a Lévy process (a Lévy white noise) is $\dot{L}_\cdot(\omega)=\langle\omega,\delta_\cdot\rangle\equiv\omega(\cdot)$, where δ is the Dirac delta-function. Therefore $\dot{L}$ is a generalized random process (in the sense of [15]) with trajectories from $\mathcal{D}'$, and μ is the measure of $\dot{L}$ in the classical sense of this notion [16].

Remark 1. *A Lévy process without Gaussian part and drift is a Poisson process if its Lévy measure is a point mass at* 1*. This measure does not satisfy the assumptions accepted above (its support does not contain an infinite number of points); nevertheless, all results of the present paper have natural analogs in the Poissonian analysis. The reader can find more information about peculiarities of the Poissonian case in [24, Subsection 1.2].*

1.2 Lytvynov's generalization of the chaotic representation property

Denote by $\widehat{\otimes}$ the symmetric tensor multiplication, by a subscript $\mathbb{C}$ — complexifications of spaces. Set $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$. Denote by $\mathcal{P}$ the set of complex-valued polynomials on $\mathcal{D}'$ that consists of zero and elements of the form

$$f(\omega)=\sum_{n=0}^{N_f}\langle\omega^{\otimes n},f^{(n)}\rangle,\quad \omega\in\mathcal{D}',\ f^{(n)}\in\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n},\ N_f\in\mathbb{Z}_+,\ f^{(N_f)}\neq 0,$$

here N_f is called the *power of a polynomial* f; $\langle\omega^{\otimes 0},f^{(0)}\rangle := f^{(0)}\in\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}0} := \mathbb{C}$. The measure μ of a Lévy white noise has a holomorphic at zero Laplace transform (this follows from (3) and properties of the measure ν, see also [32]), therefore $\mathcal{P}$ is a dense set in (L^2) [37]. Denote by $\mathcal{P}_n$, $n\in\mathbb{Z}_+$, the set of polynomials of power smaller than or equal to n, by $\overline{\mathcal{P}}_n$ the closure of $\mathcal{P}_n$ in (L^2). Let for $n\in\mathbb{N}$ $\mathbf{P}_n := \overline{\mathcal{P}}_n\ominus\overline{\mathcal{P}}_{n-1}$ (the orthogonal difference in (L^2)); put $\mathbf{P}_0 := \overline{\mathcal{P}}_0$. It is clear that

$$(L^2)=\bigoplus_{n=0}^{\infty}\mathbf{P}_n. \tag{4}$$

Let $f^{(n)}\in\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$, $n\in\mathbb{Z}_+$. Denote by $:\langle\circ^{\otimes n},f^{(n)}\rangle:\ \in(L^2)$ the orthogonal projection of a monomial $\langle\circ^{\otimes n},f^{(n)}\rangle$ onto $\mathbf{P}_n$. Let us define real (bilinear) scalar products $(\cdot,\cdot)_{ext}$ on $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$,

$n \in \mathbb{Z}_+$, by setting for $f^{(n)}, g^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$

$$(f^{(n)}, g^{(n)})_{ext} := \frac{1}{n!} \int_{\mathcal{D}'} :\langle \omega^{\otimes n}, f^{(n)} \rangle :: \langle \omega^{\otimes n}, g^{(n)} \rangle : \mu(d\omega). \tag{5}$$

The proof of the well-posedness of this definition coincides up to obvious modifications with the proof of the corresponding statement in [32].

Denote by $|\cdot|_{ext}$ the norms corresponding to scalar products (5), i.e. $|\cdot|_{ext} := \sqrt{(\cdot, \bar{\cdot})_{ext}}$. Let $\mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, be the completions of $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$ with respect to these norms. For $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ define a Wick monomial $:\langle \circ^{\otimes n}, F^{(n)} \rangle : \stackrel{\text{def}}{=} (L^2)\text{–}\lim_{k \to \infty} :\langle \circ^{\otimes n}, f_k^{(n)} \rangle :$, where $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n} \ni f_k^{(n)} \to F^{(n)}$ as $k \to \infty$ in $\mathcal{H}_{ext}^{(n)}$. The well-posedness of this definition can be proved by the method of "mixed sequences". It is easy to show that $:\langle \circ^{\otimes 0}, F^{(0)} \rangle : = \langle \circ^{\otimes 0}, F^{(0)} \rangle = F^{(0)}$ and $:\langle \circ, F^{(1)} \rangle : = \langle \circ, F^{(1)} \rangle$ (cf. [32]).

In the next statement, which follows from (4) and the fact that for each $n \in \mathbb{Z}_+$ the set $\{:\langle \circ^{\otimes n}, f^{(n)} \rangle : | f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}\}$ is dense in $\mathbf{P}_n$, Lytvynov's generalization of the chaotic representation property (CRP) is described.

Theorem 1 (cf. [32]). *A random variable $F \in (L^2)$ if and only if there exists a unique sequence of kernels $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, such that*

$$F = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, F^{(n)} \rangle : \tag{6}$$

(the series converges in (L^2)) and

$$\|F\|_{(L^2)}^2 = \int_{\mathcal{D}'} |F(\omega)|^2 \mu(d\omega) = \mathbb{E}|F|^2 = \sum_{n=0}^{\infty} n! |F^{(n)}|_{ext}^2 < \infty.$$

Remark 2. *In the present paper we do not use directly an explicit formula for the scalar products $(\cdot, \cdot)_{ext}$, and therefore we prefer not to write it down. But for the interested reader we note that such a formula is calculated in [32]; in another record form (more convenient for some calculations) it is given in, e.g., [12, 14, 20, 24, 25]. Also we note that for each $n \in \mathbb{N}$ the space $\mathcal{H}_{ext}^{(n)}$ is the symmetric subspace of the space of (classes of) complex-valued functions on $\mathbb{R}_+^n$, square integrable with respect to a certain Radon measure.*

Denote $\mathcal{H} := L^2(\mathbb{R}_+)$, then $\mathcal{H}_{\mathbb{C}} = L^2(\mathbb{R}_+)_{\mathbb{C}}$ (in what follows, this notation will be used very often). It follows from the explicit formula for $(\cdot, \cdot)_{ext}$ that $\mathcal{H}_{ext}^{(1)} = \mathcal{H}_{\mathbb{C}}$, and for $n \in \mathbb{N} \backslash \{1\}$ one can identify $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ with the proper subspace of $\mathcal{H}_{ext}^{(n)}$ that consists of "vanishing on diagonals" elements (roughly speaking, such that $F^{(n)}(u_1, \ldots, u_n) = 0$ if there exist $k, j \in \{1, \ldots, n\}$: $k \neq j$, but $u_k = u_j$). In this sense the *space $\mathcal{H}_{ext}^{(n)}$ is an extension of $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$*, this explains why we use the subscript "*ext*" in our designations.

1.3 A regular rigging of (L^2)

Denote $\mathcal{P}_W := \{f = \sum_{n=0}^{N_f} :\langle \circ^{\otimes n}, f^{(n)} \rangle :, \ f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}, \ N_f \in \mathbb{Z}_+\} \subset (L^2)$. Accept on default $\beta \in [0, 1]$, $q \in \mathbb{Z}$ in the case $\beta \in (0, 1]$ and $q \in \mathbb{Z}_+$ if $\beta = 0$. Define *real* (*bilinear*) scalar products

$(\cdot,\cdot)_{q,\beta}$ on $\mathcal{P}_W$ by setting for

$$f = \sum_{n=0}^{N_f} :\langle \circ^{\otimes n}, f^{(n)}\rangle:,\ g = \sum_{n=0}^{N_g} :\langle \circ^{\otimes n}, g^{(n)}\rangle: \in \mathcal{P}_W$$

$$(f,g)_{q,\beta} := \sum_{n=0}^{\min(N_f,N_g)} (n!)^{1+\beta} 2^{qn} (f^{(n)}, g^{(n)})_{ext}.$$

One can easily verify that the axioms of a scalar product are fulfilled (see [12]).

Denote by $(L^2)_q^\beta$ the completions of $\mathcal{P}_W$ with respect to the norms generated by scalar products $(\cdot,\cdot)_{q,\beta}$; and set $(L^2)^\beta := \mathrm{pr}\lim_{q\to+\infty}(L^2)_q^\beta$. As is easy to see, $f \in (L^2)_q^\beta$ if and only if f can be uniquely presented as series (6) (with kernels $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$) that converges in $(L^2)_q^\beta$, and

$$\|f\|^2_{(L^2)_q^\beta} = \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} |f^{(n)}|^2_{ext} < \infty; \tag{7}$$

and $f \in (L^2)^\beta$ if and only if f can be uniquely presented in form (6) with convergent series (7) for each $q \in \mathbb{Z}_+$.

Proposition 1 ([20])**.** *For any $\beta \in (0,1]$ and any $q \in \mathbb{Z}$, in the same way as for $\beta = 0$ and any $q \in \mathbb{Z}_+$, the space $(L^2)_q^\beta$ is densely and continuously embedded into $(L^2) = (L^2)_0^0$.*

Taking into account this result, we can consider a chain

$$(L^2)^{-\beta} \supset (L^2)_{-q}^{-\beta} \supseteq (L^2) \supseteq (L^2)_q^\beta \supset (L^2)^\beta, \tag{8}$$

where $(L^2)_{-q}^{-\beta}$ and $(L^2)^{-\beta} = \mathrm{ind}\lim_{q\to+\infty}(L^2)_{-q}^{-\beta}$ are the spaces dual of $(L^2)_q^\beta$ and $(L^2)^\beta$ respectively.

Definition 2. *Chain* (8) *is called a parametrized regular rigging of* (L^2)*. The spaces* $(L^2)_q^\beta$ *and* $(L^2)^\beta$ *are called parametrized Kondratiev-type spaces of regular test functions, and the spaces* $(L^2)_{-q}^{-\beta}$ *and* $(L^2)^{-\beta}$ *are called parametrized Kondratiev-type spaces of regular generalized functions.*

The next statement follows from the definition of $(L^2)_{-q}^{-\beta}$ and the general duality theory.

Proposition 2. *1) Any regular generalized function $F \in (L^2)_{-q}^{-\beta}$ can be uniquely presented as formal series* (6) *(with kernels $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$) that converges in $(L^2)_{-q}^{-\beta}$, and*

$$\|F\|^2_{(L^2)_{-q}^{-\beta}} = \sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-qn} |F^{(n)}|^2_{ext} < \infty. \tag{9}$$

Vice versa, any formal series (6) *such that series* (9) *converges, is a regular generalized function from $(L^2)_{-q}^{-\beta}$ (i.e. now series* (6) *converges in $(L^2)_{-q}^{-\beta}$).*

2) The dual pairing between $F \in (L^2)_{-q}^{-\beta}$ and $f \in (L^2)_q^\beta$ that is generated by the scalar product in (L^2), has a form $\langle\!\langle F, f\rangle\!\rangle_{(L^2)} = \sum_{n=0}^\infty n!(F^{(n)}, f^{(n)})_{ext}$, where $F^{(n)}, f^{(n)} \in \mathcal{H}_{ext}^{(n)}$ are the kernels from decompositions (6) *for F and f respectively.*

3) $F \in (L^2)^{-\beta}$ if and only if F can be uniquely presented in form (6) *and norm* (9) *is finite for some $q \in \mathbb{Z}_+$.*

Note that the term "*regular* test and generalized functions" is connected with the fact that the kernels from decompositions (6) for elements of all spaces of chain (8) belong to the same spaces $\mathcal{H}_{ext}^{(n)}$.

1.4 The extended stochastic integral

In this subsection it will be convenient to denote the spaces $(L^2)^\beta_q$, $(L^2) = (L^2)^0_0$ and $(L^2)^{-\beta}_{-q}$ from chain (8) by $(L^2)^\beta_q$, $\beta \in [-1,1]$, $q \in \mathbb{Z}$. The norms in these spaces are given, obviously, by formula (7) (cf. (7) and (9)).

Let $\mathbf{I} : (L^2)^\beta_q \to \bigoplus\limits_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} \mathcal{H}^{(n)}_{ext}$ be the generalized Wiener-Itô-Sigal isomorphism, generated by decomposition (6), $\mathbf{1} : \mathcal{H}_{\mathbb{C}} \to \mathcal{H}_{\mathbb{C}}$ be the identity operator. For each $f^{(n)}_\cdot \in \mathcal{H}^{(n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$, $n \in \mathbb{Z}_+$, define a Wick monomial

$$:\langle \circ^{\otimes n}, f^{(n)}_\cdot \rangle: \stackrel{def}{=} (\mathbf{I} \otimes \mathbf{1})^{-1}(\underbrace{0,\dots,0}_{n}, f^{(n)}_\cdot, 0, \dots) \in (L^2)^\beta_q \otimes \mathcal{H}_{\mathbb{C}}.$$

It is easy to show (see details in [12–14]) that such Wick monomials form orthogonal bases in the spaces $(L^2)^\beta_q \otimes \mathcal{H}_{\mathbb{C}}$ in the sense that any $f \in (L^2)^\beta_q \otimes \mathcal{H}_{\mathbb{C}}$ can be uniquely presented as

$$f(\cdot) = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, f^{(n)}_\cdot \rangle:, \quad f^{(n)}_\cdot \in \mathcal{H}^{(n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}} \tag{10}$$

(the series converges in $(L^2)^\beta_q \otimes \mathcal{H}_{\mathbb{C}}$), with

$$\|f\|^2_{(L^2)^\beta_q \otimes \mathcal{H}_{\mathbb{C}}} = \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} |f^{(n)}_\cdot|^2_{\mathcal{H}^{(n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}} < \infty.$$

Now we describe the construction of an extended stochastic integral with respect to a Lévy process L, that is based on decomposition (10) (a detailed presentation is given in [20,24]). Let $f^{(n)}_\cdot \in \mathcal{H}^{(n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$, $n \in \mathbb{N}$. We select a representative (a function) $\dot{f}^{(n)}_\cdot \in f^{(n)}_\cdot$ such that

$$\dot{f}^{(n)}_u(u_1, \dots, u_n) = 0 \text{ if for some } k \in \{1, \dots, n\}\ u = u_k. \tag{11}$$

Accept on default $\Delta \in \mathcal{B}(\mathbb{R}_+)$. Let $\hat{f}^{(n)}_\Delta$ be the symmetrization of a function $\dot{f}^{(n)}_\cdot 1_\Delta(\cdot)$ by $n+1$ variables. Define $\widehat{f}^{(n)}_\Delta \in \mathcal{H}^{(n+1)}_{ext}$ as the equivalence class in $\mathcal{H}^{(n+1)}_{ext}$ generated by $\hat{f}^{(n)}_\Delta$ (i.e. $\hat{f}^{(n)}_\Delta \in \widehat{f}^{(n)}_\Delta$). It is proved in [20,24] that this definition is well-posed (in particular, $\widehat{f}^{(n)}_\Delta$ does not depend on a choice of a representative $\dot{f}^{(n)}_\cdot \in f^{(n)}_\cdot$ satisfying (11)) and $|\widehat{f}^{(n)}_\Delta|_{ext} \leq |f^{(n)}_\cdot 1_\Delta(\cdot)|_{\mathcal{H}^{(n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}}$.

Definition 3. *We define the extended stochastic integral*

$$\int_\Delta \circ(u) \widehat{d} L_u : (L^2)^\beta_q \otimes \mathcal{H}_{\mathbb{C}} \to (L^2)^\beta_{q-1}$$

as

$$\int_\Delta f(u) \widehat{d} L_u := \sum_{n=0}^{\infty} :\langle \circ^{\otimes n+1}, \widehat{f}^{(n)}_\Delta \rangle:, \tag{12}$$

where $\widehat{f}^{(0)}_\Delta := f^{(0)}_\cdot 1_\Delta(\cdot) \in \mathcal{H}_{\mathbb{C}} = \mathcal{H}^{(1)}_{ext}$, *and* $\widehat{f}^{(n)}_\Delta \in \mathcal{H}^{(n+1)}_{ext}$, $n \in \mathbb{N}$, *are constructed by the kernels* $f^{(n)}_\cdot \in \mathcal{H}^{(n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$ *from decomposition* (10) *for* f.

One can show quite analogously to [20,24] that this integral is a linear *continuous* operator, and if f is integrable by Itô then $\int_\Delta f(u) \widehat{d} L_u$ coincides with the corresponding Itô stochastic integral.

It is clear that the extended stochastic integral can be defined by (12) as a linear continuous operator acting from $(L^2)^\beta \otimes \mathcal{H}_{\mathbb{C}} := \mathrm{pr}\lim_{q\to+\infty} (L^2)^\beta_q \otimes \mathcal{H}_{\mathbb{C}}$ to $(L^2)^\beta$, or from $(L^2)^{-\beta} \otimes \mathcal{H}_{\mathbb{C}} := \mathrm{ind}\lim_{q\to+\infty} (L^2)^{-\beta}_{-q} \otimes \mathcal{H}_{\mathbb{C}}$ to $(L^2)^{-\beta}$, here $\beta \in [0,1]$.

Remark 3. *As appears from the above,*

$$\int_\Delta f(u)\widehat{d}L_u = \int_{\mathbb{R}_+} f(u)1_\Delta(u)\widehat{d}L_u. \tag{13}$$

This representation can be used for an important generalization. Let a function $f : \mathbb{R}_+ \to (L^2)_q^\beta$ be such that $f(\cdot) \notin (L^2)_q^\beta \otimes \mathcal{H}_\mathbb{C}$, but for some $\Theta \in \mathcal{B}(\mathbb{R}_+)$ (for example, such that the Lebesgue measure of Θ is finite) $f(\cdot)1_\Theta(\cdot) \in (L^2)_q^\beta \otimes \mathcal{H}_\mathbb{C}$ (such functions often arise in problems). Now for any measurable $\Delta \subseteq \Theta$ one can define $\int_\Delta f(u)\widehat{d}L_u$ by formula (13)*. It is clear that similar generalization is possible for $f : \mathbb{R}_+ \to (L^2)^\beta$ and $F : \mathbb{R}_+ \to (L^2)^{-\beta}$, here $\beta \in [0, 1]$.*

Remark 4. *The operator adjoint to the extended stochastic integral is called the Hida stochastic derivative. This derivative is closely connected with so-called operators of stochastic differentiation [10, 14]; all the above-mentioned operators play an important role in the Lévy white noise analysis.*

1.5 Wick product and Wick versions of holomorphic functions on $(L^2)^{-\beta}$

Remind that we assume $\beta \in [0, 1]$. For each $F \in (L^2)^{-\beta}$ define an S-transform as a formal series

$$(SF)(\lambda) := \sum_{m=0}^\infty (F^{(m)}, \lambda^{\otimes m})_{ext} \equiv F^{(0)} + \sum_{m=1}^\infty (F^{(m)}, \lambda^{\otimes m})_{ext}, \tag{14}$$

where $F^{(m)} \in \mathcal{H}_{ext}^{(m)}$, $m \in \mathbb{Z}_+$, are the kernels from decomposition (6) for F, $\lambda \in \mathcal{D}_\mathbb{C}$ (each term in series (14) is well-defined, but the series can diverge). In particular, $(SF)(0) = F^{(0)}$, $S1 \equiv 1$.

Definition 4. *For $F, G \in (L^2)^{-\beta}$ and a holomorphic at $(SF)(0)$ function $h : \mathbb{C} \to \mathbb{C}$ we define a Wick product $F\Diamond G$ and a Wick version $h^\Diamond(F)$ by setting formally*

$$F\Diamond G := S^{-1}(SF \cdot SG), \quad h^\Diamond(F) := S^{-1}h(SF).$$

It is clear that the Wick multiplication $\Diamond$ is commutative, associative, distributive, and for any $\alpha \in \mathbb{C}$ $(\alpha F)\Diamond G = F\Diamond(\alpha G) = \alpha(F\Diamond G) \equiv \alpha F\Diamond G$.

Remark 5. *A function h from Definition 4 can be decomposed in a Taylor series*

$$h(u) = \sum_{m=0}^\infty h_m\big(u - (SF)(0)\big)^m. \tag{15}$$

Using this decomposition, it is easy to calculate that

$$h^\Diamond(F) = \sum_{m=0}^\infty h_m\big(F - (SF)(0)\big)^{\Diamond m},$$

where $F^{\Diamond m} := \underbrace{F\Diamond \cdots \Diamond F}_{m \text{ times}}$, $F^{\Diamond 0} := 1$.

It is proved in [12] that for $F_1, \dots, F_n \in (L^2)^{-\beta}$ $F_1\Diamond \cdots \Diamond F_n \in (L^2)^{-\beta}$, $n \in \mathbb{N}\backslash\{1\}$ (moreover, the Wick multiplication is *continuous* on $(L^2)^{-\beta}$); for $F \in (L^2)^{-1}$ and a function $h : \mathbb{C} \to \mathbb{C}$ holomorphic at $(SF)(0)$ $h^\Diamond(F) \in (L^2)^{-1}$; but if $\beta < 1$ then for $F \in (L^2)^{-\beta}$ $h^\Diamond(F) \notin (L^2)^{-\beta}$, generally speaking. The proof consists in direct calculation with use of "coordinate formulas" for a

Wick product and Wick versions of holomorphic functions (i.e. representations of $F_1\lozenge\cdots\lozenge F_n$ and $h^{\lozenge}(F)$ via kernels from decompositions (6) for $F_1,\dots,F_n, F$ and coefficients from decomposition (15) for h). In the present paper we do not use these formulas directly and therefore we prefer do not write it down (the interested reader can look into the papers [12, 13]), except important and necessary for further presentation particular case — the "coordinate formula" for $F\lozenge G$, $F,G\in(L^2)^{-\beta}$. In order to write out this formula, we need a preparation: it is necessary to introduce an analog of the symmetric tensor product on the spaces $\mathcal{H}_{ext}^{(n)}$, $n\in\mathbb{Z}_+$.

Let $n,m\in\mathbb{Z}_+$. Consider a function $r:\mathbb{R}_+^{n+m}\to\mathbb{C}$. Denote

$$\begin{aligned}&\widetilde{r}(u_1,\dots,u_n;u_{n+1},\dots,u_{n+m})\\ :=&\begin{cases} r(u_1,\dots,u_{n+m}), & \text{if } \forall i\in\{1,\dots,n\}, \forall j\in\{n+1,\dots,n+m\}\ u_i\neq u_j\\ 0, & \text{in other cases}\end{cases}\end{aligned} \tag{16}$$

Let $F^{(n)}\in\mathcal{H}_{ext}^{(n)}$, $G^{(m)}\in\mathcal{H}_{ext}^{(m)}$. We select representatives (functions) $\dot{f}^{(n)}\in F^{(n)}$ and $\dot{g}^{(m)}\in G^{(m)}$. Set $r(u_1,\dots,u_{n+m}):=\dot{f}^{(n)}(u_1,\dots,u_n)\cdot\dot{g}^{(m)}(u_{n+1},\dots,u_{n+m})$. Let $\widehat{f^{(n)}g^{(m)}}$ be the symmetrization of $\widetilde{r}$ by all variables, $F^{(n)}\diamond G^{(m)}\in\mathcal{H}_{ext}^{(n+m)}$ be the equivalence class in $\mathcal{H}_{ext}^{(n+m)}$ that is generated by $\widehat{f^{(n)}g^{(m)}}$ (i.e. $\widehat{f^{(n)}g^{(m)}}\in F^{(n)}\diamond G^{(m)}$). It is proved in [10] that this definition is well-posed (in particular, $F^{(n)}\diamond G^{(m)}$ does not depend on a choice of representatives from $F^{(n)}$ and $G^{(m)}$) and

$$|F^{(n)}\diamond G^{(m)}|_{ext}\leq|F^{(n)}|_{ext}|G^{(m)}|_{ext}. \tag{17}$$

Proposition 3 ([12]). *For $F,G\in(L^2)^{-\beta}$*

$$F\lozenge G=\sum_{m=0}^{\infty}:\langle\circ^{\otimes m},\sum_{k=0}^{m}F^{(k)}\diamond G^{(m-k)}\rangle:, \tag{18}$$

where $F^{(k)}\in\mathcal{H}_{ext}^{(k)}$, $G^{(m-k)}\in\mathcal{H}_{ext}^{(m-k)}$ are the kernels from decompositions (6) *for F and G respectively.*

Remark 6. *The proof of this proposition (in the same way as the proof of general "coordinate formulas") consists in direct calculation with use* (14) *and the equality*

$$(F^{(n)},\lambda^{\otimes n})_{ext}(G^{(m)},\lambda^{\otimes m})_{ext}=(F^{(n)}\diamond G^{(m)},\lambda^{\otimes n+m})_{ext},$$

$F^{(n)}\in\mathcal{H}_{ext}^{(n)}$, $G^{(m)}\in\mathcal{H}_{ext}^{(m)}$, $n,m\in\mathbb{Z}_+$, $\lambda\in\mathcal{D}_{\mathbb{C}}$, which is proved in [28].

2 Main results

2.1 On Wick calculus on $(L^2)^\beta$

As we noted in the Introduction, the parametrized Kondratiev-type spaces of regular test and generalized functions in the Lévy analysis and in the Meixner analysis have similar structure. More exactly, by decomposition (6) the spaces $(L^2)_q^\beta$ with $\beta\in[-1,1]$ and $q\in\mathbb{Z}$ are isometrically isomorphic to the extended Fock spaces $\bigoplus_{n=0}^{\infty}(n!)^{1+\beta}2^{qn}\mathcal{H}_{ext}^{(n)}$ (remind that the norm in $(L^2)_q^\beta$ is given by (7)), whereas the corresponding spaces in the Meixner analysis (e.g., [21]) are isometrically isomorphic to the spaces $\bigoplus_{n=0}^{\infty}(n!)^{1+\beta}2^{qn}\mathcal{H}_{\mu,ext}^{(n)}$, where Hilbert spaces $\mathcal{H}_{\mu,ext}^{(n)}$ do not coincide with $\mathcal{H}_{ext}^{(n)}$, generally speaking, but have similar to $\mathcal{H}_{ext}^{(n)}$ structure and properties. This gives reason to expect that in the Lévy- and Meixner analysis many properties of the spaces of regular test and generalized functions, and of similar operators and operations on them (e.g., of stochastic integrals and derivatives, of Wick multiplication, etc.), are quite similar. In particular, this applies to properties of Wick products and of Wick versions of holomorphic functions on the spaces of regular test functions. And indeed, as it turned out, not only theorems about the mentioned properties, but even proofs of these theorems coincide in the Lévy- and Meixner analysis up to simple modifications.

Now we will formulate some of the mentioned theorems and explain the above in detail. Remind that by default $\beta \in [0,1]$, $q \in \mathbb{Z}$ in the case $\beta \in (0,1]$ and $q \in \mathbb{Z}_+$ if $\beta = 0$. Since the spaces of test functions $(L^2)^\beta$ are embedded into the space $(L^2)^{-1}$, for $f, g \in (L^2)^\beta$ and a holomorphic at $(Sf)(0)$ function $h : \mathbb{C} \to \mathbb{C}$ the Wick product $f\Diamond g$ and the Wick version $h^\Diamond(f)$ are well defined as elements of $(L^2)^{-1}$ [12]. We will be interested in the question of whether $f\Diamond g$ and $h^\Diamond(f)$ belong to the spaces of test functions.

Theorem 2 (cf. [22]). *Let $f, g \in (L^2)^\beta \subset (L^2)^{-\beta}$. Then the Wick product $f\Diamond g \in (L^2)^\beta$. Moreover, the Wick multiplication is continuous in the sense that for $f_1, \ldots, f_n \in (L^2)^\beta$, $n \in \mathbb{N} \setminus \{1\}$, and $q \in \mathbb{Z}_+$*

$$\|f_1\Diamond \cdots \Diamond f_n\|_{(L^2)^\beta_q} \leq \sqrt{\max_{m\in\mathbb{Z}_+}[2^{-m}(m+1)^{n-1}]}\,\|f_1\|_{(L^2)^\beta_{q_1}} \cdots \|f_n\|_{(L^2)^\beta_{q_1}}, \tag{19}$$

where $q_1 \geq q + (1+\beta)\log_2 n + 1$.

Proof. It is clear that in order to prove the theorem it is sufficient to establish estimate (19). One can make this by direct calculation with use the "coordinate formula" for $f_1\Diamond \cdots \Diamond f_n$ [12], (6), (7) and (17), cf. [22]. $\square$

Remark 7. *In the case $n = 2$ estimate (19) reduces to*

$$\|f_1\Diamond f_2\|_{(L^2)^\beta_q} \leq \|f_1\|_{(L^2)^\beta_{q_1}} \|f_2\|_{(L^2)^\beta_{q_1}}, \tag{20}$$

$q_1 \geq q + 2 + \beta$. Using this result and the associativity of the Wick multiplication, one can prove by the mathematical induction method that for $f_1, \ldots, f_n \in (L^2)^\beta$, $n \in \mathbb{N}$, and $q \in \mathbb{Z}_+$

$$\|f_1\Diamond \cdots \Diamond f_n\|_{(L^2)^\beta_q} \leq \|f_1\|_{(L^2)^\beta_{q_1}} \|f_2\|_{(L^2)^\beta_{q_2}} \cdots \|f_{n-1}\|_{(L^2)^\beta_{q_{n-1}}} \|f_n\|_{(L^2)^\beta_{q_{n-1}}},$$

where $q_l \geq q_{l-1} + 2 + \beta$, $l \in \{1, \ldots, n-1\}$, $q_0 := q$.

It follows from Theorem 2 and Remark 5 that for a *polynomial* h and a test function $f \in (L^2)^\beta$ we have $h^\Diamond(f) \in (L^2)^\beta$. But, unfortunately, a general (holomorphic at $(Sf)(0)$) function $h : \mathbb{C} \to \mathbb{C}$ has no such a property: for $f \in (L^2)^\beta$ $h^\Diamond(f) \notin (L^2)^\beta$, generally speaking. More exactly, we have the following statement.

Proposition 4 (cf. [22]). *Let $h : \mathbb{C} \to \mathbb{C}$ be a holomorphic at $u_0 \in \mathbb{C}$ function such that all coefficients h_m from the decomposition*

$$h(u) = \sum_{m=0}^{\infty} h_m(u-u_0)^m \tag{21}$$

are non-negative and for some $K > 0$ the series $\sum_{m=1}^\infty (m!)^{1+\beta} h_m^2 K^m$ diverges. Then there exists $f \in (L^2)^\beta$ with $(Sf)(0) = u_0$ such that $h^\Diamond(f) \notin (L^2)^\beta$ (and, moreover, $h^\Diamond(f) \notin (L^2)^\beta_0$).

Proof. The proof consists in building of a relevant example of a test function — now it can be $f = u_0 + \sum\limits_{n=1}^{\infty} :\langle \circ^{\otimes n}, \frac{\varphi^{\diamond n}}{(n!)^{\frac{2+\beta}{2}}}\rangle :$, $0 \neq \varphi \in \mathcal{H}_{\mathbb{C}}$, where $\varphi^{\diamond n} := \underbrace{\varphi \diamond \cdots \diamond \varphi}_{n \text{ times}}$. By analogy with [22] one can verify by direct calculation with use (7) that for any $q \in \mathbb{Z}_+$ $\|f\|_{(L^2)^\beta_q} < \infty$, therefore $f \in (L^2)^\beta$; but $\|h^\Diamond(f)\|_{(L^2)^\beta_0} = \infty$, so, $h^\Diamond(f) \notin (L^2)^\beta_0$ and therefore $h^\Diamond(f) \notin (L^2)^\beta \subset (L^2)^\beta_0$. □

Remark 8. *By analogy with the Meixner white noise analysis one can prove the following statement (cf. [22]). Let $h : \mathbb{C} \to \mathbb{C}$ be a holomorphic at $u_0 \in \mathbb{C}$ function such that all coefficients h_m from decomposition* (21) *are non-negative. Then for each $q \in \mathbb{Z}_+$ there exists $f \in (L^2)^\beta_q$ with $(Sf)(0) = u_0$ such that $h^\Diamond(f) \notin (L^2)^\beta_0$.*

The next statement refers to cases where for a test function f $h^\Diamond(f)$ is a test function.

Theorem 3 (cf. [22]). *1) Let $f \in (L^2)^\beta$. Then for each $q \in \mathbb{Z}_+$ there exists a holomorphic at $(Sf)(0)$ not polynomial function $h : \mathbb{C} \to \mathbb{C}$ such that $h^\Diamond(f) \in (L^2)^\beta_q$.*

2) Let $f = \sum_{n=0}^{N} :\langle \circ^{\otimes n}, f^{(n)}\rangle :$, $f^{(n)} \in \mathcal{H}^{(n)}_{ext}$, and coefficients h_m from the decomposition $h(u) = \sum_{m=0}^{\infty} h_m (u - f^{(0)})^m$ for a holomorphic at $f^{(0)}$ function $h : \mathbb{C} \to \mathbb{C}$ satisfy estimates

$$|h_m| \leq \frac{K^m \min\limits_{n \in \{m,\dots,Nm\}} \alpha_n}{\big((Nm)!\big)^{\frac{1+\beta}{2}}} \tag{22}$$

with some $K > 0$, where $(\alpha_n > 0)_{n=0}^{\infty}$ — a numerical sequence such that for each $C > 0$ $\sum_{n=0}^{\infty} C^n \alpha_n < \infty$. Then $h^\Diamond(f) \in (L^2)^\beta$.

Proof. Again, the proof consists in direct calculation of $\|h^\Diamond(f)\|_{(L^2)^\beta_q}$, by analogy with [22]. □

Remark 9. *Let $h(u) = \exp(u)$, i.e. for each $m \in \mathbb{Z}_+$ we have $h_m = \frac{1}{m!}$. It is shown in [22] that now estimate* (22) *is fulfilled if there exists $\varepsilon > 0$ such that $N\big(\frac{1+\beta}{2} + \varepsilon\big) \leq 1$, i.e. $\exp^\Diamond\big(\sum_{n=0}^{N} :\langle \circ^{\otimes n}, f^{(n)}\rangle :\big) \in (L^2)^\beta$ if $\beta < 1$ and $N = 1$ (the case $N = 0$ is trivial).*

Finally we note that, as in the Meixner white noise analysis, if $f \in (L^2)^\beta$ then under some conditions $h^\Diamond(f) \in (L^2)^{\beta_1}$, where $\beta_1 < \beta$. The interested reader can formulate and prove the corresponding statements by analogy with [22], Propositions 2.5 and 2.6.

2.2 The relationship between the Wick calculus and integration on $(L^2)^\beta$

As is known, some properties of extended stochastic integrals are quite unusual. For example, for $f \in (L^2)^\beta$ and $h^{(1)} \in \mathcal{H}_{\mathbb{C}}$

$$\int_{\mathbb{R}_+} (f \otimes h^{(1)})(u) \widehat{d} L_u \equiv \int_{\mathbb{R}_+} f \cdot h^{(1)}(u) \widehat{d} L_u \neq f \cdot \int_{\mathbb{R}_+} h^{(1)}(u) \widehat{d} L_u,$$

generally speaking, although f does not depend on u. Moreover, in general, the product $f \cdot \int_{\mathbb{R}_+} h^{(1)}(u) \widehat{d} L_u$ is undefined. But if one uses the Wick multiplication instead of the pointwise multiplication, it becomes possible to take a time-independent (i.e. independent on u) multiplier out of the sign of the extended stochastic integral, as in the Lebesgue integration theory. Now we will explain this in detail.

Let us begin with a preparation (we need to introduce a Wick product of elements of $(L^2)^\beta$ and $(L^2)^\beta \otimes \mathcal{H}_{\mathbb{C}}$). Let $n, m \in \mathbb{Z}_+$. Consider a function $r : \mathbb{R}_+^{n+m+1} \to \mathbb{C}$. By analogy with (16) denote

$$\widetilde{r}_u(u_1, \dots, u_n; u_{n+1}, \dots, u_{n+m}) := \begin{cases} r_u(u_1, \dots, u_{n+m}), & \text{if } \forall i \in \{1, \dots, n\}, \forall j \in \{n+1, \dots, n+m\}\ u_i \neq u_j, \\ 0, & \text{in other cases.} \end{cases}$$

Let $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $g_\cdot^{(m)} \in \mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}_{\mathbb{C}}$. We select representatives (functions) $\dot{f}^{(n)} \in f^{(n)}$ and $\dot{g}_\cdot^{(m)} \in g_\cdot^{(m)}$. Set $r_u(u_1, \dots, u_{n+m}) := \dot{f}^{(n)}(u_1, \dots, u_n) \cdot \dot{g}_u^{(m)}(u_{n+1}, \dots, u_{n+m})$. Denote by $\widehat{f^{(n)} g_u^{(m)}}(u_1, \dots, u_{n+m})$ the symmetrization of $\widetilde{r}_u(u_1, \dots, u_n; u_{n+1}, \dots, u_{n+m})$ by variables $u_1, \dots, u_{n+m}$. Let $f^{(n)} \overline{\diamond} g_\cdot^{(m)} \in \mathcal{H}_{ext}^{(n+m)} \otimes \mathcal{H}_{\mathbb{C}}$ be the equivalence class in $\mathcal{H}_{ext}^{(n+m)} \otimes \mathcal{H}_{\mathbb{C}}$ that is generated by $\widehat{f^{(n)} g_\cdot^{(m)}}$ (i.e. $\widehat{f^{(n)} g_\cdot^{(m)}} \in f^{(n)} \overline{\diamond} g_\cdot^{(m)}$). It is proved in [14] that this definition is well-posed (in particular, $f^{(n)} \overline{\diamond} g_\cdot^{(m)}$ does not depend on a choice of representatives from $f^{(n)}$ and $g_\cdot^{(m)}$),

$$|f^{(n)} \overline{\diamond} g_\cdot^{(m)}|_{\mathcal{H}_{ext}^{(n+m)} \otimes \mathcal{H}_{\mathbb{C}}} \leq |f^{(n)}|_{\mathcal{H}_{ext}^{(n)}} |g_\cdot^{(m)}|_{\mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}_{\mathbb{C}}}, \tag{23}$$

and for $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $g^{(m)} \in \mathcal{H}_{ext}^{(m)}$ and $h^{(1)} \in \mathcal{H}_{\mathbb{C}}$

$$f^{(n)} \overline{\diamond} \big(g^{(m)} \otimes h^{(1)}(\cdot)\big) = \big(f^{(n)} \diamond g^{(m)}\big) \otimes h^{(1)}(\cdot) \in \mathcal{H}_{ext}^{(n+m)} \otimes \mathcal{H}_{\mathbb{C}}. \tag{24}$$

Now we can accept the following natural definition based on "coordinate formula" (18).

Definition 5. *Let $f \in (L^2)^\beta$, $g \in (L^2)^\beta \otimes \mathcal{H}_{\mathbb{C}}$. We define a Wick product $f \overline{\Diamond} g \in (L^2)^\beta \otimes \mathcal{H}_{\mathbb{C}}$, setting*

$$f \overline{\Diamond} g(\cdot) := \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, \sum_{k=0}^{m} f^{(k)} \overline{\diamond} g_\cdot^{(m-k)} \rangle :, \tag{25}$$

where $f^{(k)} \in \mathcal{H}_{ext}^{(k)}$ and $g_\cdot^{(m-k)} \in \mathcal{H}_{ext}^{(m-k)} \otimes \mathcal{H}_{\mathbb{C}}$ are the kernels from decompositions (6) *and* (10) *for f and g respectively.*

Using estimate (23), one can prove by analogy with [22] that this definition is well-posed and the Wick multiplication $\overline{\Diamond}$ is continuous in the sense that for all $f \in (L^2)^\beta$, $g \in (L^2)^\beta \otimes \mathcal{H}_{\mathbb{C}}$, $q, q_1 \in \mathbb{Z}_+$, $q_1 \geq q + 2 + \beta$, $\|f \overline{\Diamond} g\|_{(L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}}} \leq \|f\|_{(L^2)_{q_1}^\beta} \|g\|_{(L^2)_{q_1}^\beta \otimes \mathcal{H}_{\mathbb{C}}}$.

Remark 10. *Let $f, g \in (L^2)^\beta$, $h^{(1)} \in \mathcal{H}_{\mathbb{C}}$. Using* (25), (18) *and* (24)*, one can show that*

$$f \overline{\Diamond} (g \otimes h^{(1)}) = (f \Diamond g) \otimes h^{(1)} \in (L^2)^\beta \otimes \mathcal{H}_{\mathbb{C}}. \tag{26}$$

It is important to note that, like the Wick multiplication $\Diamond$, the Wick multiplication $\overline{\Diamond}$ is the restriction to the spaces of test functions of the Wick multiplication for elements of $(L^2)^{-\beta}$ and $(L^2)^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$, introduced and studied in [13].

Theorem 4. *Let $\Delta \in \mathcal{B}(\mathbb{R}_+)$, $f \in (L^2)^\beta$ and $g \in (L^2)^\beta \otimes \mathcal{H}_\mathbb{C}$. Then*

$$\int_\Delta (f\overline{\Diamond}g)(u)\widehat{d}L_u = f\Diamond \int_\Delta g(u)\widehat{d}L_u \in (L^2)^\beta. \tag{27}$$

Proof. It is proved in [13] that for $\Delta \in \mathcal{B}(\mathbb{R}_+)$, $f \in (L^2)^\beta \subset (L^2)^{-\beta}$ and $g \in (L^2)^\beta \otimes \mathcal{H}_\mathbb{C} \subset (L^2)^{-\beta} \otimes \mathcal{H}_\mathbb{C}$

$$\int_\Delta (f\overline{\Diamond}g)(u)\widehat{d}L_u = f\Diamond \int_\Delta g(u)\widehat{d}L_u \in (L^2)^{-\beta}. \tag{28}$$

But it follows from properties of the Wick multiplications $\Diamond$ and $\overline{\Diamond}$, and from properties of the extended stochastic integral that under the conditions of the Theorem both parts of (28) are elements of $(L^2)^\beta$. □

Remark 11. *One can interpret g as a function on $\mathbb{R}_+$ with values in $(L^2)^\beta$ and, taking into account the construction of the Wick multiplications $\Diamond$ and $\overline{\Diamond}$, rewrite equality (27) in a classical form $\int_\Delta f\Diamond g(u)\widehat{d}L_u = f\Diamond \int_\Delta g(u)\widehat{d}L_u$.*

Let us obtain an analog of property (27) for a Pettis integral (i.e. for a weak integral) on the spaces of regular test functions. Denote by ρ the Lebesgue measure on $\mathbb{R}_+$.

Definition 6. *For all $\Delta \in \mathcal{B}(\mathbb{R}_+)$ with $\rho(\Delta) < \infty$ and $g \in (L^2)^\beta \otimes \mathcal{H}_\mathbb{C}$ we define a Pettis integral $\int_\Delta g(u)du \in (L^2)^\beta$ as a unique element of $(L^2)^\beta$ such that for each $F \in (L^2)^{-\beta}$*

$$\langle\langle F, \int_\Delta g(u)du\rangle\rangle_{(L^2)} = \langle\langle F \otimes 1_\Delta, g\rangle\rangle_{(L^2)\otimes\mathcal{H}_\mathbb{C}}. \tag{29}$$

Since by the generalized Cauchy-Bunyakovsky inequality for any $q \in \mathbb{Z}_+$ such that $F \in (L^2)^{-\beta}_{-q}$

$$|\langle\langle F \otimes 1_\Delta, g\rangle\rangle_{(L^2)\otimes\mathcal{H}_\mathbb{C}}| \le \|F\|_{(L^2)^{-\beta}_{-q}} \sqrt{\rho(\Delta)}\|g\|_{(L^2)^\beta_q\otimes\mathcal{H}_\mathbb{C}},$$

this definition is well-posed and a Pettis integral

$$\int_\Delta \circ(u)du : (L^2)^\beta \otimes \mathcal{H}_\mathbb{C} \to (L^2)^\beta \tag{30}$$

is a linear *continuous* operator.

First let us show that for arbitrary $f \in (L^2)^\beta$ and $h^{(1)} \in \mathcal{H}_\mathbb{C}$

$$\int_\Delta (f \otimes h^{(1)})(u)du \equiv \int_\Delta f \cdot h^{(1)}(u)du = f \cdot \int_\Delta h^{(1)}(u)du. \tag{31}$$

In fact, for each $F \in (L^2)^{-\beta}$ by (29) we have

$$\begin{aligned}\langle\langle F, \int_\Delta f \cdot h^{(1)}(u)du\rangle\rangle_{(L^2)} &= \langle\langle F \otimes 1_\Delta, f \otimes h^{(1)}\rangle\rangle_{(L^2)\otimes\mathcal{H}_\mathbb{C}}\\ &= \langle\langle F, f\rangle\rangle_{(L^2)} \int_\Delta h^{(1)}(u)du = \langle\langle F, f \cdot \int_\Delta h^{(1)}(u)du\rangle\rangle_{(L^2)}.\end{aligned}$$

Let now $f, g \in (L^2)^\beta$ and $h^{(1)} \in \mathcal{H}_\mathbb{C}$. Using (26) and (31) we obtain

$$\begin{aligned}\int_\Delta \big(f\overline{\Diamond}(g \otimes h^{(1)})\big)(u)du &= \int_\Delta \big((f\Diamond g) \otimes h^{(1)}\big)(u)du \equiv \int_\Delta (f\Diamond g) \cdot h^{(1)}(u)du\\ &= (f\Diamond g) \cdot \int_\Delta h^{(1)}(u)du = f\Diamond\Big(g \cdot \int_\Delta h^{(1)}(u)du\Big)\\ &= f\Diamond \int_\Delta g \cdot h^{(1)}(u)du \equiv f\Diamond \int_\Delta (g \otimes h^{(1)})(u)du.\end{aligned}$$

From here, by virtue of continuity of the Wick multiplications $\overline{\Diamond}$ and $\Diamond$, and continuity of Pettis integral (30), we obtain the following statement.

Theorem 5 (cf. Theorem 4). *Let $\Delta \in \mathcal{B}(\mathbb{R}_+)$ be such that $\rho(\Delta) < \infty$, $f \in (L^2)^\beta$ and $g \in (L^2)^\beta \otimes \mathcal{H}_{\mathbb{C}}$. Then*

$$\int_\Delta (f\overline{\Diamond}g)(u)du = f\Diamond \int_\Delta g(u)du \in (L^2)^\beta. \tag{32}$$

Note that, as in the case of the extended stochastic integral, now one can interpret g as a function acting from $\mathbb{R}_+$ to $(L^2)^\beta$, and rewrite equality (32) in a classical form $\int_\Delta f\Diamond g(u)du = f\Diamond \int_\Delta g(u)du$.

Now we will introduce a Pettis integral over a set of infinite Lebesgue measure and obtain an analog of Theorem 5 for this integral.

Definition 7 (cf. Definition 6). *Let $\Delta \in \mathcal{B}(\mathbb{R}_+)$ be such that $\rho(\Delta) = \infty$, and $g \in (L^2)^\beta \otimes \mathcal{H}_{\mathbb{C}}$ satisfies the condition*

$$\forall q \in \mathbb{Z}_+ \qquad \int_\Delta \|g(u)\|_{(L^2)^\beta_q} du < \infty \tag{33}$$

(here we interpret g as a function acting from $\mathbb{R}_+$ to $(L^2)^\beta$). Define a Pettis integral $\int_\Delta g(u)du \in (L^2)^\beta$ as a unique element of $(L^2)^\beta$ such that for each $F \in (L^2)^{-\beta}$ equality (29) *is fulfilled.*

The well-posedness of this definition follows from the estimate (see (29))

$$\begin{aligned} |\langle\!\langle F\otimes 1_\Delta, g\rangle\!\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}}| &= \Big| \int_\Delta \langle\!\langle F, g(u)\rangle\!\rangle_{(L^2)} du\Big| \\ &\le \int_\Delta |\langle\!\langle F, g(u)\rangle\!\rangle_{(L^2)}| du \le \|F\|_{(L^2)^{-\beta}_{-q}} \int_\Delta \|g(u)\|_{(L^2)^\beta_q} du, \end{aligned} \tag{34}$$

where $q \in \mathbb{Z}_+$ is such that $F \in (L^2)^{-\beta}_{-q}$. Moreover, it follows from (34) that

$$\Big\| \int_\Delta g(u)du\Big\|_{(L^2)^\beta_q} \le \int_\Delta \|g(u)\|_{(L^2)^\beta_q} du. \tag{35}$$

Note that the Lebesgue integral $\int_\Delta \|g(u)\|_{(L^2)^\beta_q} du = \lim\limits_{\Delta_n\uparrow\Delta} \int_{\Delta_n} \|g(u)\|_{(L^2)^\beta_q} du$, where for each $n \in \mathbb{N}$ $\Delta_n \in \mathcal{B}(\mathbb{R}_+)$, $\Delta_n \subseteq \Delta_{n+1}$, $\rho(\Delta_n) < \infty$, and $\bigcup\limits_{n=1}^{\infty} \Delta_n = \Delta$ (see, e.g., [3] for details). Let us show that

$$\int_\Delta g(u)du = \lim_{\Delta_n\uparrow\Delta} \int_{\Delta_n} g(u)du \tag{36}$$

in $(L^2)^\beta$. In fact, by (29) for each $F \in (L^2)^{-\beta}$ we have

$$\begin{aligned} \langle\!\langle F, \int_\Delta g(u)du - \int_{\Delta_n} g(u)du\rangle\!\rangle_{(L^2)} &= \langle\!\langle F, \int_\Delta g(u)du\rangle\!\rangle_{(L^2)} - \langle\!\langle F, \int_{\Delta_n} g(u)du\rangle\!\rangle_{(L^2)} \\ &= \langle\!\langle F\otimes 1_\Delta, g\rangle\!\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} - \langle\!\langle F\otimes 1_{\Delta_n}, g\rangle\!\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} = \langle\!\langle F\otimes 1_\Delta - F\otimes 1_{\Delta_n}, g\rangle\!\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} \\ &= \langle\!\langle F\otimes (1_\Delta - 1_{\Delta_n}), g\rangle\!\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} = \langle\!\langle F\otimes 1_{\Delta\setminus\Delta_n}, g\rangle\!\rangle_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} = \langle\!\langle F, \int_{\Delta\setminus\Delta_n} g(u)du\rangle\!\rangle_{(L^2)}, \end{aligned}$$

therefore $\int_\Delta g(u)du - \int_{\Delta_n} g(u)du = \int_{\Delta\setminus\Delta_n} g(u)du$ and by (35) for each $q \in \mathbb{Z}_+$

$$\begin{aligned} \Big\| \int_\Delta g(u)du - \int_{\Delta_n} g(u)du\Big\|_{(L^2)^\beta_q} &= \Big\| \int_{\Delta\setminus\Delta_n} g(u)du\Big\|_{(L^2)^\beta_q} \le \int_{\Delta\setminus\Delta_n} \|g(u)\|_{(L^2)^\beta_q} du \\ &= \Big| \int_\Delta \|g(u)\|_{(L^2)^\beta_q} du - \int_{\Delta_n} \|g(u)\|_{(L^2)^\beta_q} du\Big| \underset{\Delta_n\uparrow\Delta}{\to} 0, \end{aligned}$$

which is what had to be proved.

Let now $f \in (L^2)^\beta$, $g \in (L^2)^\beta \otimes \mathcal{H}_\mathbb{C}$ and satisfy condition (33) (with Δ as in Definition 7). Then by construction of Wick multiplications $\overline{\Diamond}$ and $\Diamond$ for ρ-almost all $u \in \mathbb{R}_+$ $g(u) \in (L^2)^\beta$ and $(f\overline{\Diamond}g)(u) = f\Diamond(g(u))$, hence by (20) the Wick product $f\overline{\Diamond}g \in (L^2)^\beta \otimes \mathcal{H}_\mathbb{C}$ also satisfies (33). Therefore all said above about g holds true for $f\overline{\Diamond}g$. Since for each $n \in \mathbb{N}$ $\rho(\Delta_n) < \infty$, by (32) we have

$$\int_{\Delta_n} (f\overline{\Diamond}g)(u)du = f\Diamond \int_{\Delta_n} g(u)du. \tag{37}$$

But by (36)

$$\int_{\Delta_n} (f\overline{\Diamond}g)(u)du \underset{\Delta_n\uparrow\Delta}{\rightarrow} \int_{\Delta} (f\overline{\Diamond}g)(u)du, \tag{38}$$

and by (20) and (36) for each $q \in \mathbb{Z}_+$ and $q_1 \geq q+2+\beta$

$$\begin{aligned}\Big\|f\Diamond \int_{\Delta} g(u)du - f\Diamond \int_{\Delta_n} g(u)du\Big\|_{(L^2)^\beta_q} &= \Big\|f\Diamond\Big(\int_{\Delta} g(u)du - \int_{\Delta_n} g(u)du\Big)\Big\|_{(L^2)^\beta_q} \\ &\leq \|f\|_{(L^2)^\beta_{q_1}}\Big\|\int_{\Delta} g(u)du - \int_{\Delta_n} g(u)du\Big\|_{(L^2)^\beta_{q_1}} \underset{\Delta_n\uparrow\Delta}{\rightarrow} 0,\end{aligned}$$

therefore

$$f\Diamond \int_{\Delta_n} g(u)du \underset{\Delta_n\uparrow\Delta}{\rightarrow} f\Diamond \int_{\Delta} g(u)du \tag{39}$$

in $(L^2)^\beta$. Hence, by (38), (39) and (37) representation (32) holds true in the case of integration over a measurable set Δ of infinite Lebesgue measure. Let us formulate the proved statement as a theorem.

Theorem 6 (cf. Theorem 5). *Let $\Delta \in \mathcal{B}(\mathbb{R}_+)$ be such that $\rho(\Delta) = \infty$, $f \in (L^2)^\beta$ and $g \in (L^2)^\beta \otimes \mathcal{H}_\mathbb{C}$ satisfy condition* (33)*. Then representation* (32) *is fulfilled.*

Remark 12. *An analog of Theorem 6 is valed in the Lévy analysis on the spaces of regular generalized functions [13]: for $\Delta \in \mathcal{B}(\mathbb{R}_+)$ with $\rho(\Delta) = \infty$, $F \in (L^2)^{-\beta}$ and $G \in (L^2)^{-\beta} \otimes \mathcal{H}_\mathbb{C}$ such that for some $q \in \mathbb{Z}_+$ $\int_\Delta \|G(u)\|_{(L^2)^{-\beta}_{-q}}du < \infty$ we have $\int_\Delta (F\overline{\Diamond}G)(u)du = F\Diamond \int_\Delta G(u)du \in (L^2)^{-\beta}$, one can prove this statement by analogy with the proof of Theorem 6.*

As is known, in different versions of the infinite-dimensional white noise analysis an extended stochastic integral can be presented as a Pettis integral from a Wick product of the original integrand by the corresponding white noise. In particular, in the Lévy analysis this representation has a form

$$\int_\Delta F(u)\widehat{d}L_u = \int_\Delta F(u)\Diamond \dot{L}_u du,\ \Delta \in \mathcal{B}(\mathbb{R}_+), \tag{40}$$

where $\dot{L}$ is a Lévy white noise. Depending on spaces in which integration is considered, equality (40) can be formal (e.g., on the spaces of regular generalized functions, see [13]) or can have a rigorous sense (e.g., on the spaces of nonregular generalized functions, see [29]). In any case this equality is very useful for applications, in particular, for study stochastic equations with Wick type nonlinearities. Note that, in a sense, representation (40) is an analog of a formula for replacement of a measure in the Lebesgue integration theory. In particular, $\dot{L}$ is an analog of a Radon-Nikodym derivative.

As we just noted, representation (40) is valid for $F \in (L^2)^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$. In this case the equality is *formal* because $\dot{L}_u = \langle \circ, \delta_u \rangle$ (see Subsection 1.1) and $\delta_u \notin \mathcal{H}_{\mathbb{C}}$. Now the integral in the right hand side of (40) is a *formal* Pettis integral in the sense that when integrate formally, we get [13]

$$\int_\Delta F(u) \Diamond \dot{L}_u du = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n+1}, \widehat{F}_\Delta^{(n)} \rangle: = \int_\Delta F(u) \widehat{d} L_u$$

for each $\Delta \in \mathcal{B}(\mathbb{R}_+)$ (see (12)), and, of course, for each $f \in (L^2)^\beta$

$$\langle\langle \int_\Delta F(u) \Diamond \dot{L}_u du, f \rangle\rangle_{(L^2)} = \langle\langle F(\cdot) \Diamond \langle \circ, \delta_\cdot \rangle, f \otimes 1_\Delta(\cdot) \rangle\rangle_{(L^2) \otimes \mathcal{H}_{\mathbb{C}}} = \langle\langle \int_\Delta F(u) \widehat{d} L_u, f \rangle\rangle_{(L^2)}$$

(cf. (29)). Since the extended stochastic integral and the Wick multiplication $\Diamond$ on the spaces of regular test functions are the restrictions to these spaces of the corresponding stochastic integral and Wick multiplication introduced on the spaces of regular generalized functions, and for $f \in (L^2)^\beta \otimes \mathcal{H}_{\mathbb{C}}$ we have $\int_\Delta f(u) \widehat{d} L_u \in (L^2)^\beta$, from the above-described result of [13] the next statement follows.

Theorem 7. *For arbitrary $f \in (L^2)^\beta \otimes \mathcal{H}_{\mathbb{C}}$ and $\Delta \in \mathcal{B}(\mathbb{R}_+)$ the extended stochastic integral $\int_\Delta f(u) \widehat{d} L_u$ can be formally presented as*

$$\int_\Delta f(u) \widehat{d} L_u = \int_\Delta f(u) \Diamond \dot{L}_u du \equiv \int_\Delta f(u) \Diamond \langle \circ, \delta_u \rangle du \in (L^2)^\beta, \tag{41}$$

where the integral in the right hand side is a formal Pettis integral.

Remark 13. *It is easy to see that results of Theorems 4, 5, 6 and 7 hold true for integrands described in Remark 3.*

Now let us consider an example of an integral stochastic equation with Wick multiplication.

Example 1. *Let*

$$X_t = X_0 + \int_{[0,t)} f \Diamond X_u du + g \int_{[0,t)} X_u \widehat{d} L_u, \tag{42}$$

where $X_0 \in (L^2)^\beta$, $f = \sum_{n=0}^N :\langle \circ^{\otimes n}, f^{(n)} \rangle:$, $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $g \in \mathbb{C}$. Applying to this equation the S-transform with regard to (41)*, solving the obtained nonstochastic equation, and applying the inverse S-transform, one can show by analogy with [23] that the solution of* (42) *is*

$$X_t = X_0 \Diamond \exp^{\Diamond} \{ft + gL_t\} \in (L^2)^{-1}$$

(remind that $L_t = \langle \circ, 1_{[0,t)} \rangle$). But if $N \leq 1$ and $\beta \in [0,1)$ then, as it follows from Remark 9 and Theorem 2, $X_t \in (L^2)^\beta$.

Remark 14. *As we mentioned above, together with stochastic integrals and derivatives one can consider so-called operators of stochastic differentiation on the spaces of regular test and generalized functions [10, 14]. It is proved in [12] that the operator of stochastic differentiation of first order satisfies the Leibnitz rule with respect to the Wick multiplication on the space $(L^2)^{-\beta}$. Now it follows from Theorem 2 that this property holds true on the space $(L^2)^\beta$.*

Finally we will make the following observation. We noted above and repeatedly used the fact that many results of the Meixner white noise analysis can be easily reformulated for the Lévy analysis. The converse of this statement is also true: many results of the Lévy white noise analysis, in particular, all results of Subsection 2.2, can be easily reformulated for the Meixner analysis.

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Received 07.12.2021

Качановський М.О. *Про віківське числення та його зв'язок зі стохастичним інтегруванням на просторах регулярних основних функцій в аналізі білого шуму Леві* // Карпатські матем. публ. — 2022. — Т.14, №1. — С. 194–212.

Ми працюємо з просторами регулярних основних функцій в аналізі білого шуму Леві, побудованими з використанням узагальнення властивості хаотичного розкладу, запропонованого Є.В. Литвиновим. Нашою метою є вивчення властивостей віківського множення і віківських версій голоморфних функцій, а також опис зв'язку між віківським множенням та інтегруванням, на цих просторах. Більш точно, ми встановлюємо, що віківський добуток регулярних основних функцій є регулярною основною функцією; за певних умов віківська версія голоморфної функції з аргументом з простору регулярних основних функцій є регулярною основною функцією; показуємо, що, використовуючи віківське множення, можна виносити незалежний від часу множник з-під знаку розширеного стохастичного інтеграла за процесом Леві; встановлюємо аналог цього результату для інтеграла Петтіса (слабкого інтеграла); отримуємо представлення розширеного стохастичного інтеграла через формальний інтеграл Петтіса від віківського добутку вихідної підінтегральної функції на білий шум Леві. Як приклад застосування наших результатів ми розглядаємо інтегральне стохастичне рівняння з віківським множенням.

Ключові слова і фрази: процес Леві, розширений стохастичний інтеграл, інтеграл Петтіса, віківський добуток.