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Local Polya fluctuations of Riesz gravitational fields and the Cauchy problem

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We consider a pseudodifferential equation of parabolic type with a fractional power of the Laplace operator of order $\alpha \in (0;1)$ acting with respect to the spatial variable. This equation naturally generalizes the well-known fractal diffusion equation. It describes the local interaction of moving objects in the Riesz gravitational field. A simple example of such system of objects is stellar galaxies, in which interaction occurs according to Newton's gravitational law. The Cauchy problem for this equation is solved in the class of continuous bounded initial functions. The fundamental solution of this problem is the Polya distribution of probabilities $\mathcal{P}_\alpha(F)$ of the force F of local interaction between these objects. With the help of obtained solution estimates the correct solvability of the Cauchy problem on the local field fluctuation coefficient under certain conditions is determined. In this case, the form of its classical solution is found and the properties of its smoothness and behavior at the infinity are studied. Also, it is studied the possibility of local strengthening of convergence in the initial condition. The obtained results are illustrated on the α-wandering model of the Lévy particle in the Euclidean space $\mathbb{R}^3$ in the case when the particle starts its motion from the origin. The probability of this particle returning to its starting position is investigated. In particular, it established that this probability is a descending to zero function, and the particle "leaves" the space $\mathbb{R}^3$.

Key words and phrases: gravitational field, Riesz potential, Polya distribution, symmetric stable random Lévy process, Lévy's flight, fractal diffusion equation, fractional Laplacian, fundamental solution, Cauchy problem.

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Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the scalar product $(\cdot,\cdot)$ and the norm $|r| = (r,r)^{1/2}$; $\mathbb{Z}_+^n$ be the set of all n-dimensional multi-indices; $\mathbb{R} = \mathbb{R}^1$ and $\mathbb{Z}_+ = \mathbb{Z}_+^1$. The Fourier transform operator is denoted by the symbol $\mathbb{F}$.

In the space $\mathbb{R}^3$, we consider a system of moving objects Z_j with masses m_j. We believe that the interaction between objects is subject to Riesz potential [26]. This means that the gravitational influence F between any two objects of masses M and m is described by the law

$$F = G\frac{Mm}{|r|^\beta}r^0, \qquad \beta > 0, \tag{1}$$

where G is the corresponding gravitational constant, r is the vector of the distance between these objects, and $r^0 = r/|r|$, $r \in \mathbb{R}^3$. A simple example of such systems are stellar galaxies, in

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which the interaction between star objects Z_j is described by the well-known Newton's law (1) with $\beta = 2$.

In this system we fix some object Z_0 and assume that it is at the origin. We are concerned with the force $F(t)$ of local influence on the unit of mass of the object Z_0 at time t, which is caused by the close environment of this object. Since this environment is constantly and unpredictably changing, it is convenient to consider F as a random variable.

In [18], it is established that the nonstationary probabilities distribution $W_\beta(F;t)$ of the force $F(t)$ is determined by equality

$$W_\beta(F;t) = \mathbb{F}^{-1}\left[e^{-a_\beta(t)|\xi|^{3/\beta}}\right](F;t), \qquad \beta > 3/2, \tag{2}$$

where $a_\beta(\cdot)$ is the so-called the coefficient of local fluctuation of the system's gravitational field, which is determined by the distribution of objects in the system and their average mass.

Under certain conditions on $a_\beta(\cdot)$, the distribution W_β on the set $\mathbb{R}^3 \times (0;T]$ is a fundamental solution of the Cauchy problem for the pseudodifferential equation ($\overline{\text{PDE}}$) [18]

$$\partial_t u(x;t) + a'_\beta(t) A_\nu u(x;t) = 0, \qquad t \in (0;T],\ x \in \mathbb{R}^n. \tag{3}$$

Here $n = 3$, $\nu = 3/\beta$, $T \in (0;+\infty]$; A_ν is the Riesz operator of fractional differentiation of ν order, i.e. $A_\nu = (-\Delta)^{\nu/2}$, where Δ is the Laplace operator [29], and

$$a'_\beta(t) = \frac{da_\beta(t)}{dt}.$$

In the simplest case $a'_\beta(t) \equiv \text{const}$, equation (3) is known as "fractal diffusion equation" [12, p. 324] or "isotropic superdiffusion equation" [35, p. 251]. An important example for motivating the study of the fractal diffusion equation is given in [4, p. 2]. Here a probabilistic model of a random walk of the particle X in long jumps is proposed and it is shown that the probability $u(x;t)$ of the presence particle X at the time t at the spatial point x is the solution to equation (3) for $a'_\beta(t) \equiv 1$. Processes of this type occur in nature quite often, see in particular the biological observations in [25,36] and the mathematical discussions in [9,23].

The fractal diffusion equation is the source of many random processes [13]. In further generality, it is known that the Riesz operator A_ν (the fractional Laplacian) is an infinitesimal generator of the Lévy process, see e.g. [1,2] for further details. In this regard, we note that each distribution $W_\beta(\cdot;t)$, $\beta > 3/2$, with a fixed $t \in [0;T]$ belongs to the class of the Lévy distributions of symmetric stable random processes [16,37]:

$$\mathcal{L}_\nu(\cdot) = \mathbb{F}^{-1}\left[e^{-b|\xi|^\nu}\right](\cdot), \qquad \nu \in (0;2]. \tag{4}$$

In particular, W_2 is the known Holtsmark distribution [5,11].

Obviously, $W_\beta = \mathcal{L}_\nu$ for $\nu = 3/\beta$ and $b = a_\beta(t)$, $t \in (0;T]$. This equality characterizes the general nature of symmetric stable random Lévy processes. Each of such processes $\mathcal{L}_\nu$ for $\nu \in (0;2)$ can be regarded as a process of local influence of moving objects in the corresponding gravitational field of M. Riesz.

In his fundamental work [16], P. Lévy proved that the function $\mathcal{L}_\nu(\cdot)$ is the probability density only for $\nu \in (0;2]$. This study was preceded by the research of the Hungarian mathematician G. Polya [24], who established this fact for the case $\nu \in (0;1)$. Thus Lévy distributions $\mathcal{L}_\nu(\cdot)$ of order $\nu \in (0;1)$ are also called Polya $\mathcal{P}_\nu(\cdot)$ distributions in the literature.

For convenience, the fundamental solution of the problems Cauchy for $\overline{\text{PDE}}$ (3) we denote by

$$G_\nu(x;t) = \mathbb{F}^{-1}\left[e^{-\hat{a}_\beta(t)|\xi|^\nu}\right](x;t), \qquad x \in \mathbb{R}^n,\ t \in (0;T], \tag{5}$$

where $\hat{a}_\beta(\cdot) = a_\beta(\cdot) - a_\beta(0)$.

Investigation of the Cauchy problem for $\overline{\text{PDE}}$ (3) and the corresponding function $G_\nu(x;t)$ in the case when the coefficient $a_\beta(\cdot)$ is a strictly increasing function on the interval $(0;T]$, was conducted in many works [3, 6–8, 30] (see the detailed review in [18]). There, for $\nu \in [1;2]$, various methods were developed to study the properties of the fundamental solution $G_\nu(x;t)$, and statements were formulated about the correct solvability of the Cauchy problem in classes of Hölder functions. Also, the typical properties of the classical solutions of $\overline{\text{PDE}}$ (3) were clarified, in particular, an analogue of the maximum principle was established.

At the same time, the case of $\nu \in (0;1)$ appeared to be much more problematic and had remained little-studied for a long time. Recently, new results have been obtained [14, 15, 21]. Here, in a slightly different form than that in [7], a parametrix was proposed for constructing the structure of the fundamental solution of the Lévy-type operator L with a variable symbol of order $\nu \in (0;1)$. Gradient estimates of this solution are also established, which are important in the study of the corresponding Markov processes. In addition, for $\overline{\text{PDE}}$ (3) the correct solution of the Cauchy problem in the class of unbounded, discontinuous with integrative singularity of initial functions is proved in [17]. Also an analogue of the maximum principle is established, by means of which the uniqueness of the solution of this problem is substantiated.

The subject of our research is the properties of the Polya distribution density related to the problem of local influence of moving objects in the Riesz gravitational field, i.e. the properties of fundamental solutions G_ν of the Cauchy problem for $\overline{\text{PDE}}$ (3) of purely fractional order ν, and the correct solvability of this problem in the class of bounded continuous initial data. The results obtained here harmoniously complement the results of research conducted in [7, 17].

The contents of the work is as follows. Section 1 contains the necessary information about the operator A_ν and the properties of the function G_ν. The Cauchy problem for $\overline{\text{PDE}}$ (3) of order $\nu \in (0;1)$ in the class of continuous bounded initial functions is solved in Section 2. Here the classical solution of this problem is obtained, the form of the image of a solution is found and properties of its smoothness and behavior at infinity are investigated. Section 3 clarifies the question of the uniqueness of the solution of this Cauchy problem under certain conditions on the coefficient of local fluctuation $a_\beta(\cdot)$. The possibility of local increasing the convergence of the solution of the Cauchy problem to its limit value when approaching the initial hyperplane is clarified in Section 4. The obtained result are illustrated in Section 5 by the example of solving the problem of finding the time of return of a wandering Lévy particle, to the place of its start. Section 6 presents conclusions.

1 Preliminary information

We assume that $\mathbb{C}^l(Q)$ is the class of all continuously differentiable to order l functions on the set Q, $\mathbb{S} = \mathbb{S}\left(\mathbb{R}^3\right)$ is the Schwartz space defined on $\mathbb{R}^3$ infinitely differentiable rapidly decreasing functions [31], and $\Pi_Q = \left\{(x;t):\ x \in \mathbb{R}^3, t \in Q\right\}$.

As it was mentioned above, the Riesz operator of fractional differentiation is the fractional power of the Laplace operator, taken with the "minus" sign: $A_\nu = (-\triangle)^{\nu/2}$. On the elements of the Schwartz space of rapidly decreasing functions, this operator is determined by equality

$$(A_\nu f)(\cdot) = \mathbb{F}^{-1}\left[|\xi|^\nu \mathbb{F}[f]\right](\cdot), \qquad f \in \mathbb{S}. \tag{6}$$

However, the classical form of fractional differentiation (6) is not suitable for extending the operator A_ν to wider classes of functions. The following form is more convenient for our research [29, p.367]:

$$(A_\nu f)(x) = c(\nu)\int_{\mathbb{R}^3} \frac{f(x)-f(x+y)}{|y|^{3+\nu}}dy, \quad x \in \mathbb{R}^3,\ \nu \in (0;1), \tag{7}$$

where

$$c(\nu) = \frac{\nu(1+\nu)}{4\pi\Gamma(1-\nu)\cos(\nu\pi/2)}$$

(here $\Gamma(\cdot)$ is gamma function).

It should be noted that the theory of Riesz potential and the corresponding fractional differentiation originates from [10, 26, 27]. G. Thorin, S. Sobolev, S. Stein, P. Lizorkin, S. Samko and others made a significant contribution to its development (see [22, 28, 32–34]).

Note that the integral from equality (7) converges absolutely, for example, for bounded Hölder functions with an order greater than ν, so formula (7) allows us to apply the operator A_ν to functions of wider classes than the space $\mathbb{S}$. The set of all functions f defined on $\mathbb{R}^3$, for which the right-hand part of relation (7) has meaning, is denoted by $\mathcal{D}(A_\nu)$. It is obvious that the constant function $f(x) \equiv \text{const}$ belongs to the set $\mathcal{D}(A_\nu)$ for every $\nu \in (0;1)$, and $A_\nu f = 0$. Further, we assume that the coefficient $a_\beta(\cdot) \in \mathbb{C}^1([0;T])$ and

$$\hat{a}_\beta(t) \equiv a_\beta(t) - a_\beta(0) > 0 \qquad \forall t \in (0;T]. \tag{8}$$

Under such conditions, the following statement holds.

Theorem 1. *The density $W_\beta(x;t)$ of probability distribution on the set $\Pi_{(0;T]}$ is infinitely differentiable with respect to the variable x and once differentiable with respect to the variable t. The following estimates are correct:*

$$\left|\partial_x^k W_\beta(x;t)\right| \le c_1 a_\beta(t)\left(\left(a_\beta(t)\right)^{1/\nu} + |x|\right)^{-3-|k|-\nu}, \tag{9}$$

$$\left|\partial_t\partial_x^k W_\beta(x;t)\right| \le c_2\left|a'_\beta(t)\right|\left(\left(a_\beta(t)\right)^{1/\nu} + |x|\right)^{-3-|k|-\nu}, \tag{10}$$

where c_1 and c_2 are positive constants.

This theorem is easily proved according to the scheme of the proof of Lemma 2 from [19]. Hence, taking into account (2) and (5), the following consequence becomes obvious.

Corollary 1. *For the derivatives of the fundamental solution G_ν the following estimates are correct:*

$$\left|\partial_x^k G_\nu(x;t)\right| \le c_1 \hat{a}_\beta(t)\left(\left(\hat{a}_\beta(t)\right)^{1/\nu} + |x|\right)^{-3-|k|-\nu}, \tag{11}$$

$$\left|\partial_t\partial_x^k G_\nu(x;t)\right| \le c_2\left|a'_\beta(t)\right|\left(\left(\hat{a}_\beta(t)\right)^{1/\nu} + |x|\right)^{-3-|k|-\nu}, \tag{12}$$

for all $(x;t) \in \Pi_{(0;T]}$ and $k \in \mathbb{Z}_+^3$.

Note that estimates (11), (12) for the case when the coefficient $a_\beta(\cdot)$ is a strictly increasing function, were obtained in [19, 20].

Estimates (11), (12) will allow us to establish the correct solvability of the Cauchy problem for $\overline{\text{PDE}}$ (3) in the class of continuous bounded initial functions and to study some properties of its solutions.

2 The Cauchy problem

For $\overline{\text{PDE}}$ (3) we consider the Cauchy problem

$$u(\cdot;t)_{|t=0} = f, \tag{13}$$

in which f is a bounded continuous function on $\mathbb{R}^3$.

Definition 1. *The solution of the Cauchy problem* (3), (13) *on the set* $\Pi_{(0;T]}$ *is called the function* $u(x;t)$*, which on this set is differentiable by the variable* t *and* $u(\cdot;t) \in \mathcal{D}(A_\nu)$*,* $t \in (0;T]$*. In this case the function* u *on* $\Pi_{(0;T]}$ *satisfies the equation* (3) *in the usual sense, and the initial condition* (13) *in the sense of the boundary relation*

$$u(x;t) \underset{t\to+0}{\to} f(x), \quad x \in \mathbb{R}^3. \tag{14}$$

This auxiliary statement holds.

Lemma 1. *Let* $a_\beta(\cdot) \in \mathbb{C}^1\big([0;T]\big)$ *satisfy condition* (8)*. Then the function*

$$u(x;t) = (f * G_\nu)(x;t), \quad (x;t) \in \Pi_{(0;T]} \tag{15}$$

is:

1) on $\mathbb{R}^3$ *– infinitely differentiable with respect to the variable* x *with a fixed* $t \in (0;T]$ *and bounded together with all its derivatives;*

2) on $(0;T]$ *– differentiable at* t *for a fixed* $x \in \mathbb{R}^3$*.*

In this case, we have

$$\partial_x^k u(x;t) = \Big(f * \partial_x^k G_\nu\Big)(x;t), \quad \partial_t u(x;t) = (f * \partial_t G_\nu)(x;t), \quad (x;t) \in \Pi_{(0;T]}. \tag{16}$$

If

$$\exists\{c,\alpha\} \subset (0;+\infty)\ \forall\, x \in \mathbb{R}^3: \quad \big|f(x)\big| \le \frac{c}{\big(1+|x|\big)^{\alpha}}, \tag{17}$$

then

$$\lim_{|x|\to+\infty} u(x;t) = 0 \quad \forall\, t \in (0;T]. \tag{18}$$

Proof. First, we note that

$$(f * G_\nu)(x;t) = \int_{\mathbb{R}^3} f(y) G_\nu(x-y;t)dy, \quad (x;t) \in \Pi_{(0;T]}. \tag{19}$$

Having taken into account the conditions of the function f and estimate (11), we obtain that for all $k \in \mathbb{Z}_+^3$ and $(x;t) \in \Pi_{(0;T]}$ the following inequalities hold true

$$\begin{aligned}\left|\int_{\mathbb{R}^3} f(y)\partial_x^k G_\nu(x-y;t)dy\right| &\le \int_{\mathbb{R}^3} \big|f(y)\big| \left|\partial_{x-y}^k G_\nu(x-y;t)\right| dy \\ &\le \int_{\mathbb{R}^3} \frac{c\hat{a}_\beta(t)dy}{\Big(\big(\hat{a}_\beta(t)\big)^{1/\nu} + |x-y|\Big)^{3+|k|+\nu}} \\ &= \int_{\mathbb{R}^3} \frac{c\big(\hat{a}_\beta(t)\big)^{-|k|/\nu}dz}{\big(1+|z|\big)^{3+|k|+\nu}} \equiv c_k\big(\hat{a}_\beta(t)\big)^{-|k|/\nu},\end{aligned}$$

where c_k is a positive value that depends only on k. These estimates ensure that the following equality

$$\partial_x^k (f * G_\nu)(x;t) = \left(f * \partial_x^k G_\nu\right)(x;t), \quad (x;t) \in \Pi_{(0;T]},$$

holds for each $k \in \mathbb{Z}_+^3$. Hence we obtain that the function $u(x;t)$ is infinitely differentiable with respect to the variable x on $\Pi_{(0;T]}$ and the derivatives $\partial_x^k u$ are bounded. Similarly, using estimate (12), we verify the differentiability of $u(x;t)$ by the variable t and the fulfillment of the second equality with (16).

We are going to set the boundary relation (18). To do this, we use the estimate

$$\left|u(x;t)\right| \le c \left(\int_{2|y|\ge|x|} \frac{G_\nu(x-y;t)dy}{\left(1+|y|\right)^\alpha} + \int_{2|y|<|x|} \frac{G_\nu(x-y;t)dy}{\left(1+|y|\right)^\alpha}\right) \qquad \forall\,(x;t) \in \Pi_{(0;T]}.$$

According to the equality

$$\int_{\mathbb{R}^3} G_\nu(x;t)dx = 1, \qquad t \in (0;T], \tag{20}$$

we have

$$\int_{2|y|\ge|x|} \frac{G_\nu(x-y;t)dy}{\left(1+|y|\right)^\alpha} \le \frac{2^\alpha}{\left(1+|x|\right)^\alpha} \int_{\mathbb{R}^3} G_\nu(x-y;t)dy \equiv \frac{2^\alpha}{\left(1+|x|\right)^\alpha} \underset{|x|\to+\infty}{\longrightarrow} 0.$$

Further, if $2|y| < |x|$, then $|x-y| \ge \big||x|-|y|\big| = |x|\big|1-|y|/|x|\big| \ge |x|/2$.

Considering this and estimate (11), we find

$$\begin{aligned}
\int_{2|y|<|x|} \frac{G_\nu(x-y;t)}{\left(1+|y|\right)^\alpha} dy &\le \int_{2|y|<|x|} \frac{c_1 \hat{a}_\beta(t)dy}{|x-y|^{\nu/2}\left(\left(\hat{a}_\beta(t)\right)^{1/\nu} + |x-y|\right)^{3+\nu/2}} \\
&\le \frac{\sqrt{2^\nu}}{|x|^{\nu/2}} \int_{\mathbb{R}^3} \frac{c_1 \hat{a}_\beta(t)dy}{\left(\left(\hat{a}_\beta(t)\right)^{1/\nu} + |x-y|\right)^{3+\nu/2}} \\
&= \frac{\sqrt{2^\nu}}{|x|^{\nu/2}} \int_{\mathbb{R}^3} \frac{c_1\sqrt{\hat{a}_\beta(t)}dz}{\left(1+|z|\right)^{3+\nu/2}} \underset{|x|\to+\infty}{\longrightarrow} 0 \qquad \forall\, t \in (0;T].
\end{aligned}$$

Thus, the fulfillment of the boundary relation (18) is substantiated. ☐

Theorem 2. *Let* $a_\beta(\cdot) \in \mathbb{C}^1\left([0;T]\right)$ *satisfy condition* (8)*, then formula* (15) *determines the solution of Cauchy problem* (3)*,* (13).

Proof. We write formally

$$A_\nu u(x;t) = c(\nu) \int_{\mathbb{R}^3} \frac{u(x;t) - u(x+y;t)}{|y|^{3+\nu}} dy, \quad (x;t) \in \Pi_{(0;T]}.$$

By Lemma 1, the function $u(\cdot;t)$ is infinitely differentiable and bounded together with all derivatives by $\mathbb{R}^3$, so the integral from the previous equality is absolutely convergent on $\Pi_{(0;T]}$. This means that $u(\cdot;t) \in \mathcal{D}(A_\nu)$, $t \in (0;T]$.

Further, we find that

$$\int_{\mathbb{R}^3} \frac{u(x;t) - u(x+y;t)}{|y|^{3+\nu}} dy = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(z) \frac{G_\nu(x-z;t) - G_\nu(x+y-z;t)}{|y|^{3+\nu}} dzdy$$

directly from equations (15) and (19). The absolute convergence of the integral from the left-hand side of the equality makes it possible to change the order of integration in the integral in the right-hand side of this equality and obtain

$$A_\nu u(x;t) = (f * A_\nu G_\nu)(x;t), \quad (x;t) \in \Pi_{(0;T]}.$$

Hence, taking into account that the function G_ν is a solution of equation (3), as well as formula (16), we find

$$A_\nu u(x;t) = -(f * \partial_t G_\nu)(x;t) = -\partial_t u(x;t), \quad (x;t) \in \Pi_{(0;T]}.$$

Therefore, on the set $\Pi_{(0;T]}$ equation (15) determines the classical solution of $\overline{\text{PD}}$E (3).

We now show that this solution satisfies the initial condition (13), i.e. the boundary relation (14). To do this, we use equality (20), according to which

$$\big|(f * G_\nu)(x;t) - f(x)\big| \le \int_{\mathbb{R}^3} \big|G_\nu(\xi;t)\big|\big|f(x-\xi) - f(x)\big| d\xi \equiv \mathfrak{I}(x;t).$$

Since f is a continuous function on $\mathbb{R}^3$, for every $x \in \mathbb{R}^3$ and arbitrary $\varepsilon > 0$ there exists such t_0 that $t_0^{\frac{1}{2\nu}} < \varepsilon$ and $\big|f(x-\xi) - f(x)\big| < \varepsilon$, if $|\xi| < t_0^{\frac{1}{2\nu}}$. Then

$$\mathfrak{I}(x;t) < \varepsilon \int_{|\xi|<t_0^{\frac{1}{2\nu}}} \big|G_\nu(\xi;t)\big| d\xi + \int_{|\xi|\ge t_0^{\frac{1}{2\nu}}} \big|G_\nu(\xi;t)\big|\big|f(x-\xi) - f(x)\big| d\xi \le \varepsilon \mathfrak{I}_1(t) + \mathfrak{I}_2(x;t),$$

where

$$\mathfrak{I}_1(t) = \int_{\mathbb{R}^3} \big|G_\nu(\xi;t)\big| d\xi, \qquad \mathfrak{I}_2(x;t) = \int_{|\xi|\ge t_0^{\frac{1}{2\nu}}} \big|G_\nu(\xi;t)\big|\big|f(x-\xi) - f(x)\big| d\xi.$$

Further, considering estimate (11) and the boundedness of the function f in $\mathbb{R}^3$, for all $t \in (0;T]$ and $x \in \mathbb{R}^3$ we find

$$\mathfrak{I}_1(t) \le c_1 \hat{a}_\beta(t) \int_{\mathbb{R}^3} \frac{d\xi}{\Big(\big(\hat{a}_\beta(t)\big)^{1/\nu} + |\xi|\Big)^{3+\nu}} = c_1 \int_{\mathbb{R}^3} \frac{dz}{\big(1+|z|\big)^{3+\nu}} \equiv c_2;$$

$$\mathfrak{I}_2(x;t) \le c_3 \hat{a}_\beta(t) \int_{|\xi|\ge t_0^{\frac{1}{2\nu}}} |\xi|^{-(3+\nu)} d\xi = c_3 \hat{a}_\beta(t) \int_{t_0^{\frac{1}{2\nu}}}^{+\infty} \rho^{-(1+\nu)} d\rho = c_4 \hat{a}_\beta(t) t_0^{-1/2}. \tag{21}$$

It should be noted that the functions $a_\beta(\cdot)$ and $\hat{a}_\beta(\cdot)$ on $(0;T]$ are positive, while $a_\beta(\cdot)$ is continuously differentiable, so according to mean value theorem, there is a constant $\delta > 0$ such that for all $t \in (0;T]$ the following estimate

$$\hat{a}_\beta(t) \le \delta t.$$

is performed. Hence it follows from the above and from inequality (21) that for all $x \in \mathbb{R}^3$ and $t \le t_0$ we have

$$\mathfrak{I}_2(x;t) \le c_4 \delta t_0^{1/2} < c_4 \delta \varepsilon^\nu \equiv c_5 \varepsilon^\nu.$$

So, for each $x \in \mathbb{R}^3$ and arbitrary $\varepsilon > 0$ there is $t_0 < \varepsilon^{2\nu}$ such that for all $t \le t_0$ the inequality

$$\mathfrak{I}(x;t) < c(\varepsilon + \varepsilon^\nu)$$

holds, i.e. the boundary relation (14) is true. □

Taking into account the non-negativity of the function G_ν and equality (20), directly from formula (15), we arrive at the following statement.

Corollary 2. *If Cauchy problem* (3), (13) *has a unique solution* u*, then*

$$\inf_{x\in\mathbb{R}^3} f(x) \le u(x;t) \le \sup_{x\in\mathbb{R}^3} f(x) \qquad \forall\,(x;t)\in\Pi_{[0;T]}.$$

In the next section, the question of uniqueness of the Cauchy problem solution (3), (13) is clarified.

3 Conditions for the uniqueness of the solution

We previously assumed that on the set $[0;T]$ the fluctuation coefficient $a_\beta(\cdot)$ is such continuously differentiable function that satisfy the condition (8), i.e.

$$a_\beta(t) > a_\beta(0) \qquad \forall\, t\in(0;T].$$

Fulfillment of this condition causes the increase of the function $a_\beta(\cdot)$, even if not on the whole interval $(0;T]$, then at least on some part of it $(0;t_0)$, $t_0 < T$. However, on $[t_0;T]$ the function $a_\beta(\cdot)$ can be non-increasing. In this case, we have

$$a'_\beta(t_0) = 0; \quad a'_\beta(t) > 0, \quad t\in(0;t_0); \quad a'_\beta(t)\le 0, \quad t\in[t_0;T].$$

The following statement holds.

Theorem 3. *Let* $a_\beta(\cdot)\in\mathbb{C}^1([0;T])$ *satisfy condition* (8) *and* t_0 *is a fixed point with* $[0;T]$ *such that*

$$a'_\beta(t)\ge 0, \quad t\in(0;t_0].$$

Then on $\Pi_{(0;t_0]}$ *the Cauchy problem* (3), (13) *has only one solution for which the boundary relation* (18) *holds.*

Proof. Suppose that for the Cauchy problem (3), (13) on $\Pi_{(0;t_0]}$ there are two different solutions u_1 and u_2 with property (18). Consider the function $v = u_1 - u_2$, which on $\Pi_{(0;t_0]}$ is also a solution of $\overline{\text{PDE}}$ (3) with property (18). The zero initial condition (13) is obviously satisfied for v, i.e.

$$v(\cdot;t)_{|t=0} = 0.$$

We have to show that

$$v(x;t)\equiv 0 \qquad \forall\,(x;t)\in\Pi_{(0;t_0]}. \tag{22}$$

We apply the method of proof by contradiction. Suppose that condition (22) is not satisfied. This means that

$$\lambda = \inf_{(x;t)\in\Pi_{[0;t_0]}} v(x;t) < 0 \quad \text{or} \quad \mu = \sup_{(x;t)\in\Pi_{[0;t_0]}} v(x;t) > 0.$$

Let $\lambda < 0$ and

$$Lw(x;t) = \partial_t w(x;t) + a'_\beta(t)A_\nu w(x;t).$$

We consider the auxiliary function

$$\hat{v}(x;t) = v(x;t) + t\chi, \quad (x;t) \in \Pi_{(0;t_0]},$$

where χ is such fixed number that $0 < \chi < -\lambda/t_0$. Obviously

$$\inf_{(x;t)\in\Pi_{[0;t_0]}} \hat{v}(x;t) < 0.$$

It should be noted that $\hat{v}(x;t)$ is a continuous function on $\Pi_{[0;t_0]}$ for a set of variables, in addition,

$$\hat{v}(x;t)_{|t=0} = v(x;t)_{|t=0} = 0, \quad x \in \mathbb{R}^3,$$

and

$$\hat{v}(x;t) \underset{|x|\to\infty}{\to} t\chi > 0, \quad t \in (0;t_0].$$

Therefore, $v(x;t)$ has a negative global minimum in some point $(x_*;t_*) \in \Pi_{(0;t_0]}$. Then

$$\partial_t \hat{v}(x_*;t_*) = 0.$$

Beside this

$$\hat{v}(x_*;t_*) - \hat{v}(x_* + y;t_*) \le 0 \qquad \forall\, y \in \mathbb{R}^3,$$

i.e.

$$A_\nu \hat{v}(x_*;t_*) = c(\nu) \int_{\mathbb{R}^3} \frac{\hat{v}(x_*;t_*) - \hat{v}(x_* + y;t_*)}{|y|^{3+\nu}} dy \le 0.$$

Hence, we find that

$$L\hat{v}(x_*;t_*) = \partial_t \hat{v}(x_*;t_*) + a'_\beta(t_*) A_\nu \hat{v}(x_*;t_*) = a'_\beta(t_*) A_\nu \hat{v}(x_*;t_*) \le 0.$$

On the other hand, for all $(x;t) \in \Pi_{(0;T]}$ we have

$$L\hat{v}(x;t) = L\big(v(x;t) + t\chi\big) = Lv(x;t) + L(t\chi) = L(t\chi) = \chi + ta'_\beta(t) A_\nu \chi = \chi > 0.$$

Here a contradiction arises. Therefore,

$$\inf_{(x;t)\in\Pi_{[0;t_0]}} v(x;t) = 0.$$

The falseness of the condition $\mu > 0$ is established similarly using the function

$$\check{v}(x;t) = v(x;t) - t\chi, \qquad (x;t) \in \Pi_{(0;t_0]},$$

(here χ is a fixed constant such that $0 < \chi < \mu/t_0$). Thus, the fulfillment of condition (22) is justified. □

4 The principle of the solution localization on the initial hyperplane

In this section we clarify the question of the possibility of increasing convergence in the initial condition (13) on that part of the space $\mathbb{R}^3$, where the initial function f is smooth.

The below statement holds.

Theorem 4. *Let f be a continuous function bounded on $\mathbb{R}^3$, and let u be a corresponding solution of the Cauchy problem* (3), (13). *If $f \in \mathrm{C}^l(Q)$, $Q \subset \mathbb{R}^3$, then*

$$\partial_x^k u(x;t) \underset{t\to+0}{\overset{\mathbb{K}\subset Q}{\rightrightarrows}} \partial_x^k f(x), \quad 0 \le |k| \le l,$$

(this refers to uniform convergence on each compact set $\mathbb{K}$ from the set Q).

Proof. Obviously, it will be enough to prove the fulfillment of the boundary relation

$$\partial_x^k u(x;t) \underset{t\to+0}{\overset{\mathbb{K}\subset Q}{\rightrightarrows}} 0, \quad k \in \mathbb{Z}_+^3,$$

for $f(x) = 0$, $x \in Q$.

Let $\mathbb{K} \subset \mathbb{K}_1 \subset Q$, where $\mathbb{K}_1$ is some compact set of $\mathbb{R}^3$, such that

$$\forall x \in \mathbb{K}\ \forall \xi \in \mathbb{R}^3 \setminus \mathbb{K}_1 : \quad |x-\xi| \ge b > 0. \tag{23}$$

We consider the finite function $\eta \in \mathrm{C}^\infty(\mathbb{R}^3)$, such that $\mathrm{supp}\eta \subset Q$, $\eta(x) = 1$ on $\mathbb{K}_1$ and we put $\mu = 1 - \eta$.

According to Lemma 1, for all $k \in \mathbb{Z}_+^3$ and $(x;t) \in \Pi_{(0;T]}$ we have the relation

$$\partial_x^k u(x;t) = \int_{\mathbb{R}^3} \partial_x^k G_\nu(x-\xi;t)\eta(\xi)f(\xi)d\xi + \int_{\mathbb{R}^3} \partial_x^k G_\nu(x-\xi;t)\mu(\xi)f(\xi)d\xi$$

from which, considering the equality $f = 0$, $\mu = 0$ on the sets Q and $\mathbb{K}_1$, respectively, and that

$$\mathrm{supp}\left(\partial_x^k G_\nu(x-\cdot;t)\eta(\cdot)\right) \subset Q,$$

we find

$$\partial_x^k u(x;t) = \int_{\mathbb{R}^3\setminus\mathbb{K}_1} \partial_x^k G_\nu(x-\xi;t)\mu(\xi)f(\xi)d\xi, \quad (x;t) \in \Pi_{(0;T]}.$$

Hence, using estimate (11) and taking into account the boundedness of the functions μ, f on $\mathbb{R}^3$ as well as condition (23), for $x \in \mathbb{K}$ and $0 < t << 1$ we obtain

$$\begin{aligned}\left|\partial_x^k u(x;t)\right| &\le \int_{\mathbb{R}^3\setminus\mathbb{K}_1} \left|\partial_x^k G_\nu(x-\xi;t)\right| \left|\mu(\xi)f(\xi)\right| d\xi \\ &\le c\hat{a}_\beta(t) \int_{\mathbb{R}^3\setminus\mathbb{K}_1} |x-\xi|^{-3-|k|-\nu} d\xi \le c\hat{a}_\beta(t) \int_{|z|>b} |z|^{-3-\nu} dz \equiv c_0\hat{a}_\beta(t) \underset{t\to+0}{\to} 0\end{aligned}$$

(here c_0 is a positive constant). □

From Theorem 4, considering that the initial function f is continuous on $\mathbb{R}^3$, we arrive at the following statement.

Corollary 3. *Let u be the solution of Cauchy problem* (3), (13). *Then for each compact set $\mathbb{K} \subset \mathbb{R}^3$ the following boundary relation*

$$u(x;t) \underset{t\to+0}{\overset{\mathbb{K}}{\rightrightarrows}} f(x).$$

is fulfilled.

5 Example

As an example, consider the model "Lévy ν-wandering" of the particle X in $\mathbb{R}^3$ [4, p. 2]. The probability $u(x;t)$ of finding the particle X in point $(x;t)$ is a solution of the following equation

$$\partial_t u(x;t) + A_\nu u(x;t) = 0, \quad (x;t) \in \Pi_{(0;+\infty)}. \tag{24}$$

Suppose that at the initial time $t = 0$ the probabilistic location of the particle X in $\mathbb{R}^3$ is characterized by the function

$$f(x) = (1 + x^2)^{-2}, \quad x \in \mathbb{R}^3.$$

Then the mathematical model of the "ν-wandering" of the particle X is the Cauchy problem for $\overline{\text{PD}}$E (24) with the initial condition

$$u(x;t)_{|t=0} = (1 + x^2)^{-2}, \quad x \in \mathbb{R}^3. \tag{25}$$

The function f is continuous on $\mathbb{R}^3$ and satisfies condition (17), therefore, according to Theorems 2 and 3, the only solution for Cauchy problem (24), (25) is

$$u(x;t) = \int_{\mathbb{R}^3} G_\nu(x - \xi; t) f(\xi) d\xi, \quad (x;t) \in \Pi_{(0;+\infty)},$$

where

$$G_\nu(\cdot;t) = F^{-1}\left[e^{-t|y|^\nu}\right](\cdot;t), \quad t > 0.$$

In views of Corollary 2 and

$$\sup_{x\in\mathbb{R}^3} f(x) = 1, \ \inf_{x\in\mathbb{R}^3} f(x) = 0,$$

we get the following estimates

$$0 \le u(x;t) \le 1 \qquad \forall\, (x;t) \in \Pi_{(0;+\infty)}.$$

According to condition (25), at the initial moment of time $t = 0$ the particle X with probability 1 starts its motion from point $O(0)$.

Let us investigate the probability $u(0;t)$ of X returning to its original position. Using

$$f(x) = \pi^2 \mathbb{F}^{-1}\left[e^{-|y|}\right](x), \quad x \in \mathbb{R}^3$$

and

$$\mathbb{F}^{-1}[f * g] = \mathbb{F}^{-1}[f]\mathbb{F}^{-1}[g],$$

we find

$$\begin{aligned} u(0;t) &= \pi^2 \int_{\mathbb{R}^3} \mathbb{F}^{-1}\left[e^{-t|y|^\nu}\right](\xi;t)\mathbb{F}^{-1}\left[e^{-|y|}\right](\xi)d\xi \\ &= \pi^2 \int_{\mathbb{R}^3} \mathbb{F}^{-1}\left[e^{-t|y|^\nu} * e^{-|y|}\right](\xi;t)d\xi \\ &= \pi^2 \mathbb{F}\left[\mathbb{F}^{-1}\left[e^{-t|y|^\nu} * e^{-|y|}\right]\right](0;t)d\xi \\ &= \pi^2 \int_{\mathbb{R}^3} e^{-t|y|^\nu - |y|} dy, \qquad t > 0. \end{aligned}$$

Hence it follows that

$$\forall\{t_1, t_2\} \subset (0; +\infty),\ t_1 < t_2: \quad 1 = u(0;0) > u(0;t_1) > u(0;t_2)$$

and

$$\lim_{t\to+\infty} u(0;t) = 0.$$

This means that over time, the probability of the return of the particle X to its original position tends to zero.

It turns out that for $t \to +\infty$ the particle X leaves the space $\mathbb{R}^3$ altogether, since

$$\lim_{t\to+\infty} u(x;t) = 0 \qquad \forall x \in \mathbb{R}^3. \tag{26}$$

Let us prove boundary relation (26). Taking into account that equation (24) is a special case of (3) for $a_\beta(t) = t + a$, $a \geq 0$, from (11) we obtain the estimates

$$G_\nu(x;t) \leq \frac{c_1 t}{\left(t^{1/\nu} + |x|\right)^{3+\nu}}, \quad t > 0,\ x \in \mathbb{R}^3.$$

Hence, we have

$$u(x;t) \leq c \int_{\mathbb{R}^3} \frac{t d\xi}{\left(1 + |x-\xi|^2\right)^2 \left(t^{1/\nu} + |\xi|\right)^{3+\nu}} \leq \frac{c}{t^{3/\nu}} \int_{\mathbb{R}^3} \frac{dz}{\left(1+|z|^2\right)^2} \equiv \frac{c_0}{t^{3/\nu}} \underset{t\to+\infty}{\to} 0, \quad x \in \mathbb{R}^3.$$

In conclusion, we note the following. If we assume that the particle X is a hungry shark, then the considered model (24), (25) turns into the model "About a yawing shark in search of a prey". This problem becomes more natural in the context of the problem of the local influence of moving objects in the corresponding Riesz gravitational field generated by the force F of the predator's gravitation to the prey, which obeys the law (1) for $\nu = 3/\beta$.

6 Conclusions

In this research the important estimates of the derivatives of the nonstationary probability distribution Polya W_β for the force F of the local influence of moving objects in the Riesz gravitational field are found. The problem case of studying the Cauchy problem for the corresponding $\overline{\text{PDE}}$ with the Riesz operator of fractional differentiation is considered. The correct solvability of this problem in the class of bounded continuous initial functions is determined.

The obtained results are important for further studies of P. Lévy symmetric stable random processes, the Riesz gravitational fields in particular. The estimates of the derivatives of the W_β function found here reveal wide possibilities for studying these processes in areas with boundary conditions by means of the theory of boundary value problems for $\overline{\text{PDE}}$s with point-nonsmooth symbols.

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Розглядається псевдодиференціальне рівняння параболічного типу з дробовим степенем оператора Лапласа порядку $\alpha \in (0;1)$, що діє за просторовою змінною. Це рівняння природньо узагальнює відоме рівняння фрактальної дифузії. Воно описує локальний вплив рухомих об'єктів у гарвітаційному полі Рісса. Простішим прикладом такої системи об'єктів є зоряні галактики, в яких взаємодія відбувається згідно з гравітаційним законом Ньютона. Для цього рівняння розв'язується задача Коші в класі неперервних обмежених початкових функцій. Фундаментальний розв'язок цієї задачі є розподілом Пойа $\mathcal{P}_\alpha(F)$ ймовірностей для сили F локальної взаємодії між цими об'єктами. Одержано оцінки похідних цього розв'язку, за допомогою яких встановлено коректну розв'язність задачі Коші за певних умов на коефіцієнт локальної флуктуації гравітаційного поля. При цьому знайдено форму класичного розв'язку цієї задачі та досліджено властивості його гладкості й поведінку на нескінченності. Також з'ясовано можливість локального посилення збіжності в початковій умові. Одержані результати проілюстровано на моделі α-блукання частинки Леві в евклідовому просторі $\mathbb{R}^3$ у випадку, коли частинка починає свій рух з початку координат. Досліджено ймовірність повернення цієї частинки у своє вихідне положення. Зокрема, встановлено, що ця ймовірність є спадною функцією, яка з плином часу прямує до нуля, а сама частинка "покидає" простір $\mathbb{R}^3$.

Ключові слова і фрази: гравітаційне поле, потенціал Рісса, розподіл Пойя, симетричний стійкий випадковий процес Леві, політ Леві, рівняння фрактальної дифузії, дробовий лапласіан, фундаментальний розв'язок, задача Коші.