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On weighted statistical convergence in gradual normed linear spaces

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During the last few years, enormous works by different researchers have been carried out in summability theory through investigating various notions of convergence of sequences. In the present paper, we introduce the concept of weighted g-statistical convergence in the gradual normed linear spaces. We investigate a few fundamental properties and establish some implication relations. Finally, we introduce the concept of gradual weighted g-statistical Cauchy sequences and show that every gradual weighted g-statistically convergent sequence is a gradual weighted g-statistical Cauchy sequence and vice versa.

Key words and phrases: gradual number, gradual normed linear space, g-density, g-statistical convergence.

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1 Introduction and background

In 1965, the concept of fuzzy sets [25] was introduced by L.A. Zadeh as one of the extensions of the classical set-theoretical concept. Nowadays, it has wide applications in different branches of science and engineering. The term "fuzzy number" plays a vital role in the study of fuzzy set theory. Fuzzy numbers were essentially the generalization of intervals, not numbers. Indeed fuzzy numbers do not obey a couple of algebraic properties of the classical numbers. So the term "fuzzy number" is debatable to many researchers due to its different behavior. The term "fuzzy intervals" is often used by many authors instead of fuzzy numbers. To overcome the confusion among the researchers, in 2008, J. Fortin et. al. [11] introduced the notion of gradual real numbers as elements of fuzzy intervals. Gradual real numbers are mainly known by their respective assignment function, which is defined in the interval $(0, 1]$. So, in some sense, every real number can be viewed as a gradual number with a constant assignment function. The gradual real numbers also obey all the algebraic properties of the classical real numbers and have been used in computation and optimization problems.

In 2011, I. Sadeqi and F.Y. Azari [18] were the first to introduce the concept of gradual normed linear space. They studied various properties from both the algebraic and topological points of view. Further development in this direction has been occurred due to M. Ettefagh et. al. [8, 9], C. Choudhury and S. Debnath [4], and many others. For an extensive study on gradual real numbers, one may refer to [2, 7, 16, 23], where many more references can be found.

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On the other hand, in 1951, H. Fast [10] and H. Steinhaus [22] introduced the idea of statistical convergence independently with the aim of providing deeper insights into the summability theory. Thereafter, it was further investigated from the sequence space point of view by J.A. Fridy [13, 14], T. Šalát [19], and many mathematicians across the globe. Statistical convergence has motivated many researchers to develop various notions of density and study the behavior of the sequences, which are divergent from the ordinary or statistical sense, but convergent with respect to some other density. Several investigations and generalizations in this direction can be found in the works of J.S. Connor [5], B.C. Tripathy [24], and many others [17]. Statistical convergence has become one of the most active areas of research mainly because of its wide applicability in various branches of mathematics such as number theory, mathematical analysis, probability theory, etc.

Since statistical convergence is not a matrix method, its extension is only possible with some modifications to the asymptotic density definitions.

In 2015, M. Balcerzak et. al. [3] generalized the notion of natural density to g-density by using a function $g : \mathbb{N} \to [0, \infty)$ that satisfies

$$\lim_{n\to\infty} \frac{n}{g(n)} \neq 0 \quad \text{and} \quad \lim_{n\to\infty} g(n) = \infty. \tag{1}$$

Here $\mathbb{N}$ represents the set of all positive integers. For $g(n) = n$, the definition of g-density turns to the definition of natural density.

Definition 1 ([3]). *Let E be a subset of $\mathbb{N}$ and E_n denotes the set $\{k \in E : k \leq n\}$. The density of weight g (in short g-density) of the set E is denoted and defined by $\delta_g(E) = \lim\limits_{n\to\infty} \frac{|E_n|}{g(n)}$, provided that the limit exists.*

Later on, g-density was applied by many researchers in several areas of summability theory. For more details we refer the reader to [6, 15, 20, 21], where many more references can be found. Recently, A.A. Adem and M. Altinok [1] investigated the notion of weighted statistical convergence of real valued sequences. Let us recall the definition.

Definition 2 ([1]). *A sequence (x_k) is said to be weighted g-statistically convergent to l, if for each $\varepsilon > 0$ we have*

$$\delta_g\big(\{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}\big) = 0.$$

In this case, l is called the weighted g-statistical limit of the sequence (x_k) and symbolically it is expressed as $x_k \xrightarrow{st_g} l$.

Definition 3 ([1]). *A sequence (x_k) is said to be weighted g-statistical Cauchy, if for every $\varepsilon > 0$ there exists a natural number $N = N(\varepsilon)$ such that*

$$\delta_g\big(\{k \leq n : |x_k - x_N| \geq \varepsilon\}\big) = 0.$$

For $g(n) = n$, Definition 1, Definition 2, and Definition 3 turns to the definition of natural density [12], statistical convergence [13], and statistical Cauchy sequence [13], respectively.

The main purpose of this work is to examine the weighted g-statistical convergence of sequences in gradual normed linear spaces.

2 Definitions and preliminaries

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ will be used to denote the set of all positive integers and the set of all real numbers, respectively, and g will denote a function from $\mathbb{N}$ to $[0, \infty)$ that satisfies (1).

Definition 4 ([11]). *A gradual real number $\tilde{r}$ is defined by an assignment function $A_{\tilde{r}} : (0,1] \to \mathbb{R}$. The set of all gradual real numbers is denoted by $G(\mathbb{R})$.*

A gradual real number $\tilde{r}$ is said to be non-negative, if for every $\beta \in (0,1]$ we have $A_{\tilde{r}}(\beta) \geq 0$. The set of all non-negative gradual real numbers is denoted by $G^*(\mathbb{R})$.

The gradual operations between the elements of $G(\mathbb{R})$ are defined as follows.

Definition 5 ([11]). *Let $\circ$ be any operation in $\mathbb{R}$ and suppose $\tilde{r}_1, \tilde{r}_2 \in G(\mathbb{R})$ with assignment functions $A_{\tilde{r}_1}$ and $A_{\tilde{r}_2}$, respectively. Then $\tilde{r}_1 \circ \tilde{r}_2 \in G(\mathbb{R})$ is defined with the assignment function $A_{\tilde{r}_1 \circ \tilde{r}_2}$ given by $A_{\tilde{r}_1 \circ \tilde{r}_2}(\beta) = A_{\tilde{r}_1}(\beta) \circ A_{\tilde{r}_2}(\beta)$ for all $\beta \in (0,1]$. Then the gradual addition $\tilde{r}_1 + \tilde{r}_2$ and the gradual scalar multiplication $\lambda\tilde{r}$, $\lambda \in \mathbb{R}$, are respectively defined by*

$$A_{\tilde{r}_1+\tilde{r}_2}(\beta) = A_{\tilde{r}_1}(\beta) + A_{\tilde{r}_2}(\beta) \quad \text{and} \quad A_{\lambda\tilde{r}}(\beta) = \lambda A_{\tilde{r}}(\beta) \quad \forall \beta \in (0,1].$$

For any real number $s \in \mathbb{R}$, the constant gradual real number $\tilde{s}$ is defined by the constant assignment function $A_{\tilde{s}}(\beta) = s$ for any $\beta \in (0,1]$. In particular, $\tilde{0}$ and $\tilde{1}$ are the constant gradual numbers defined by $A_{\tilde{0}}(\beta) = 0$ and $A_{\tilde{1}}(\beta) = 1$, respectively. It is easy to verify that the set $G(\mathbb{R})$ with the gradual addition and multiplication forms a real vector space [11].

Definition 6 ([18]). *Let X be a real vector space. The function $\|\cdot\|_G : X \to G^*(\mathbb{R})$ is said to be a gradual norm on X if for every $\beta \in (0,1]$, following three conditions are true for any $x, y \in X$:*

(G_1) *$A_{\|x\|_G}(\beta) = A_{\tilde{0}}(\beta)$ if $x = 0$,*

(G_2) *$A_{\|\lambda x\|_G}(\beta) = |\lambda| A_{\|x\|_G}(\beta)$ for every $\lambda \in \mathbb{R}$,*

(G_3) *$A_{\|x+y\|_G}(\beta) \leq A_{\|x\|_G}(\beta) + A_{\|y\|_G}(\beta)$.*

The pair $(X, \|\cdot\|_G)$ is called a gradual normed linear space (GNLS for short).

Example 1 ([18]). *Let $X = \mathbb{R}^m$. For $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m, \beta \in (0,1]$, define $\|\cdot\|_G$ by*

$$A_{\|x\|_G}(\beta) = e^{\beta} \sum_{i=1}^{m} |x_i|.$$

Then $\|\cdot\|_G$ is a gradual norm on $\mathbb{R}^m$ and $(\mathbb{R}^m, \|\cdot\|_G)$ is a gradual normed linear space.

Definition 7 ([18]). *Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. Then (x_k) is said to be gradual convergent to $x \in X$, if for every $\beta \in (0,1]$ and $\varepsilon > 0$ there exists $N = N(\beta, \varepsilon) \in \mathbb{N}$ such that $A_{\|x_k - x\|_G}(\beta) < \varepsilon$ holds for every $k \geq N$.*

Definition 8 ([9]). *Let $(X, \|\cdot\|_G)$ be a GNLS. Then a sequence (x_k) in X is said to be gradual bounded, if for every $\beta \in (0,1]$ there exists $M = M(\beta) > 0$ such that $A_{\|x_k\|_G}(\beta) < M$ holds for all $k \in \mathbb{N}$.*

Definition 9 ([18]). *Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. Then (x_k) is said to be gradual Cauchy, if for every $\beta \in (0,1]$ and $\varepsilon > 0$ there exists $N = N(\beta, \varepsilon) \in \mathbb{N}$ such that $A_{\|x_k - x_j\|_G}(\beta) < \varepsilon$ holds for all $k, j \geq N$.*

3 Main results

In this section, we first define gradual weighted g-statistical convergence as follows.

Definition 10. *Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. Then (x_k) is said to be gradual weighted g-statistically convergent to $x \in X$, if for every $\beta \in (0,1]$ and $\varepsilon > 0$ we have*

$$\delta_g\left(\{k \in \mathbb{N} : A_{\|x_k - x\|_G}(\beta) \geq \varepsilon\}\right) = 0.$$

In this case, we write $x_k \xrightarrow{st_g\text{-}\|\cdot\|_G} x$.

Let us point out that the condition in the above definition can be expressed as follows

$$\delta_g\left(\{k \in \mathbb{N} : A_{\|x_k - x\|_G}(\beta) < \varepsilon\}\right) = \delta_g\left(\mathbb{N}\right).$$

If $g(n) = n$, then the above definition turns to the definition of gradual statistical convergence, which is denoted by $x_k \xrightarrow{st\text{-}\|\cdot\|_G} x$.

Theorem 1. *Let $(X, \|\cdot\|_G)$ be a GNLS. If a sequence (x_k) is gradual convergent to $x \in X$, then (x_k) is gradual weighted g-statistically convergent to $x \in X$.*

Proof. Since (x_k) is gradual convergent to x, then the set $\left\{k \in \mathbb{N} : A_{\|x_k - x\|_G}(\beta) \geq \varepsilon\right\}$ contains a finite number of element and consequently, its g-density is zero. So, the proof is completed. □

But the converse of Theorem 1 is not true. Following example will illustrate the fact.

Example 2. *Let $X = \mathbb{R}^m$ and $||\cdot||_G$ be the norm defined in Example 1. Consider the sequence (x_k) in $\mathbb{R}^m$ defined as*

$$x_k = \begin{cases} (0, 0, \ldots, 0, m), & k = t^3,\ t \in \mathbb{N}, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

where $\mathbf{0}$ denotes the element $(0, 0, \ldots, 0, 0) \in \mathbb{R}^m$. Let the weight $g : \mathbb{N} \to [0, \infty)$ be the function defined as $g(n) = n$, $n \in \mathbb{N}$. Then for any $\varepsilon > 0$ and $\beta \in (0,1]$ we have

$$\left\{k \in \mathbb{N} : A_{\|x_k - \mathbf{0}\|_G}(\beta) \geq \varepsilon\right\} \subseteq \{1, 8, 27, \ldots\}$$

and eventually $\delta_g\left(\{k \in \mathbb{N} : A_{\|x_k - \mathbf{0}\|_G}(\beta) \geq \varepsilon\}\right) = 0$. This implies that $x_k \xrightarrow{st_g\text{-}\|\cdot\|_G} \mathbf{0}$ in $\mathbb{R}^m$.

But it is clear that (x_k) is not gradual convergent to $\mathbf{0}$.

In the following theorem, we will show that the gradual weighted g-statistical limit of a sequence is unique, if exists.

Theorem 2. *Let (x_k) be any sequence in the GNLS $(X, \|\cdot\|_G)$ such that $x_k \xrightarrow{st_g\text{-}\|\cdot\|_G} x$ in X. Then x is uniquely determined.*

Proof. Suppose $x_k \xrightarrow{st_g\text{-}\|\cdot\|_G} x$ and $x_k \xrightarrow{st_g\text{-}\|\cdot\|_G} y$ for some $x \neq y$ in X. Then, by definition, for any $\varepsilon > 0$ and $\beta \in (0,1]$, we have

$$\delta_g\left(P_1(\beta, \varepsilon)\right) = \delta_g\left(P_2(\beta, \varepsilon)\right) = \delta_g(\mathbb{N}),$$

where $P_1(\beta,\varepsilon) = \{k \in \mathbb{N} : A_{\|x_k - x\|_G}(\beta) < \varepsilon\}$ and $P_2(\beta,\varepsilon) = \{k \in \mathbb{N} : A_{\|x_k - y\|_G}(\beta) < \varepsilon\}$.

Choose an arbitrary point p such that $p \in P_1(\beta,\varepsilon) \cap P_2(\beta,\varepsilon)$, then

$$A_{\|x_p - x\|_G}(\beta) < \varepsilon \quad \text{and} \quad A_{\|x_p - y\|_G}(\beta) < \varepsilon$$

hold. Hence,

$$A_{\|x-y\|_G}(\beta) \leq A_{\|x_p - x\|_G}(\beta) + A_{\|x_p - y\|_G}(\beta) < \varepsilon + \varepsilon = 2\varepsilon.$$

Since ε is arbitrary, the equality $A_{\|x-y\|_G}(\beta) = A_{\tilde{0}}(\beta)$ holds. This implies that $x = y$. □

In the next theorem, we are going to prove that the set of all sequences (x_k), which are gradual weighted g-statistically convergent, is closed under addition and scalar multiplication operations.

Theorem 3. *Let (x_k) and (y_k) be two arbitrary sequences in the GNLS $(X, \|\cdot\|_G)$ such that $x_k \xrightarrow{st_g\text{-}\|\cdot\|_G} x$ and $y_k \xrightarrow{st_g\text{-}\|\cdot\|_G} y$. Then*

(i) $x_k + y_k \xrightarrow{st_g\text{-}\|\cdot\|_G} x + y$

and

(ii) $\lambda x_k \xrightarrow{st_g\text{-}\|\cdot\|_G} \lambda x$ *for* $\lambda \in \mathbb{R}$.

Proof. *(i)* Suppose $x_k \xrightarrow{st_g\text{-}\|\cdot\|_G} x$ and $y_k \xrightarrow{st_g\text{-}\|\cdot\|_G} y$. Then for given $\varepsilon > 0$ and $\beta \in (0,1]$ we have

$$\delta_g\left(B_1(\beta,\varepsilon)\right) = \delta_g\left(B_2(\beta,\varepsilon)\right) = 0,$$

where $B_1(\beta,\varepsilon) = \{k \in \mathbb{N} : A_{\|x_k - x\|_G}(\beta) \geq \frac{\varepsilon}{2}\}$ and $B_2(\beta,\varepsilon) = \{k \in \mathbb{N} : A_{\|y_k - y\|_G}(\beta) \geq \frac{\varepsilon}{2}\}$. Now the following inclusion

$$\left(\mathbb{N} \setminus B_1(\beta,\varepsilon)\right) \cap \left(\mathbb{N} \setminus B_2(\beta,\varepsilon)\right) \subseteq \left\{k \in \mathbb{N} : A_{\|x_k + y_k - x - y\|_G}(\beta) < \varepsilon\right\}$$

implies the inequality

$$\delta_g\left(\{k \in \mathbb{N} : A_{\|x_k + y_k - x - y\|_G}(\beta) \geq \varepsilon\}\right) \leq \delta_g\left(B_1(\beta,\varepsilon) \cup B_2(\beta,\varepsilon)\right) = 0.$$

Consequently, $x_k + y_k \xrightarrow{st_g\text{-}\|\cdot\|_G} x + y$.

(ii) If $\lambda = 0$, then there is nothing to prove. So let us assume $\lambda \neq 0$. Since $x_k \xrightarrow{st_g\text{-}\|\cdot\|_G} x$, for given $\varepsilon > 0$ and $\beta \in (0,1]$ we have $\delta_g\left(B_1(\beta,\varepsilon)\right) = 0$, where

$$B_1(\beta,\varepsilon) = \left\{k \in \mathbb{N} : A_{\|x_k - x\|_G}(\beta) \geq \frac{\varepsilon}{|\lambda|}\right\}.$$

Now, since $A_{\|\lambda x_k - \lambda x\|_G}(\beta) = |\lambda| A_{\|x_k - x\|_G}(\beta)$ holds for any $\lambda \in \mathbb{R}$, we must have $B_2(\beta,\varepsilon) \subseteq B_1(\beta,\varepsilon)$, where

$$B_2(\beta,\varepsilon) = \left\{k \in \mathbb{N} : A_{\|\lambda x_k - \lambda x\|_G}(\beta) \geq \varepsilon\right\},$$

which as a consequence implies $\delta_g\left(B_2(\beta,\varepsilon)\right) = 0$. This completes the proof. □

Theorem 4. *Let (x_k) be any sequence in the GNLS $(X, \|\cdot\|_G)$. If every subsequence of (x_k) is gradual weighted g-statistically convergent to x, then (x_k) is also gradual weighted g-statistically convergent to x.*

Proof. If possible suppose (x_k) is not gradual weighted g-statistically convergent to x. Then there exists some $\varepsilon > 0$ and $\beta \in (0,1]$ such that $\delta_g(C) \neq 0$, where

$$C = \Big\{k \in \mathbb{N} : A_{\|x_k - x\|_G}(\beta) \geq \varepsilon\Big\}.$$

Since the g-density of a finite set is zero, then the set C must be an infinite set. Let $C = \{k_1 < k_2 < \cdots < k_n < \cdots\}$. Now define a sequence (y_n) as $y_n = x_{k_n}$ for $n \in \mathbb{N}$. Hence, (y_n) is a subsequence of (x_k), which is not gradual weighted g-statistically convergent to x. This is a contradiction to assumption of the theorem. □

Remark 1. *The converse of Theorem 4 is not true. One can easily verify this from Example 2.*

Theorem 5. *Let $(X, \|\cdot\|_G)$ be a GNLS. A subsequence (x_{k_n}) of a gradual weighted g-statistically convergent sequence (x_k) in X is gradual weighted g-statistically convergent if and only if*

$$\delta_g\big(\{k_n : n \in \mathbb{N}\}\big) = \delta_g(\mathbb{N}).$$

Proof. The proof can be obtained easily. So it is omitted here. □

Theorem 6. *Let (x_k) and (y_k) be two sequences in the GNLS $(X, \|\cdot\|_G)$, such that (y_k) is gradual convergent to $y \in X$ and $\delta_g\big(\{k \in \mathbb{N} : x_k \neq y_k\}\big) = 0$. Then (x_k) is gradual weighted g-statistically convergent to the same point.*

Proof. Suppose $\delta_g\big(\{k \in \mathbb{N} : x_k \neq y_k\}\big) = 0$ holds and $y_k \xrightarrow{\|\cdot\|_G} y$. Then, by definition, for every $\varepsilon > 0$ and $\beta \in (0,1]$ the set

$$\Big\{k \in \mathbb{N} : A_{\|y_k - y\|_G}(\beta) \geq \varepsilon\Big\}$$

contains a finite number of elements and consequently we have

$$\delta_g\Big(\{k \in \mathbb{N} : A_{\|y_k - y\|_G}(\beta) \geq \varepsilon\}\Big) = 0. \tag{2}$$

Now, since the inclusion

$$\{k \in \mathbb{N} : A_{\|x_k - y\|_G}(\beta) \geq \varepsilon\} \subseteq \{k \in \mathbb{N} : A_{\|y_k - y\|_G}(\beta) \geq \varepsilon\} \cap \{k \in \mathbb{N} : x_k \neq y_k\}$$

holds, by using (2) and the hypothesis of the theorem we get

$$\delta_g\Big(\{k \in \mathbb{N} : A_{\|x_k - y\|_G}(\beta) \geq \varepsilon\}\Big) = 0.$$

Hence, $x_k \xrightarrow{st_g\text{-}\|\cdot\|_G} y$ and the proof is complete. □

In [9, Theorem 3.5], M. Ettefagh et. al. proved that in a GNLS every gradual convergent sequence is gradual bounded. But this fact is not true if we consider gradual weighted g-statistical convergence instead of gradual convergence.

Let $g(n) = n$ for $n \in \mathbb{N}$. Consider the gradual norm $\|\cdot\|_G$ defined by

$$A_{\|x\|_G}(\beta) = e^{\beta} \sum_{i=1}^{2} |x_i| .$$

Let (x_k) be sequence in $\mathbb{R}^2$ defined by

$$x_k = \begin{cases} (0,k), & \text{if } k = p^3,\ p \in \mathbb{N}, \\ (0,0), & \text{otherwise.} \end{cases}$$

It is easy to verify that (x_k) is not gradual bounded, but gradual weighted g-statistically convergent to $(0,0) \in \mathbb{R}^2$.

One of the most natural questions here is whether the unboundedness of a gradual weighted g-statistically convergent sequence can be controlled.

Definition 11. *Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. Then (x_k) is said to be gradual weighted g-statistical bounded if there exists $M > 0$ such that*

$$\delta_g\big(\{k \in \mathbb{N} : A_{\|x_k\|_G}(\beta) > M\}\big) = 0$$

holds for every $\beta \in (0,1]$.

Theorem 7. *Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. Then (x_k) is weighted g-statistical bounded if and only if there exists a gradual bounded sequence (y_k) such that*

$$\delta_g\big(\{k \in \mathbb{N} : x_k \neq y_k\}\big) = 0.$$

Proof. Firstly, let (x_k) be weighted g-statistical bounded. Then, by definition, there exists $M > 0$ such that $\delta_g(B) = 0$ for every $\beta \in (0,1]$, where $B = \big\{k \in \mathbb{N} : A_{\|x_k\|_G}(\beta) > M\big\}$.

Consider the seqence (y_k) defined by

$$y_k = \begin{cases} x_k, & k \in \mathbb{N} \setminus B, \\ \theta, & \text{otherwise,} \end{cases}$$

where θ is the zero element of the GNLS $(X, \|\cdot\|_G)$.

Then clearly (y_k) is gradual bounded and $\delta_g\big(\,\{k \in \mathbb{N} : x_k \neq y_k\}\,\big) = 0$, because of the inclusion $\{k \in \mathbb{N} : x_k \neq y_k\} \subseteq B$ holds.

For the converse part, let $\delta_g\big(\{k \in \mathbb{N} : x_k \neq y_k\}\big) = 0$. Since (y_k) is gradual bounded, there exists some $M > 0$ such that $A_{\|y_k\|_G}(\beta) \leq M$ for all $k \in \mathbb{N}$. Then the inclusion

$$\big\{k \in \mathbb{N} : x_k \neq y_k\big\} \supseteq \big\{k \in \mathbb{N} : A_{\|x_k\|_G}(\beta) > B\big\}$$

holds. From this inclusion, we have

$$\delta_g\Big(\big\{k \in \mathbb{N} : A_{\|x_k\|_G}(\beta) > B\big\}\Big) = 0.$$

Hence, (x_k) is gradual weighted g-statistical bounded. □

Theorem 8. *Let $(X, \|\cdot\|_G)$ be a GNLS and suppose (x_k) be a gradual weighted g-statistically convergent sequence. Then (x_k) is gradual weighted g-statistical bounded.*

Proof. The proof is easy and so is omitted. □

Theorem 9. *Let $(X, \|\cdot\|_G)$ be a GNLS and suppose (x_k) be a sequence such that $x_k \xrightarrow{st\text{-}\|\cdot\|_G} x$. Then*

$$x_k \xrightarrow{st_g\text{-}\|\cdot\|_G} x.$$

Proof. The proof is easy and so is omitted. □

Remark 2. *The converse of Theorem 9 is not necessarily true.*

The following example illustrates this fact.

Example 3. *Let $g : \mathbb{N} \to [0, \infty)$ be a weight function defined by*

$$g(n) = \begin{cases} 1, & n \le 3, \\ 2^{2^{2k}}, & n \in \left[2^{2^k}, 2^{2^{k+1}}\right),\ k \in \mathbb{N}. \end{cases}$$

Let C_k and C denote the sets $\left\{n \in \mathbb{N} : 2^{2^k} \le n < 2 \cdot 2^{2^k}\right\}$ and $\bigcup_{k=1}^{\infty} C_k$, respectively.

Define the sequence (x_k) in $\mathbb{R}^m$ as follows

$$x_n = \begin{cases} (1, 0, \dots, 0), & n \in C, \\ \mathbf{0}, & \text{otherwise.} \end{cases}.$$

Then, with respect to the gradual norm given in Example 1, the sequence (x_k) is gradual weighted g-statistically convergent to $\mathbf{0}$, but not gradual statistically convergent to $\mathbf{0}$.

Justification. Let $m_k = \max C_k$. Then for any $\varepsilon > 0$ and $\beta \in [0, 1)$ we have

$$\begin{aligned} \frac{1}{g(n)}\left|\left\{k \le n : A_{\|x_k - \mathbf{0}\|_G}(\beta) \ge \varepsilon \cdot e^{\beta}\right\}\right| &= \frac{1}{g(n)}\left|\ \{k \le n : x_k \in C\}\ \right| \\ &= \frac{1}{g(n)}\left|\ \{k \le m_k : x_k \in C\}\ \right| \le \frac{|C_k|}{2^{2^{2k}}} \le \frac{2^{2^k}}{2^{2^{2k}}}. \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we obtain $x_k \xrightarrow{st_g\text{-}\|\cdot\|_G} \mathbf{0}$.

Let us show that (x_k) is not gradual statistically convergent to $\mathbf{0}$. From the construction of C_k, it is clear that

$$\frac{2^{2^k}}{2} \le |C_k| < 2 \cdot 2^{2^k}.$$

Then for $\varepsilon = \frac{e^{\beta}}{2}$ the following inequality

$$\frac{1}{n}\left|\left\{k \le n : A_{\|x_k - \mathbf{0}\|_G}(\beta) \ge \frac{e^{\beta}}{2}\right\}\right| = \frac{1}{n}\left|\ \{k \le n : x_k \in C\}\ \right| = \frac{|C|}{m_k} \ge \frac{|C_k|}{m_k} \ge \frac{\frac{2^{2^k}}{2}}{2 \cdot 2^{2^k}} = \frac{1}{4}$$

holds for all $k \ge 1$, and this shows that (x_k) is not gradual statistically convergent to $\mathbf{0}$. □

Theorem 10. *A sequence $x = (x_k)$ in the GNLS $(X, \|\cdot\|_G)$ is gradual weighted g-statistically convergent if and only if for any $\varepsilon > 0$ and $\beta \in (0,1]$ there exists $N = N(\beta, \varepsilon) \in \mathbb{N}$ such that*

$$\delta_g\big(\{k \in \mathbb{N} : A_{\|x_k - x_N\|_G}(\beta) < \varepsilon\}\big) = \delta_g(\mathbb{N}).$$

Proof. Let $x_k \xrightarrow{st_g\text{-}\|\cdot\|_G} x_0$. Then, by definition, for any $\varepsilon > 0$ and $\beta \in (0,1]$ we have

$$\delta_g\big(B(\beta,\varepsilon)\big) = \delta_c(\mathbb{N}),$$

where

$$B(\beta,\varepsilon) = \Big\{k \in \mathbb{N} : A_{\|x_k - x_0\|_G}(\beta) < \frac{\varepsilon}{2}\Big\}.$$

Fix an element $N = N(\beta,\varepsilon) \in B(\beta,\varepsilon)$. Then for any $k \in B(\beta,\varepsilon)$ the following inequality

$$A_{\|x_k - x_N\|_G}(\beta) = A_{\|x_k - x_0 + x_0 - x_N\|_G}(\beta) \le A_{\|x_k - x_0\|_G}(\beta) + A_{\|x_N - x_0\|_G}(\beta) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

holds. This implies that

$$B(\beta,\varepsilon) \subseteq \big\{k \in \mathbb{N} : A_{\|x_k - x_N\|_G}(\beta) < \varepsilon\big\}$$

and consequently, we have

$$\delta_g\big(\{k \in \mathbb{N} : A_{\|x_k - x_N\|_G}(\beta) < \varepsilon\}\big) = \delta_g(\mathbb{N}).$$

Conversely, suppose that for any $\varepsilon > 0$ and $\beta \in (0,1]$ there exists $N = N(\beta,\varepsilon) \in \mathbb{N}$ such that

$$\delta_g\big(\{k \in \mathbb{N} : A_{\|x_k - x_N\|_G}(\beta) < \varepsilon\}\big) = \delta_g(\mathbb{N})$$

is true. Then for any $\varepsilon > 0$ we have

$$\delta_g\big(C(\beta,\varepsilon)\big) = \delta_g(\mathbb{N}),$$

where

$$C(\beta,\varepsilon) = \Big\{k \in \mathbb{N} : A_{\|x_k\|_G}(\beta) \in \big[A_{\|x_N\|_G}(\beta) - \varepsilon, A_{\|x_N\|_G}(\beta) + \varepsilon\big]\Big\}.$$

Denote the interval $\big[A_{\|x_N\|_G}(\beta) - \varepsilon, A_{\|x_N\|_G}(\beta) + \varepsilon\big]$ by $I_{(\beta,\varepsilon)}$ for a fixed $\varepsilon > 0$. Then $\delta_g\big(C(\beta,\varepsilon)\big) = \delta_c(\mathbb{N})$ and $\delta_g\left(C\left(\beta, \frac{\varepsilon}{2}\right)\right) = \delta_g(\mathbb{N})$ holds and consequently we have,

$$\delta_g\Big(C(\beta,\varepsilon) \cap C\Big(\beta, \frac{\varepsilon}{2}\Big)\Big) = \delta_g(\mathbb{N}).$$

But this implies that $I = I_{(\beta,\varepsilon)} \cap I_{(\beta,\frac{\varepsilon}{2})} \neq \varnothing$, $\delta_g\big(\{k \in \mathbb{N} : A_{\|x_k\|_G}(\beta) \in I\}\big) = \delta_g(\mathbb{N})$, and $\operatorname{diam} I \le \frac{1}{2} \operatorname{diam} I(\beta,\varepsilon)$, where diam I denote the length of the interval I.

In this way, by induction, we can construct a sequence of closed intervals satisfying

$$I_\varepsilon = J_0 \supseteq J_1 \supseteq \ldots J_n \supseteq \ldots,$$

such that $\operatorname{diam} J_n \le \frac{1}{2} \operatorname{diam} J_{n-1}$ for $n = 2, 3, \ldots$ and $\delta_g\big(\{k \in \mathbb{N} : A_{\|x_k\|_G}(\beta) \in J_n\}\big) = \delta_g(\mathbb{N})$. Then there exists an $\eta \in \bigcap_{n \in \mathbb{N}} J_n$ and it is routine work to verify that $x_k \xrightarrow{st_g\text{-}\|\cdot\|_G} \eta$. □

Definition 12. *Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. Then (x_k) is said to be gradual weighted g-statistical Cauchy sequence if for every $\varepsilon > 0$ and $\beta \in (0,1]$ there exists $N = N(\beta,\varepsilon) \in \mathbb{N}$ such that*

$$\delta_g\big(\{k \in \mathbb{N} : A_{\|x_k - x_N\|_G}(\beta) \geq \varepsilon\}\big) = 0$$

holds.

Theorem 11. *Let (x_k) be a sequence in the GNLS $(X, \|\cdot\|_G)$. Then the following statements are equivalent:*

(i) $x_k \xrightarrow{st_g\text{-}\|\cdot\|_G} x$ *as* $k \to \infty$,

(ii) (x_k) *is a gradual weighted g-statistical Cauchy sequence,*

(iii) *there exists a gradual convergent sequence (y_k) to x such that*

$$\delta_g\big(\{k \in \mathbb{N} : x_k \neq y_k\}\big) = 0.$$

Proof. *(i) $\Rightarrow$ (ii)* Let (x_k) be a sequence in X such that $x_k \xrightarrow{st_g\text{-}\|\cdot\|_G} x$. Then for every $\varepsilon > 0$ and $\beta \in (0,1]$ we have

$$\delta_g\left(B_1(\beta,\varepsilon)\right) = 0, \tag{3}$$

where

$$B_1(\beta,\varepsilon) = \Big\{k \in \mathbb{N} : A_{\|x_k - x\|_G}(\beta) \geq \varepsilon\Big\}.$$

Let us choose $N \in \mathbb{N} \setminus B_1(\beta,\varepsilon)$. Then we have $A_{\|x_N - x\|_G}(\beta) < \varepsilon$.

Let $B_2(\beta,\varepsilon) = \big\{k \in \mathbb{N} : A_{\|x_k - x_N\|_G}(\beta) \geq 2\varepsilon\big\}$. Let us prove that the following inclusion

$$B_2(\beta,\varepsilon) \subseteq B_1(\beta,\varepsilon) \tag{4}$$

is true. For $p \in B_2(\beta,\varepsilon)$ we have

$$2\varepsilon \leq A_{\left\|x_p - x_N\right\|_G}(\beta) \leq A_{||x_p - x||_G}(\beta) + A_{\|x - x_N\|_G}(\beta) < A_{\left\|x_p - x\right\|_G}(\beta) + \varepsilon,$$

which implies $p \in B_1(\beta,\varepsilon)$ and so (4) is true.

From the equations (3) and (4) we conclude that $\delta_g\left(B_2(\beta,\varepsilon)\right) = 0$, which means that (x_k) is gradual weighted g-statistical Cauchy sequence.

(ii) $\Rightarrow$ (iii) Let (x_k) be a gradual weighted g-statistical Cauchy sequence. Choose N so that

$$\delta_g\Big(\{k \in \mathbb{N} : A_{\|x_k\|_G}(\beta) \notin I\}\Big) = 0,$$

where

$$I = \big[A_{\|x_N\|_G}(\beta) - 1, A_{\|x_N\|_G}(\beta) + 1\big].$$

Again choose M so that $\delta_g\big(\{k \in \mathbb{N} : A_{\|x_k\|_G}(\beta) \notin I'\}\big) = 0$, where

$$I' = \Big[A_{\|x_M\|_G}(\beta) - \frac{1}{2}, A_{\|x_M\|_G}(\beta) + \frac{1}{2}\Big].$$

Now as the equality

$$\left\{k \le n : A_{\|x_k\|_G}(\beta) \notin I \cap I'\right\} = \left\{k \le n : A_{\|x_k\|_G}(\beta) \notin I\right\} \cup \left\{k \le n : A_{\|x_k\|_G}(\beta) \notin I'\right\}$$

holds, so we must have

$$\delta_g\Big(\left\{k \in \mathbb{N} : A_{\|x_k\|_G}(\beta) \notin I \cap I'\right\}\Big) = 0.$$

Denote $I \cap I'$ by I_1. Then it is clear that I_1 is a closed interval with $diam\,(I_1) \le 1$, where $diam\,(I_1)$ represents the length of the interval I_1. Proceeding like this, we choose $N(2)$ so that $\delta_g\big(\{k \in \mathbb{N} : A_{\|x_k\|_G}(\beta) \notin I''\}\big) = 0$, where

$$I'' = \Big[A_{\|x_{N(2)}\|_G}(\beta) - \frac{1}{4}, A_{\|x_{N(2)}\|_G}(\beta) + \frac{1}{4}\Big].$$

Let us denote $I_1 \cap I''$ by I_2. Then by the previous argument we can say that I_2 is a closed interval with $diam\,(I_2) \le \frac{1}{2}$ satisfying

$$\delta_g\Big(\left\{k \in \mathbb{N} : A_{\|x_k\|_G}(\beta) \notin I_2\right\}\Big) = 0.$$

Continuing in this way, we obtain a sequence (I_m) of closed intervals such that

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_m \supseteq I_{m+1} \supseteq \ldots$$

and

$$diam(I_m) \le 2^{1-m}.$$

By Nested Interval Theorem, there exists a $\lambda \in \mathbb{R}$ such that

$$\bigcap_{m=1}^{\infty} I_m = \{\lambda\}.$$

Now we choose an increasing sequence of natural numbers (T_m) such that

$$\frac{1}{g(n)}\Big|\left\{k \le n : A_{\|x_k\|_G}(\beta) \notin I_m\right\}\Big| < \frac{1}{g(m)} \tag{5}$$

if $n > T_m$. Define a subsequence (z_k) of (x_k) consisting of all terms x_k such that $k > T_1$ and if $T_m < k \le T_{m+1}$, then $A_{\|x_k\|_G}(\beta) \notin I_m$. Define the sequence (y_k) as follows

$$y_k = \begin{cases} \tilde{\lambda}, & \text{if } x_k \text{ is a term of } (z_k), \\ x_k, & \text{otherwise,} \end{cases}$$

where $A_{\tilde{\lambda}}(\beta) = \lambda, \forall\, \beta \in (0,1]$. Then $y_k \to \tilde{\lambda}$. If $\varepsilon > \frac{1}{g(m)} > 0$ and $k > T_m$, then either x_k is a term of (z_k), which means $y_k = \tilde{\lambda}$, or $y_k = x_k$, $A_{\|x_k\|_G}(\beta) \in I_m$ and

$$A_{\|y_k - \tilde{\lambda}\|_G}(\beta) \le diam\,(I_m) \le 2^{1-m}.$$

We also claim that $\delta_g\big(\{k \in \mathbb{N} : x_k \ne y_k\}\big) = 0$. If $T_m < n < T_{m+1}$, then the inclusion

$$\left\{k \le n : x_k \ne y_k\right\} \subseteq \left\{k \le n : A_{\|x_k\|_G}(\beta) \notin I_m\right\}$$

holds and consequently by (5) we have

$$\frac{1}{g(n)}\Big|\{k \leq n : x_k \neq y_k\}\Big| < \frac{1}{g(n)}.$$

Letting $n \to \infty$ on both sides of the above inequality, we obtain

$$\lim_{n\to\infty} \frac{1}{g(n)}\Big|\{k \leq n : x_k \neq y_k\}\Big| = 0,$$

i.e.

$$\delta_g\Big(\big\{k \in \mathbb{N} : x_k \neq y_k\big\}\Big) = 0.$$

(iii) $\Rightarrow$ *(i)* Finally we assume that $\delta_g\big(\{k \in \mathbb{N} : x_k \neq y_k\}\big) = 0$ and $y_k \to x(G)$. Then, by definition, for any $\varepsilon > 0$ and $\beta \in (0,1]$ the set

$$\Big\{k \leq n : A_{\|y_k - x\|_G}(\beta) \geq \varepsilon\Big\}$$

contains a finite number of elements, say N_0. Now as the inclusion

$$\big\{k \leq n : A_{\|x_k - x\|_G}(\beta) \geq \varepsilon\big\} \subseteq \big\{k \leq n : x_k \neq y_k\big\} \cup \big\{k \leq n : A_{\|y_k - x\|_G}(\beta) \geq \varepsilon\big\}$$

holds, so we must have

$$\frac{1}{g(n)}\Big|\big\{k \leq n : A_{\|x_k - x\|_G}(\beta) \geq \varepsilon\big\}\Big| \leq \frac{1}{g(n)}\Big|\big\{k \leq n : x_k \neq y_k\big\}\Big| + \frac{N_0}{g(n)}.$$

Letting $n \to \infty$ on both sides of the above inequality and using the fact that

$$\delta_g\Big(\big\{k \in \mathbb{N} : x_k \neq y_k\big\}\Big) = 0,$$

we obtain $x_k \xrightarrow{st_g\text{-}\|\cdot\|_G} x$. This completes the proof. □

Conclusions

In this paper, we have investigated a few fundamental properties of weighted g-statistical convergence in the gradual normed linear spaces. Theorem 8 establishes the connection between weighted g-statistical convergence and weighted g-statistical boundedness. Theorem 10 gives the Cauchy condition for weighted g-statistical convergence. Theorem 11 establishes the relationship between gradual weighted g-statistically convergent and gradual weighted g-statistical Cauchy sequences.

Summability theory and the convergence of sequences have wide applications in various branches of mathematics particularly, in mathematical analysis. Research in this direction based on gradual normed linear spaces has not yet gained much ground and it is still in its infant stage. For example, it can be seen from Theorem 8 that the weighted g-statistically convergent sequence has a weighted g-statistically convergent subsequence.

Depending on this result, the following question can be asked: which subsequences of the weighted g-statistically convergent sequence are weighted g-statistically convergent to the same point?

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Чоудхурі Ч., Кючюкаслан М. *Про зважену статистичну збіжність у поступово нормованих лінійних просторах* // Карпатські матем. публ. — 2025. — Т.17, №1. — С. 317–330.

Упродовж останніх кількох років було виконано значну кількість досліджень у теорії сумовності шляхом вивчення різних понять збіжності послідовностей. У цій статті ми вводимо поняття зваженої g-статистичної збіжності в поступово нормованих лінійних просторах. Ми досліджуємо деякі фундаментальні властивості та встановлюємо певні співвідношення між поняттями. Нарешті, ми вводимо поняття поступово зважених g-статистичних послідовностей Коші та показуємо, що кожна поступово зважена g-статистично збіжна послідовність є поступово зваженою g-статистичною послідовністю Коші і навпаки.

Ключові слова і фрази: поступове число, поступово нормований лінійний простір, g-щільність, g-статистична збіжність.