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Coefficient inverse problem for the strongly degenerate parabolic equation

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The coefficient inverse problem for the degenerate parabolic equation is investigated. The minor coefficient of this equation is the polynomial of the first power with respect to the space variable with two unknown time-dependent functions. The investigation is carried out under given inhomogeneous initial condition, Dirichlet boundary conditions and integral overdetermination conditions. We establish the conditions of the unique solvability to the named problem for the case of strong degeneration.

Key words and phrases: inverse problem, minor coefficient, parabolic equation, strong degeneration.

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Introduction

In this paper, we consider the coefficient inverse problem for the parabolic equation with strong power degeneration.

Those problems are called inverse in which knowing consequences we need to find the reasons causing them. These problems are widely studied within the past decades (see [2, 10, 21]) due to its applications in medicine, geophysics, tomography, acoustics, ecology, financial mathematics, electrodynamics, etc. Various statements of coefficient inverse problems for parabolic equation are investigated in [1, 4, 7, 9, 11–13, 18, 24–27, 31]. In these papers, the authors studied both the inverse problems of recovering of the time-dependent major coefficients in the parabolic equations without degenerations and the minor coefficients or source terms in them. They also differ in boundary and overdetermination conditions. The conditions of existence and uniqueness of the solutions are established in these works.

Degenerate parabolic problems arise in the mathematical models of the flow in the porous media, climate models, population genetics, propagation of the thermal waves in plasma, financial mathematics and others [3, 6, 14, 22]. Despite their fundamental importance and practical application, the literature on inverse degenerate problems for parabolic equations is rather recent and scarce. The conditions of unique solvability to the inverse problems of recovering of the time-dependent major coefficient $a = a(t), a(t) > 0, t \in [0, T]$ (thermal diffusivity) in the one-dimensional degenerate parabolic equation

$$w_t = a(t)t^\beta w_{xx} + b(x,t)w_x + c(x,t)w + f(x,t)$$

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are established in [20, 30] for both cases of weak ($0 < \beta < 1$) and strong ($\beta \geq 1$) degeneration. The inverse problems of identification of the time-dependent minor coefficient $b = b(t)$ (convection coefficient) in the degenerate parabolic equation

$$w_t = a(t)t^\beta w_{xx} + b(t)w_x + c(x,t)w + f(x,t)$$

are studied in [15, 16] for both cases of degenerations. In these papers, the unknown coefficients of the equations depend on time and the degeneration of the equation is caused by the power function with respect to time variable. The coefficient inverse problems for the parabolic equations with degeneration caused by the function with respect to space variable are studied in [5, 23, 28, 29] and to time variable – in [8, 14] (see also the bibliography in them).

The inverse problems for identification of the coefficients which depend simultaneously on both spatial and time variables remain unexplored in spite of their importance. In the present paper, we consider a coefficient inverse problem for the one-dimensional degenerate parabolic equation. It is known that the major coefficient of the equation is the product of the power function t^β which caused degeneration and a known positive time-dependent function. The minor convection coefficient of the equation is the polynomial of the first power with respect to the space variable with two unknown functions which depend on time variable. We investigate this inverse problem under given nonhomogeneous initial condition, the Dirichlet boundary conditions and the values of the heat moments as the overdetermination conditions. The case of strong degeneration is studied.

1 Statement of the problem and the main result

In a rectangle $Q_T = \{(x,t) : 0 < x < l, 0 < t < T\}$, we consider an inverse problem of identification of the time-dependent functions $b_1 = b_1(t),\ \ b_2 = b_2(t)$ in a minor coefficient of the parabolic equation

$$v_t = a(t)t^\beta v_{xx} + (b_1(t)x + b_2(t))v_x + c(x,t)v + f(x,t) \tag{1}$$

with initial condition

$$v(x,0) = \varphi(x), \quad x \in [0,l], \tag{2}$$

boundary conditions

$$v(0,t) = \mu_1(t),\ v(l,t) = \mu_2(t),\ t \in [0,T] \tag{3}$$

and integral overdetermination conditions

$$\int_0^l v(x,t)dx = \mu_3(t), \quad t \in [0,T], \tag{4}$$

$$\int_0^l xv(x,t)dx = \mu_4(t), \quad t \in [0,T]. \tag{5}$$

It is known that $a(t) > 0, t \in [0,T]$, and a degeneration of the equation (1) is caused by the power function t^β. The case of weak degeneration is studied in [17]. In this paper, the case of strong degeneration ($\beta \geq 1$) is investigated. Our purpose is to find the conditions of existence and uniqueness of the solution to this inverse problem. The main result of this work is contained in the following theorem.

Theorem 1. *Suppose that the following conditions hold:*

A1) $\varphi \in C^2[0,l]$, $\mu_i \in C^1[0,T]$, $i \in \overline{1,4}$, $a \in C[0,T]$, $a(t) > 0$, $t \in [0,T]$, *and functions* $c, f \in C(\overline{Q}_T)$ *satisfy the Hölder condition with respect to* x *uniformly to* t *with indicator* α, $0 < \alpha < 1$;

A2) $\big(l\mu_2(t) - \mu_3(t)\big)^2 - \big(\mu_2(t) - \mu_1(t)\big)\big(l^2\mu_2(t) - 2\mu_4(t)\big) \neq 0$, $t \in [0,T]$;

A3) $|f(x,t)| \leq A_1 t^\gamma$, $|c(x,t)| \leq A_2 t^\gamma$, $(x,t) \in \overline{Q}_T$, $|\mu_3'(t)| \leq A_3 t^\gamma$, $|\mu_4'(t)| \leq A_4 t^\gamma$, $t \in [0,T]$, *where* A_i, $i = 1,2,3,4$, *are some positive known constants and* $\gamma > \frac{\beta-1}{2}$ *is an arbitrary fixed number;*

A4) $\mu_1(0) = \varphi(0)$, $\mu_2(0) = \varphi(l)$, $\int_0^l \varphi(x)dx = \mu_3(0)$, $\int_0^l x\varphi(x)dx = \mu_4(0)$.

Then there exist T_0, $0 < T_0 \leq T$, *and a unique solution* $(b_1, b_2, v) \in \big(C[0,T_0]\big)^2 \times C^{2,1}(Q_{T_0}) \cap C(\overline{Q}_{T_0})$, $|b_1(t)| \leq M_1 t^\delta$, $|b_2(t)| \leq M_2 t^\delta$ *with* $\delta = \min\left\{\gamma, \frac{\beta+1}{2}\right\}$ *to the problem* (1) – (5), *where the numbers* T_0 *and* $M_1 > 0, M_2 > 0$ *are defined by the problem data.*

To prove the existence of the solution to the problem (1) – (5) we use the apparatus of Green functions for the heat equation and Schauder fixed point theorem. The uniqueness of the solution is based on the properties of the solutions to the homogeneous integral equations with integrable kernels.

2 Existence of the solution

First, let us replace the inverse problem (1) – (5) by a system of equations that is equivalent to it. For this purpose we make in the named problem the substitution

$$v(x,t) = \widetilde{v}(x,t) + v_0(x,t). \tag{6}$$

The function $v_0(x,t)$ in (6) is satisfied given initial and boundary conditions (2), (3). Taking into account the compatibility conditions *A4)*, it is easy to verify that this function is equal to

$$v_0(x,t) = \varphi(x) - \varphi(0) + \mu_1(t) + \frac{x}{l}\Big(\mu_2(t) - \mu_1(t) - \mu_2(0) + \mu_1(0)\Big). \tag{7}$$

The substitution (6) yields for the function $\widetilde{v} = \widetilde{v}(x,t)$ nonhomogeneous equation

$$\begin{aligned}\widetilde{v}_t = a(t)t^\beta \widetilde{v}_{xx} + \big(b_1(t)x + b_2(t)\big)\widetilde{v}_x + c(x,t)\widetilde{v} + f(x,t) - \mu_1'(t) - \frac{x}{l}\big(\mu_2'(t) - \mu_1'(t)\big)\\ + a(t)t^\beta\varphi''(x) + \big(b_1(t)x + b_2(t)\big)\Big(\varphi'(x) + \frac{1}{l}\big(\mu_2(t) - \mu_1(t) - \mu_2(0) + \mu_1(0)\big)\Big)\\ + c(x,t)\Big(\varphi(x) - \varphi(0) + \mu_1(t) + \frac{x}{l}\big(\mu_2(t) - \mu_1(t) - \mu_2(0) + \mu_1(0)\big)\Big)\end{aligned} \tag{8}$$

with homogeneous initial and boundary conditions

$$\widetilde{v}(x,0) = 0, \quad x \in [0,l], \tag{9}$$

$$\widetilde{v}(0,t) = \widetilde{v}(l,t) = 0, \quad t \in [0,T]. \tag{10}$$

Now we replace the problem (8) – (10) by the following equivalent integro-differential equation (see [19, p. 49]):

$$\widetilde{v}(x,t) = \int_0^t \int_0^l G_1(x,t,\eta,\tau)\Big(\big(b_1(\tau)\eta + b_2(\tau)\big)\widetilde{v}_\eta(\eta,\tau) + c(\eta,\tau)\widetilde{v}(\eta,\tau) + a(\tau)\tau^\beta \varphi''(\eta) - \mu_1'(\tau)$$
$$- \frac{\eta}{l}\big(\mu_2'(\tau) - \mu_1'(\tau)\big) + \big(b_1(\tau)\eta + b_2(\tau)\big)\Big(\varphi'(\eta) + \frac{1}{l}\big(\mu_2(\tau) - \mu_1(\tau) - \mu_2(0) + \mu_1(0)\big)\Big)$$
$$+ c(\eta,\tau)\Big(\varphi(\eta) - \varphi(0) + \mu_1(\tau) + \frac{\eta}{l}\big(\mu_2(\tau) - \mu_1(\tau) - \mu_2(0) + \mu_1(0)\big)\Big) + f(\eta,\tau)\Big)\, d\eta\, d\tau. \tag{11}$$

For this purpose we use the Green function $G_1 = G_1(x,t,\eta,\tau)$ for the first boundary value problem for the heat equation

$$w_t = a(t)t^\beta w_{xx}. \tag{12}$$

It is known [19, p. 12] that Green functions of the first ($k = 1$) or the second ($k = 2$) boundary value problem for the equation (12) can be written in the form

$$G_k(x,t,\eta,\tau) = \frac{1}{2\sqrt{\pi(\theta(t) - \theta(\tau))}} \times \sum_{n=-\infty}^{+\infty} \left(\exp\left(-\frac{(x - \eta + 2nl)^2}{4\big(\theta(t) - \theta(\tau)\big)} \right) + (-1)^k \exp\left(-\frac{(x + \eta + 2nl)^2}{4\big(\theta(t) - \theta(\tau)\big)} \right) \right), \tag{13}$$

where $k = 1, 2$, $0 \le x, \eta \le l$, $0 \le \tau < t \le T$, $\theta(t) = \int_0^t a(\tau)\tau^\beta d\tau$. These functions possess the properties

$$\int_0^l \big|G_k(x,t,\eta,\tau)\big| d\eta d\tau \le 1, \quad \int_0^h \big|G_{kx}(x,t,\eta,\tau)\big|\, d\eta \le \frac{C_1}{\sqrt{\theta(t) - \theta(\tau)}}, \quad k = 1, 2, \tag{14}$$

where C_1 is known positive constant.

Let us consider the behavior of the integral $I \equiv \int\limits_0^t \frac{d\tau}{\sqrt{\theta(t) - \theta(\tau)}}$ as $t \to 0$. Using the definition of the function $\theta = \theta(t)$, we deduce

$$I = \int_0^t \frac{d\tau}{\sqrt{\int_\tau^t a(\sigma)\sigma^\beta d\sigma}} \le \sqrt{\frac{1+\beta}{A_0}} \int_0^t \frac{d\tau}{\sqrt{t^{1+\beta} - \tau^{1+\beta}}},$$

where $A_0 \equiv \min\limits_{[0,T]} a(t)$. By the substitution $z = \frac{\tau}{t}$ we obtain

$$I \le \sqrt{\frac{1+\beta}{A_0}} t^{\frac{1-\beta}{2}} \int_0^1 \frac{dz}{\sqrt{1 - z^{1+\beta}}} \le \sqrt{\frac{1+\beta}{A_0}} t^{\frac{1-\beta}{2}} \int_0^1 \frac{dz}{\sqrt{1 - z}} \le C_2 t^{\frac{1-\beta}{2}}. \tag{15}$$

Put $u(x,t) \equiv v_x(x,t)$. Taking into account (6), (11), we replace the direct problem (1) – (3) by the system of integral equations for unknowns $v = v(x,t), u = u(x,t)$:

$$v(x,t) = \int_0^t \int_0^l G_1(x,t,\eta,\tau)\Big(\Big(b_1(\tau)\eta + b_2(\tau)\Big)u(\eta,\tau) + c(\eta,\tau)v(\eta,\tau) - \mu_1'(\tau)$$
$$- \frac{\eta}{l}\Big(\mu_2'(\tau) - \mu_1'(\tau)\Big) + a(\tau)\tau^\beta\varphi''(\eta) + f(\eta,\tau)\Big)\, d\eta\, d\tau + v_0(x,t), \quad (x,t) \in \overline{Q}_T, \tag{16}$$

$$u(x,t) = \int_0^t \int_0^l G_{1x}(x,t,\eta,\tau)\Big(\Big(b_1(\tau)\eta + b_2(\tau)\Big)u(\eta,\tau) + c(\eta,\tau)v(\eta,\tau) - \mu_1'(\tau) - \frac{\eta}{l}\Big(\mu_2'(\tau) - \mu_1'(\tau)\Big) + a(\tau)\tau^\beta\varphi''(\eta) + f(\eta,\tau)\Big)\, d\eta\, d\tau + v_{0x}(x,t), \quad (x,t) \in Q_T. \tag{17}$$

Note, that we obtain the equation (17) differentiating (16) with respect to the space variable.

To find the equations for the functions $b_1 = b_1(t), b_2 = b_2(t)$ we multiply (1) by x^k, $k = 0,1$, and integrate it with respect to x from 0 to l:

$$\begin{aligned} b_1(t) = \triangle^{-1}(t)\Bigg(&\Big(\mu_3'(t) - a(t)t^\beta\big(u(l,t) - u(0,t)\big) \\ &- \int_0^l \big(c(x,t)v(x,t) + f(x,t)\big)dx\Big)\Big(l\mu_2(t) - \mu_3(t)\Big) \\ &-\Big(\mu_4'(t) - a(t)t^\beta\big(lu(l,t) - \mu_2(t) + \mu_1(t)\big) \\ &- \int_0^l x\big(c(x,t)v(x,t) + f(x,t)\big)dx\Big)\Big(\mu_2(t) - \mu_1(t)\Big)\Bigg), \ t \in [0,T], \end{aligned} \tag{18}$$

$$\begin{aligned} b_2(t) = \triangle^{-1}(t)\Bigg(&\Big(\mu_4'(t) - a(t)t^\beta\big(lu(l,t) - \mu_2(t) + \mu_1(t)\big) \\ &- \int_0^l x\big(c(x,t)v(x,t) + f(x,t)\big)dx\Big)\Big(l\mu_2(t) - \mu_3(t)\Big) \\ &-\Big(\mu_3'(t) - a(t)t^\beta\big(u(l,t) - u(0,t)\big) \\ &- \int_0^l \big(c(x,t)v(x,t) + f(x,t)\big)dx\Big)\Big(l^2\mu_2(t) - 2\mu_4(t)\Big)\Bigg), \ t \in [0,T]. \end{aligned} \tag{19}$$

Note that the expression

$$\triangle(t) = \Big(l\mu_2(t) - \mu_3(t)\Big)^2 - \Big(\mu_2(t) - \mu_1(t)\Big)\Big(l^2\mu_2(t) - 2\mu_4(t)\Big) \tag{20}$$

never becomes zero according to the condition *A2)* of the Theorem.

Now we denote $U(t) = \max\limits_{(x,\tau)\in[0,l]\times[0,t]} |u(x,\tau)|$, $V(t) = \max\limits_{(x,\tau)\in[0,l]\times[0,t]} |v(x,\tau)|$, $t \in [0,T]$. Using the condition *A3)* of the Theorem and (14), (15) we derive from (16) – (19) respectively

$$V(t) \le C_3 + C_4\int_0^t \Big(\Big(|b_1(\tau)| + |b_2(\tau)|\Big)U(\tau) + \tau^\gamma V(\tau)\Big)d\tau, \quad t \in [0,T], \tag{21}$$

$$U(t) \le \frac{C_5}{t^{\frac{\beta-1}{2}}} + C_6\int_0^t \frac{\Big(|b_1(\tau)| + |b_2(\tau)|\Big)U(\tau) + \tau^\gamma V(\tau)}{\sqrt{t^{\beta+1} - \tau^{\beta+1}}}d\tau, \quad t \in (0,T], \tag{22}$$

$$|b_1(t)| \le C_7 t^\gamma + C_8 t^\beta U(t) + C_9 t^\gamma V(t), \quad t \in [0,T], \tag{23}$$

$$|b_2(t)| \le C_{10} t^\gamma + C_{11} t^\beta U(t) + C_{12} t^\gamma V(t), \quad t \in [0,T]. \tag{24}$$

We conclude from (21) – (24) that the function $v = v(x,t)$ is continuous in $\overline{Q}_T$, the function $u = u(x,t)$ has the behavior $t^{\frac{1-\beta}{2}}$ as $t \to 0$ accordingly to (15) and the functions $b_1(t), b_2(t)$ tend to zero as $t \to 0$ like a power function t^δ, where $\delta = \min\{\gamma, \frac{\beta+1}{2}\}$.

Thus, the problem (1) – (5) is reduced to the equivalent system of equations (16) – (19). The term "equivalence" means that: if the triplet of functions (b_1, b_2, v) is a global solution to the problem (1) – (5), that the functions $(v, u, b_1, b_2) \in C(\overline{Q}_T) \times C\big([0,l] \times (0,T]\big) \times \big(C[0,T]\big)^2$, $|b_1(t)| \leq M_1 t^\delta$, $|b_2(t)| \leq M_2 t^\delta$, $t \in [0,T]$, satisfy (16) – (19). And conversely, if a triplet $(v, u, b_1, b_2) \in C(\overline{Q}_T) \times C\big([0,l] \times (0,T]\big) \times \big(C[0,T]\big)^2$ is a solution to the system of equations (16) – (19), then (b_1, b_2, v) belongs to $\big(C[0,T]\big)^2 \times C^{2,1}(Q_T) \cap C(\overline{Q}_T)$, verifies (1) – (5) and satisfies the estimates $|b_1(t)| \leq M_1 t^\delta$, $|b_2(t)| \leq M_2 t^\delta$, $t \in [0,T]$.

The fact that the first part of the statement is true follows from how the system of equations (16) – (19) is obtained. Now we prove the contrary statement. For this aim we suppose that $(v, u, b_1, b_2) \in C(\overline{Q}_T) \times C\big([0,l] \times (0,T]\big) \times \big(C[0,T]\big)^2$ is the solution to the system of equation (16) – (19). The condition *A1)* of the Theorem allows us to differentiate the equation (16) with respect to x. We get

$$v_x(x,t) = \int_0^t \int_0^l G_{1x}(x,t,\eta,\tau)\bigg(\Big(b_1(\tau)\eta + b_2(\tau)\Big)u(\eta,\tau) + c(\eta,\tau)v(\eta,\tau) - \mu_1'(\tau) \\ - \frac{\eta}{l}\Big(\mu_2'(\tau) - \mu_1'(\tau)\Big) + a(\tau)\tau^\beta \varphi''(\eta) + f(\eta,\tau)\bigg) d\eta d\tau + v_{0x}(x,t).$$

The right-hand sides of this equality and (17) coincide, so $u(x,t) \equiv v_x(x,t)$, $(x,t) \in Q_T$. Furthermore, taking into account the behaviors of the functions $b_1 = b_1(t), b_2 = b_2(t)$, $u = u(x,t)$, we deduce that $b_i(t)u(x,t)$, $i = 1,2$, are continuous in $\overline{Q}_T$. Let us look at the equation (16) as an integro-differential one with respect to the function $v = v(x,t)$. Now we can state that the function v belongs to $C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ and satisfies (1) – (3) (see [19, p. 49]).

Then we multiply the equations (18) and (19) by $l\mu_2(t) - \mu_3(t)$, and $\mu_2(t) - \mu_1(t)$ respectively. Adding obtained equalities we get

$$b_1(t)\big(l\mu_2(t) - \mu_3(t)\big) + b_2(t)\big(\mu_2(t) - \mu_1(t)\big) = \mu_3'(t) - a(t)t^\beta\big(v_x(h,t) - v_x(0,t)\big) \\ - \int_0^h \big(c(x,t)v(x,t) + f(x,t)\big)dx.$$

Taking into account (1) – (3), this expression we rewrite in the form

$$b_1(t)\bigg(\int_0^l v(x,t)dx - \mu_3(t)\bigg) = -\bigg(\int_0^l v_t(x,t)dx - \mu_3'(t)\bigg).$$

Put $z(t) \equiv \int_0^l v(x,t)dx - \mu_3(t)$. Then $z'(t) = -b_1(t)z(t)$, and by integration we obtain $z(t) = z(0)e^{-\int_0^t b_1(\tau)d\tau}$. Taking into account that $z(0) = 0$, we conclude $z(t) \equiv 0$. This means that the condition (4) holds.

Similarly, multiplying the equations (18) and (19) by $l^2\mu_2(t) - 2\mu_4(t)$ and $l\mu_2(t) - \mu_3(t)$ respectively, and adding them we deduce

$$b_1(t)\big(l^2\mu_2(t) - 2\mu_4(t)\big) + b_2(t)\big(l\mu_2(t) - \mu_3(t)\big) = \mu_4'(t) - a(t)t^\beta\big(lv_x(h,t) - \mu_2(t) + \mu_1(t)\big) \\ - \int_0^l x\big(c(x,t)v(x,t) + f(x,t)\big)dx,$$

and

$$2b_2(t)\bigg(\int_0^l xv(x,t)dx - \mu_4(t)\bigg) = -\bigg(\int_0^l xv_t(x,t)dx - \mu_4'(t)\bigg).$$

Then (5) follows from (1) – (3) and compatibility condition *A4)*. We show that the inverse problem (1) – (5) and the system of equations (16) – (19) are equivalent.

Denote $p_1(t) = b_1(t)t^{-\delta}, p_2(t) = b_2(t)t^{-\delta}, w(x,t) = t^{\frac{\beta-1}{2}}u(x,t)$. The system of equations (16) – (19) we rewrite in the form

$$v(x,t) = \int_0^t \int_0^l G_1(x,t,\eta,\tau)\Big(\Big(p_1(\tau)\eta + p_2(\tau)\Big)\tau^{\delta-\frac{\beta-1}{2}}w(\eta,\tau) + c(\eta,\tau)v(\eta,\tau) - \mu_1'(\tau) - \frac{\eta}{l}\Big(\mu_2'(\tau) - \mu_1'(\tau)\Big) + a(\tau)\tau^\beta\varphi''(\eta) + f(\eta,\tau)\Big)\, d\eta\, d\tau + v_0(x,t), \quad (x,t) \in \overline{Q}_T, \tag{25}$$

$$w(x,t) = t^{\frac{\beta-1}{2}} \int_0^t \int_0^l G_{1x}(x,t,\eta,\tau)\Big(\Big(p_1(\tau)\eta + p_2(\tau)\Big)\tau^{\delta-\frac{\beta-1}{2}}w(\eta,\tau) + c(\eta,\tau)v(\eta,\tau) - \mu_1'(\tau) - \frac{\eta}{l}\Big(\mu_2'(\tau) - \mu_1'(\tau)\Big) + a(\tau)\tau^\beta\varphi''(\eta) + f(\eta,\tau)\Big)\, d\eta\, d\tau + t^{\frac{\beta-1}{2}}v_{0x}(x,t), \quad (x,t) \in \overline{Q}_T, \tag{26}$$

$$p_1(t) = \triangle^{-1}(t)t^{-\delta}\Bigg(\Big(\mu_3'(t) - a(t)t^{\frac{\beta+1}{2}}(w(l,t) - w(0,t)) - \int_0^l (c(x,t)v(x,t) + f(x,t))dx\Big)\Big(l\mu_2(t) - \mu_3(t)\Big) - \Big(\mu_4'(t) - a(t)t^\beta\Big(lt^{\frac{1-\beta}{2}}w(l,t) - \mu_2(t) + \mu_1(t)\Big) - \int_0^l x\big(c(x,t)v(x,t) + f(x,t)\big)dx\Big)\Big(\mu_2(t) - \mu_1(t)\Big)\Bigg), \quad t \in [0,T], \tag{27}$$

$$p_2(t) = \triangle^{-1}(t)t^{-\delta}\Bigg(\Big(\mu_4'(t) - a(t)t^\beta(lt^{\frac{1-\beta}{2}}w(l,t) - \mu_2(t) + \mu_1(t)) - \int_0^l x\big(c(x,t)v(x,t) + f(x,t)\big)dx\Big)\Big(l\mu_2(t) - \mu_3(t)\Big) - \Big(\mu_3'(t) - a(t)t^{\frac{\beta+1}{2}}\Big(w(l,t) - w(0,t)\Big) - \int_0^l (c(x,t)v(x,t) + f(x,t))dx\Big)\Big(l^2\mu_2(t) - 2\mu_4(t)\Big)\Bigg), \quad t \in [0,T]. \tag{28}$$

Now we start studying the system of equations (25) – (28). We consider this system as an operator equation

$$\omega = P\omega, \tag{29}$$

where $\omega = (v, w, p_1, p_2)$ and the operator $P = (P_1, P_2, P_3, P_4)$ is defined by the right-hand sides of the equations (25) – (28).

Assume that $|v(x,t)| \le M_3, |w(x,t)| \le M_4$, $(x,t) \in \overline{Q}_T$, where M_3, M_4 are some positive constants. We define the values of these constants below. Taking into account these estimates and definition of δ in (27), (28) we find

$$\begin{aligned} |P_3\omega| &\le \frac{C_{13}\left(t^{\gamma-\delta}+t^{\frac{\beta+1}{2}-\delta}M_4+t^{\gamma-\delta}M_3\right)}{\min\limits_{t\in[0,T]}|\triangle(t)|} \\ &\le \frac{C_{13}}{\min\limits_{t\in[0,T]}|\triangle(t)|}\max\left\{\left(1+T^{\frac{\beta+1}{2}-\delta}M_4+M_3\right),\left(T^{\gamma-\delta}+M_4+T^{\gamma-\delta}M_3\right)\right\}:=M_1, \end{aligned} \tag{30}$$

$$\begin{aligned} |P_4\omega| &\le \frac{C_{14}\left(t^{\gamma-\delta}+t^{\frac{\beta+1}{2}-\delta}M_4+t^{\gamma-\delta}M_3\right)}{\min\limits_{t\in[0,T]}|\triangle(t)|} \\ &\le \frac{C_{14}}{\min\limits_{t\in[0,T]}|\triangle(t)|}\max\left\{\left(1+T^{\frac{\beta+1}{2}-\delta}M_4+M_3\right),\left(T^{\gamma-\delta}+M_4+T^{\gamma-\delta}M_3\right)\right\}:=M_2, \end{aligned} \tag{31}$$

where $t\in[0,T]$ and the constants C_{13}, C_{14} depend on the problem data. Let us consider the equations (25), (26). Using (30), (31), we derive

$$\begin{aligned} |P_1\omega| \le \bigg| \int_0^t\int_0^l & G_1(x,t,\eta,\tau)\bigg(\left(M_1 l+M_2\right)t^{\delta-\frac{\beta-1}{2}}M_4+\max_{(\eta,\tau)\in\overline{Q}_T}\left|c(\eta,\tau)\right|M_3 \\ &+\max_{(\eta,\tau)\in\overline{Q}_T}\left|f(\eta,\tau)-\mu_1'(\tau)-\frac{\eta}{l}\left(\mu_2'(\tau)-\mu_1'(\tau)\right)+a(\tau)\tau^\beta\varphi''(\eta)\right|\bigg)\,d\eta\,d\tau\bigg| \\ &+\max_{(x,t)\in\overline{Q}_T}\left|v_0(x,t)\right|\le C_{15}t^{\delta-\frac{\beta-3}{2}}+C_{16}t^{\gamma+1}+C_{17}t+\max_{(x,t)\in\overline{Q}_T}\left|v_0(x,t)\right|, \end{aligned} \tag{32}$$

$$\begin{aligned} |P_2\omega| \le \bigg| t^{\frac{\beta-1}{2}}\int_0^t\int_0^l & G_{1x}(x,t,\eta,\tau)\bigg(\left(M_1 l+M_2\right)t^{\delta-\frac{\beta-1}{2}}M_4+\max_{(\eta,\tau)\in\overline{Q}_T}\left|c(\eta,\tau)\right|M_3 \\ &+\max_{(\eta,\tau)\in\overline{Q}_T}\left|f(\eta,\tau)-\mu_1'(\tau)-\frac{\eta}{l}\left(\mu_2'(\tau)-\mu_1'(\tau)\right)+a(\tau)\tau^\beta\varphi''(\eta)\right|\bigg)\,d\eta\,d\tau\bigg| \\ &+\max_{(x,t)\in\overline{Q}_T}\left|t^{\frac{\beta-1}{2}}v_{0x}(x,t)\right|\le C_{18}t^{\eta-\frac{\beta-1}{2}}+C_{19}t^{\gamma}+C_{20}+\max_{(x,t)\in\overline{Q}_T}\left|t^{\frac{\beta-1}{2}}v_{0x}(x,t)\right|. \end{aligned} \tag{33}$$

Now we choose the constants M_3, M_4 such that

$$M_3>\max_{(x,t)\in\overline{Q}_T}\left|v_0(x,t)\right|,\quad M_4>C_{20}+\max_{(x,t)\in\overline{Q}_T}\left|t^{\frac{\beta-1}{2}}v_{0x}(x,t)\right|.$$

Then fix number T_0, $0<T_0\le T$, such that

$$C_{15}T_0^{\delta-\frac{\beta-3}{2}}+C_{16}T_0^{\gamma+1}+C_{17}T_0+\max_{(x,t)\in\overline{Q}_T}\left|v_0(x,t)\right|\le M_3, \tag{34}$$

$$C_{18}T_0^{\delta-\frac{\beta-1}{2}}+C_{19}T_0^{\gamma}+C_{20}+\max_{(x,t)\in\overline{Q}_T}\left|t^{\frac{\beta-1}{2}}v_{0x}(x,t)\right|\le M_4. \tag{35}$$

As a result we find

$$|P_1\omega|\le M_3,\quad |P_2\omega|\le M_4,\quad (x,t)\in\overline{Q}_{T_0}. \tag{36}$$

Note that the estimates (34), (35) are the basis for choosing the number T_0 and show its dependence on the input data of the problem.

We consider the operator equation (29) on the closed and convex set

$$\mathcal{N} \equiv \Big\{ (v, w, p_1, p_2) \in \big(C(\overline{Q}_{T_0})\big)^2 \times \big(C[0, T_0]\big)^2 : |v(x,t)| \le M_3, |w(x,t)| \le M_4, \\ |p_1(t)| \le M_1, |p_2(t)| \le M_2 \Big\}$$

in a Banach space $\mathcal{B} \equiv \big(C(\overline{Q}_{T_0})\big)^2 \times \big(C[0, T_0]\big)^2$. The estimates (30), (31), (36) guarantee that P maps $\mathcal{N}$ into $\mathcal{N}$. To prove the compactness of the operator P we have to show that the set $P\mathcal{N}$, by Arzela-Ascoli theorem, is uniformly bounded and equicontinuous. It means that for any ϵ there exists ϱ such that $\Big|P\omega(x_2, t_2) - P\omega(x_1, t_1)\Big| < \epsilon$ for arbitrary $|x_2 - x_1| < \varrho$, $|t_2 - t_1| < \varrho$, $\omega(x, t) \in \mathcal{N}$.

Consider the operator

$$P_2\omega = t^{\frac{\beta-1}{2}} \int_0^t \int_0^l G_{1x}(x, t, \eta, \tau) \bigg(\Big(p_1(\tau)\eta + p_2(\tau) \Big) \tau^{\delta - \frac{\beta-1}{2}} w(\eta, \tau) + c(\eta, \tau) v(\eta, \tau) - \mu_1'(\tau) \\ - \frac{\eta}{l} \Big(\mu_2'(\tau) - \mu_1'(\tau) \Big) + a(\tau)\tau^\beta \varphi''(\eta) + f(\eta, \tau) \bigg) \, d\eta \, d\tau + t^{\frac{\beta-1}{2}} v_{0x}(x, t) \\ \equiv t^{\frac{\beta-1}{2}} \int_0^t \int_0^l G_{1x}(x, t, \eta, \tau) z(\eta, \tau) \, d\eta \, d\tau + t^{\frac{\beta-1}{2}} v_{0x}(x, t).$$

It follows from (36) that the set $P_2\mathcal{N}$ is uniformly bounded. We prove that it is also equicontinuous. For this aim fix an arbitrary $\epsilon > 0$ and consider the difference

$$\Big| P_2\omega(x_1, t_1) - P_2\omega(x_2, t_2) \Big| \\ \le \left| t_1^{\frac{\beta-1}{2}} \int_0^{t_1} \int_0^l G_{1x}(x_1, t_1, \eta, \tau) z(\eta, \tau) \, d\eta \, d\tau - t_2^{\frac{\beta-1}{2}} \int_0^{t_2} \int_0^l G_{1x}(x_2, t_2, \eta, \tau) z(\eta, \tau) \, d\eta \, d\tau \right| \\ + t_1^{\frac{\beta-1}{2}} \Big| v_{0x}(x_1, t_1) - v_{0x}(x_2, t_2) \Big| + \Big| t_1^{\frac{\beta-1}{2}} - t_2^{\frac{\beta-1}{2}} \Big| \Big| v_{0x}(x_2, t_2) \Big| \equiv I_1 + I_2 + I_3.$$

Taking into account the continuity of the input data we can state that $I_2 + I_3 < \frac{2\epsilon}{3}$, when $|x_2 - x_1| < \varrho$, $|t_2 - t_1| < \varrho$.

Since $\lim\limits_{t\to 0} \int_0^t \int_0^l t^{\frac{\beta-1}{2}} G_{1x}(x, t, \eta, \tau) z(\eta, \tau) \, d\eta \, d\tau = \kappa_1$, then

$$\left| \int_0^t \int_0^l t^{\frac{\beta-1}{2}} G_{1x}(x, t, \eta, \tau) z(\eta, \tau) \, d\eta \, d\tau - \kappa_1 \right| < \frac{\epsilon}{15}, \quad \text{when} \quad t < \delta_1, \ \ x \in [0, l]. \tag{37}$$

As a result we obtain $I_1 < \frac{2\epsilon}{15}$, when $t_1, t_2 \le \delta_1$.

Let now $t_1, t_2 > \delta_1$ and, to be definitive, $t_2 > t_1$. Represent I_1 in the form

$$I_1 = \left| t_1^{\frac{\beta-1}{2}} \int_0^{\delta_1} \int_0^l G_{1x}(x_1, t_1, \eta, \tau) z(\eta, \tau) d\eta d\tau \right| + \left| t_2^{\frac{\beta-1}{2}} \int_0^{\delta_1} \int_0^l G_{1x}(x_2, t_2, \eta, \tau) z(\eta, \tau) d\eta d\tau \right| \\ + \left| \left(t_1^{\frac{\beta-1}{2}} - t_2^{\frac{\beta-1}{2}} \right) \int_{\delta_1}^{t_1} \int_0^l G_{1x}(x_1, t_1, \eta, \tau) z(\eta, \tau) \, d\eta d\tau \right| \\ + t_2^{\frac{\beta-1}{2}} \left| \int_{\delta_1}^{t_1} \int_0^l \big(G_{1x}(x_1, t_1, \eta, \tau) - G_{1x}(x_1, t_2, \eta, \tau) \big) z(\eta, \tau) d\eta d\tau \right| \\ + t_2^{\frac{\beta-1}{2}} \left| \int_{\delta_1}^{t_1} \int_0^l \big(G_{1x}(x_1, t_2, \eta, \tau) - G_{1x}(x_2, t_2, \eta, \tau) \big) z(\eta, \tau) \, d\eta d\tau \right| \le \frac{2\epsilon}{15} + I_{1,1} + I_{1,2} + I_{1,3}.$$

Since $\left|\int_{\delta_1}^{t_1}\int_0^l G_{1x}(x_1,t_1,\eta,\tau)z(\eta,\tau)\,d\eta d\tau\right| \le C_{21}$ then from the continuity of the function t^β we get $I_{1,1} < \frac{\epsilon}{15}$, when $|t_1-t_2|<\delta_2$. Using (13), we represent $I_{1,2}$ in the form

$$I_{1,2} \le C_{22}\int_{\delta_1}^{t_1}\int_0^l \Bigg|\frac{1}{4\sqrt{\pi}\big(\theta(t_1)-\theta(\tau)\big)^{\frac{3}{2}}}\sum_{n=-\infty}^{+\infty}\Bigg(\exp\Bigg(-\frac{(x_1-\eta+2nl)^2}{\theta(t_1)-\theta(\tau)}\Bigg)(x_1-\eta+2nl)$$
$$+\exp\left(-\frac{(x_1+\eta+2nl)^2}{\theta(t_1)-\theta(\tau)}\right)(x_1+\eta+2nl)\Bigg)$$
$$-\frac{1}{4\sqrt{\pi}\Big(\theta(t_2)-\theta(\tau)\Big)^{\frac{3}{2}}}\sum_{n=-\infty}^{+\infty}\left(\exp\left(-\frac{(x_1-\eta+2nl)^2}{\theta(t_2)-\theta(\tau)}\right)(x_1-\eta+2nl)\right.$$
$$\left.+\exp\left(-\frac{(x_1+\eta+2nl)^2}{\theta(t_2)-\theta(\tau)}\right)(x_1+\eta+2nl)\right)\Bigg|\,d\eta\,d\tau$$
$$\le C_{23}\int_{\delta_1}^{t_1}\left|\int_{\theta(t_2)-\theta(\tau)}^{\theta(t_1)-\theta(\tau)}\frac{\partial}{\partial s}\left(\int_0^l\frac{1}{s^{\frac{3}{2}}}\sum_{n=-\infty}^{+\infty}\left((x_1-\eta+2nl)\exp\left(-\frac{(x_1-\eta+2nl)^2}{4s}\right)\right.\right.\right.$$
$$\left.\left.\left.+(x_1+\eta+2nl)\exp\left(-\frac{(x_1+\eta+2nl)^2}{4s}\right)\right)d\eta\right)ds\right|d\sigma$$
$$\le C_{24}\int_{\delta_1}^{t_1}\left|\int_{\theta(t_2)-\theta(\tau)}^{\theta(t_1)-\theta(\tau)}\frac{d\sigma}{\sqrt{\sigma}}\right|d\tau \le C_{25}\Big|\theta(t_2)-\theta(t_1)\Big| \le C_{26}\Big|t_2^{\beta+1}-t_1^{\beta+1}\Big| < \frac{\epsilon}{15}$$

when $|t_1-t_2|<\delta_2$. The summand $I_{1,3}$ we reduce to the form

$$I_{1,3} \le \left|\int_{\delta_1}^{t_1}\int_0^l\Big(G_{1x}(x_1,t_2,\eta,\tau)-G_{1x}(x_2,t_2,\eta,\tau)\Big)z(\eta,\tau)d\eta d\tau\right|$$
$$+\left|\int_{t_1}^{t_2}\int_0^l G_{1x}(x_2,t_2,\eta,\tau)z(\eta,\tau)d\eta d\tau\right| = I_{1,3,1}+I_{1,3,2}.$$

To estimate $I_{1,3,2}$ we use (14) and find $I_{1,3,2} \le C_{27}\int_{t_1}^{t_2}\frac{d\tau}{\sqrt{\theta(t_2)-\theta(\tau)}} \le C_{28}\left|t_2^{\beta+1}-t_1^{\beta+1}\right| < \frac{\epsilon}{30}$, when $|t_1-t_2|<\delta_3$. Using (13) for $I_{1,3,1}$ we obtain

$$I_{1,3,1} = \left|\int_{\delta_1}^{t_1}\int_0^l\int_{x_1}^{x_2}G_{1xx}(x,t_2,\eta,\tau)dx\,d\eta\,d\tau\right| \le C_{29}|x_2-x_1| \le \frac{\epsilon}{30}, \quad \text{when} \quad |x_1-x_2|<\delta_4.$$

Choosing $\varrho=\min\{\delta_1,\delta_2,\delta_3,\delta_4\}$ and unifying the obtained estimates we obtain the equicontinuity of the set $P_2\mathcal{N}$. Obviously, the same arguments work for the operator P_1. Let us consider $P_3\omega(t)$. Using the equation (27), we rewrite it in the form $P_3\omega = \frac{F(t)}{\triangle(t)t^\delta}$, where $\triangle(t)$ is defined by (20) and

$$F(t) \equiv \Bigg(\mu_3'(t)-a(t)t^{\frac{\beta+1}{2}}\Big(w(l,t)-w(0,t)\Big)$$
$$-\int_0^l\Big(c(x,t)v(x,t)+f(x,t)\Big)dx\Bigg)\Big(l\mu_2(t)-\mu_3(t)\Big)$$
$$-\Bigg(\mu_4'(t)-a(t)t^\beta\left(lt^{\frac{1-\beta}{2}}w(l,t)-\mu_2(t)+\mu_1(t)\right)$$
$$-\int_0^l x\Big(c(x,t)v(x,t)+f(x,t)\Big)dx\Bigg)\Big(\mu_2(t)-\mu_1(t)\Big).$$

Due to the condition *A2)* of the Theorem and the definition of the set $\mathcal{N}$ we conclude that $F(t)$ is a continuous on $[0,T_0]$ and $F(t)\le C_{30}t^{\delta}$, where the constant C_{30} is defined by the problem data. Since $\lim\limits_{t\to 0}\frac{F(t)}{\triangle(t)t^{\delta}}=\kappa_2$, then we can indicate such number t^* that $\left|\frac{F(t)}{\triangle(t)t^{\delta}}-\kappa_2\right|<\frac{\epsilon}{2}$ for $0<t\le t^*$. As a result we obtain

$$\Big|P_3\omega(t_2)-P_3\omega(t_1)\Big|\le\Big|P_3\omega(t_2)-\kappa_2\Big|+\Big|\kappa_2-P_3\omega(t_1)\Big|<\epsilon$$

for $0<t_1,t_2\le t^*$. In the case $t_1,t_2>t^*$ we get

$$\begin{aligned}\Big|P_3\omega(t_2)-P_3\omega(t_1)\Big|&\le\left|\frac{F(t_1)}{t_1^{\delta}}\left(\frac{1}{\triangle(t_1)}-\frac{1}{\triangle(t_2)}\right)\right|+\left|\frac{F(t_1)}{\triangle(t_2)}\left(\frac{1}{t_1^{\delta}}-\frac{1}{t_2^{\delta}}\right)\right|+\left|\frac{F(t_1)-F(t_2)}{\triangle(t_2)t_2^{\delta}}\right|\\&\le\frac{|F(t_1)||\triangle(t_2)-\triangle(t_1)|}{(t^*)^{\delta}\Big(\min\limits_{\tau\in[0,T_0]}|\triangle(\tau)|\Big)^2}+\frac{|F(t_1)||t_1^{\delta}-t_2^{\delta}|}{(t^*)^{2\delta}\min\limits_{\tau\in[0,T_0]}|\triangle(\tau)|}+\frac{|F(t_1)-F(t_2)|}{(t^*)^{\delta}\min\limits_{\tau\in[0,T_0]}|\triangle(\tau)|}.\end{aligned}$$

Taking into account the continuity of the input data and the mean value theorem we deduce $\Big|P_3\omega(t_2)-P_3\omega(t_1)\Big|\le\epsilon$, $|t_2-t_1|<\varrho$. The case $t_1<t^*, t_2>t^*$ combines two previous ones because

$$\Big|P_3\omega(t_2)-P_3\omega(t_1)\Big|\le\Big|P_3\omega(t_2)-P_3\omega(t^*)\Big|+\Big|P_3\omega(t^*)-P_3\omega(t_1)\Big|$$

and equicontinuous of the operator P_3 is proved. The equicontinuity of the operator P_4 can be showed in a similar way.

Applying the Schauder fixed-point theorem we obtain the existence of the continuous solution to the system (25)–(28) on $[0,l]\times[0,T_0]$. It yields the existence of the solution (b_1,b_2,v) to the inverse problem (1)–(5) on $[0,l]\times[0,T_0]$. It means that we prove the existence of the solution to the problem (1)–(5).

3 Uniqueness of the solution

Assume that the system (25)–(28) has two solutions (v_i,w_i,p_{1i},p_{2i}), $i=1,2$. Denote $v(x,t)=v_1(x,t)-v_2(x,t)$, $w(x,t)=w_1(x,t)-w_2(x,t)$, $p_1(t)=p_{11}(t)-p_{12}(t)$, $p_2(t)=p_{21}(t)-p_{22}(t)$. Using (25)–(28), we deduce

$$\begin{aligned}v(x,t)=\int_0^t\!\!\int_0^l G_1(x,t,\eta,\tau)\bigg(\Big(p_{11}(\tau)\eta+p_{21}(\tau)\Big)\tau^{\delta-\frac{\beta-1}{2}}w(\eta,\tau)\\+\Big(p_1(\tau)\eta+p_2(\tau)\Big)\tau^{\delta-\frac{\beta-1}{2}}w_2(\eta,\tau)+c(\eta,\tau)v(\eta,\tau)\bigg)\,d\eta\,d\tau,\end{aligned}\tag{38}$$

$$\begin{aligned}w(x,t)=t^{\frac{\beta-1}{2}}\int_0^t\!\!\int_0^l G_{1x}(x,t,\eta,\tau)\bigg(\Big(p_{11}(\tau)\eta+p_{21}(\tau)\Big)\tau^{\delta-\frac{\beta-1}{2}}w(\eta,\tau)\\+\Big(p_1(\tau)\eta+p_2(\tau)\Big)\tau^{\delta-\frac{\beta-1}{2}}w_2(\eta,\tau)+c(\eta,\tau)v(\eta,\tau)\,d\eta\,d\tau\end{aligned}\tag{39}$$

for $(x,t)\in\overline{Q}_{T_0}$ and

$$p_1(t) = \triangle(t)^{-1}t^{-\delta}\Big(\Big(a(t)lt^{\frac{1+\beta}{2}}w(l,t) + \int_0^l xc(x,t)v(x,t)dx\Big)\Big(\mu_2(t) - \mu_1(t)\Big)$$
$$- \Big(a(t)t^{\frac{\beta+1}{2}}(w(l,t) - w(0,t)) + \int_0^l c(x,t)v(x,t)dx\Big)\Big(l\mu_2(t) - \mu_3(t)\Big)\Big), \tag{40}$$

$$p_2(t) = \triangle(t)^{-1}t^{-\delta}\Big(\Big(a(t)t^{\frac{\beta+1}{2}}(w(l,t) - w(0,t)) + \int_0^l c(x,t)v(x,t)dx\Big)\Big(l^2\mu_2(t) - 2\mu_4(t)\Big)$$
$$- \Big(a(t)lt^{\frac{1+\beta}{2}}w(l,t) + \int_0^l xc(x,t)v(x,t)dx\Big)\Big(l\mu_2(t) - \mu_3(t)\Big)\Big) \tag{41}$$

for $t \in [0,T]$.

Substituting (40) – (41) into (38) – (39), we reduce these equations to the form

$$v(x,t) = \int_0^t \Big(K_{11}(t,\tau)v(x,\tau) + K_{12}(t,\tau)w(x,\tau)\Big)d\tau, \quad t \in [0,T_0], \tag{42}$$

$$w(x,t) = \int_0^t \Big(K_{21}(t,\tau)v(x,\tau) + K_{22}(t,\tau)w(x,\tau)\Big)d\tau, \quad t \in [0,T_0]. \tag{43}$$

The equations (42) – (43) form a system of homogeneous integral Voltera equations of the second kind with respect to unknowns $v = v(x,t), w = w(x,t)$ for every $x \in [0,l]$. Using (14), (15), we can state that the kernels of this system have integrable singularities. It means that this system has only trivial solution

$$v(x,t) \equiv 0,\ w(x,t) \equiv 0, \quad (x,t) \in \overline{Q}_{T_0}. \tag{44}$$

Substituting (44) into (40), (41) we get

$$p_1(t) \equiv 0,\ p_2(t) \equiv 0, \quad t \in [0,T_0]. \tag{45}$$

It means that the Theorem is proved.

Conclusions

The inverse problem for identification two time-dependent functions in the minor coefficient in a strongly degenerate parabolic equation is investigated.

1. The sufficient conditions of existence and uniqueness of the solution to this inverse problem are established. Note that both existence and uniqueness are local in time.

2. It is proved, that for the strongly degenerate parabolic equation in contrast to the case of weak degeneration the first derivative of unknown function $u = u(x,t)$ has a singularity at the point $t = 0$ as $t^{\frac{1-\beta}{2}}$ and unknown functions $b_1 = b_1(t)$, $b_2 = b_2(t)$ tend to zero as $t \to 0$ like t^δ with $\delta = \min\left\{\gamma, \frac{\beta+1}{2}\right\}, \gamma > \frac{\beta-1}{2}$.

3. The system of equations (16) – (19) which is equivalent to the inverse problem (1) – (5) can serve as a base to application of some numerical methods to this problem.

4. This problem is the first step to solve the more complex inverse problems of finding the minor coefficient in a parabolic equation that depends on both spatial and time variables using the approximation of a continuous function by an algebraic polynomial. Furthermore, it is the base in investigation of inverse problems for multidimensional parabolic equations.

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Гузик Н.М., Пукач П.Я., Вовк М.І. *Коефіцієнтна обернена задача для сильно виродженого параболічного рівняння* // Карпатські матем. публ. — 2023. — Т.15, №1. — С. 52–65.

Досліджується коефіцієнтна обернена задача для виродженого параболічного рівняння. Молодший коефіцієнт цього рівняння є многочленом першого степеня за просторовою змінною з двома невідомими залежними від часу функціями. Дослідження проведено при заданих неоднорідних початковій умові, крайових умовах Діріхле та інтегральних умовах перевизначення. Встановлено умови однозначної розв'язності вказаної задачі у випадку сильного виродження.

Ключові слова і фрази: обернена задача, молодший коефіцієнт, параболічне рівняння, сильне виродження.