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Completeness of the systems of Bessel functions of index $-5/2$

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Let $L^2((0;1);x^4dx)$ be the weighted Lebesgue space of all measurable functions $f:(0;1)\to\mathbb{C}$, satisfying $\int_0^1 t^4|f(t)|^2\,dt<+\infty$. Let $J_{-5/2}$ be the Bessel function of the first kind of index $-5/2$ and $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers. Necessary and sufficient conditions for the completeness of the system $\{\rho_k^2\sqrt{x\rho_k}J_{-5/2}(x\rho_k):k\in\mathbb{N}\}$ in the space $L^2((0;1);x^4dx)$ are found in terms of an entire function with the set of zeros coinciding with the sequence $(\rho_k)_{k\in\mathbb{N}}$. In this case, we study an integral representation of some class $E_{4,+}$ of even entire functions of exponential type $\sigma\le 1$. This complements similar results on approximation properties of the systems of Bessel functions of negative half-integer index less than -1, due to B. Vynnyts'kyi, V. Dilnyi, O. Shavala and the author.

Key words and phrases: Bessel function, Paley-Wiener theorem, Phragmén-Lindelöf theorem, Fubini's theorem, Hurwitz's theorem, Hahn-Banach theorem, Jensen's formula, entire function of exponential type, complete system.

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Introduction

Let $L^2(X)$ be the space of all measurable functions $f:X\to\mathbb{C}$ on a measurable set $X\subseteq\mathbb{R}$ with the norm

$$\|f\|^2_{L^2(X)}:=\int_X|f(x)|^2\,dx.$$

Let $\gamma\in\mathbb{R}$ and $L^2\big((0;1);t^\gamma dt\big)$ be the weighted Lebesgue space of all measurable functions $f:(0;1)\to\mathbb{C}$, satisfying

$$\int_0^1 t^\gamma|f(t)|^2\,dt<+\infty.$$

Let (see, for example, [2, p. 4], [12, p. 345], [22, p. 40])

$$J_\nu(z)=\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)},\qquad z=x+iy=re^{i\varphi},$$

be the Bessel function of the first kind of index $\nu\in\mathbb{R}$, where Γ is the gamma function. By Hurwitz's theorem (see [2, p. 59], [22, p. 483]), for $\nu>1$ the function $J_{-\nu}$ has infinitely many real zeros and also $2[\nu]$ pairwise conjugate complex zeros, among them two pure imaginary zeros, when $[\nu]$ is an odd integer. Let ρ_k, $k\in\mathbb{N}$, be the zeros of the function $J_{-\nu}$ for which $\operatorname{Im}\rho_k>0$ if $\rho_k\in\mathbb{C}$ or $\rho_k>0$ if $\rho_k\in\mathbb{R}$.

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Since (see [12, p. 350], [22, p. 55]) $\sqrt{z}J_{-5/2}(z) = -\sqrt{2/\pi}z^{-2}\left(z^2\cos z - 3z\sin z - 3\cos z\right)$, we have that the function $\rho^2\sqrt{x\rho}J_{-5/2}(x\rho)$ belongs to the space $L^2\left((0;1);x^4dx\right)$ for every $\rho \in \mathbb{C}$. A system of elements $\{e_k : k \in \mathbb{N}\}$ in a separable Hilbert space $\mathcal{H}$ is called complete (see [8, p. 131], [9, p. 4258]) if $\overline{\text{span}}\ \{e_k : k \in \mathbb{N}\} = \mathcal{H}$.

Various approximation properties of the systems of Bessel functions has been studied in many papers (see, for example, [1–7, 10–22]). In particular, it is well known that the system $\left\{\sqrt{x}J_\nu\left(x\widetilde{\rho}_k\right) : k \in \mathbb{N}\right\}$ is an orthogonal basis for the space $L^2(0;1)$ if $\nu > -1$ and $(\widetilde{\rho}_k)_{k\in\mathbb{N}}$ is a sequence of positive zeros of J_ν (see [1, 2, 4, 12, 22]). It follows that if $\nu > -1$ and $(\widetilde{\rho}_k)_{k\in\mathbb{N}}$ is a sequence of positive zeros of J_ν, then the system $\left\{x^{-\nu}J_\nu\left(x\widetilde{\rho}_k\right) : k \in \mathbb{N}\right\}$ is complete and minimal in $L^2\left((0;1);x^{2\nu+1}dx\right)$. The system $\left\{\sqrt{x}J_\nu\left(x\widetilde{\rho}_k\right) : k \in \mathbb{N}\right\}$ is also complete (see [12, pp. 347, 356]) in $L^2(0;1)$ if $(\widetilde{\rho}_k)_{k\in\mathbb{N}}$ is a sequence of zeros of the function J'_ν. Besides, from [3] it follows that if $\nu > -1/2$ and $(\widetilde{\rho}_k)_{k\in\mathbb{N}}$ is a sequence of distinct positive numbers such that $\widetilde{\rho}_k \le \pi(k+\nu/2)$ for all sufficiently large $k \in \mathbb{N}$, then the system $\left\{\sqrt{x}J_\nu\left(x\widetilde{\rho}_k\right) : k \in \mathbb{N}\right\}$ is complete in $L^2(0;1)$.

Basis properties (completeness, minimality, basicity) of the above systems of Bessel functions and more general systems $\left\{x^{-p-1}\sqrt{x\widetilde{\rho}_k}J_\nu(x\widetilde{\rho}_k) : k \in \mathbb{N}\right\}$ in the space $L^2\left((0;1);x^{2p}dx\right)$, where $\nu \ge 1/2$, $p \in \mathbb{R}$ and $(\widetilde{\rho}_k)_{k\in\mathbb{N}}$ is a sequence of distinct nonzero complex numbers, have been studied in [6, 7, 15–19]. Those results are formulated in terms of sequences of zeros of functions from certain classes of entire functions.

Approximation properties of the systems of Bessel functions for $\nu < -1$, $\nu \notin \mathbb{Z}$, were investigated in [5,10,11,13,14,20,21]. In particular, at studying of one boundary value problem, in [20] (see also [21]) it was proven that the system $\left\{\rho_k\sqrt{x\rho_k}J_{-3/2}\left(x\rho_k\right) : k \in \mathbb{N}\right\}$ is complete in the space $L^2\left((0;1);x^2dx\right)$, and the system $\left\{\rho_k\sqrt{x\rho_k}J_{-3/2}\left(x\rho_k\right) : k \in \mathbb{N}\setminus\{1\}\right\}$ is complete, minimal and is not a basis in this space, where $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$, $\rho_{-k} := -\rho_k$, is a sequence of zeros of the function $J_{-3/2}$.

In addition, in [10] it was shown that the system $\left\{\rho_k^2\sqrt{x\rho_k}J_{-5/2}\left(x\rho_k\right) : k \in \mathbb{N}\setminus\{1;2\}\right\}$ is complete and minimal in $L^2\big((0;1);x^4dx\big)$, where $(\rho_k)_{k\in\mathbb{N}}$ is a sequence of zeros of $J_{-5/2}$. Besides, in [11] it was established that the system $\left\{\rho_k^{\nu-1/2}\sqrt{x\rho_k}J_{-\nu}\left(x\rho_k\right) : k \in \mathbb{N}\setminus\{1;2;\ldots;l\}\right\}$ is complete in $L^2\big((0;1);x^{2\nu-1}dx\big)$ if $\nu = l+1/2$, $l \in \mathbb{N}$ and $(\rho_k)_{k\in\mathbb{N}}$ is a sequence of zeros of $J_{-\nu}$. However, the problem of completeness of this system in $L^2\left((0;1);x^{2\nu-1}dx\right)$, when $\nu = l+1/2$, $l \in \mathbb{N}$ and $(\rho_k)_{k\in\mathbb{N}}$ is an arbitrary sequence of distinct nonzero complex numbers, remain open. In this direction, in [13] the authors obtained a criterion for the completeness and minimality of the system $\left\{\rho_k\sqrt{x\rho_k}J_{-3/2}(x\rho_k) : k \in \mathbb{N}\right\}$ in $L^2\left((0;1);x^2dx\right)$ with an arbitrary sequence of nonzero complex numbers $(\rho_k)_{k\in\mathbb{N}}$. In addition, in [14] it was proven that the system $\left\{x^{-2}(\rho_k\sqrt{x\rho_k}J_{-3/2}(x\rho_k) - \rho_1\sqrt{x\rho_1}J_{-3/2}(x\rho_1)) : k \in \mathbb{N}\setminus\{1\}\right\}$ is also complete and minimal in $L^2\left((0;1);x^2dx\right)$, where $(\rho_k)_{k\in\mathbb{N}}$ is a sequence of distinct nonzero complex numbers such that $\rho_k^2 \ne \rho_m^2$ for $k \ne m$.

In this paper, using methods of [5–7, 9, 13–16], we study an integral representation of some class $E_{4,+}$ of even entire functions of exponential type $\sigma \le 1$ (see Theorem 1) and find necessary and sufficient conditions for the completeness of the system $\{\psi_k : k \in \mathbb{N}\}$, where $\psi_k(x) := \rho_k^2\sqrt{x\rho_k}J_{-5/2}\left(x\rho_k\right)$, in the space $L^2\left((0;1);x^4dx\right)$ in terms of an entire function with the set of zeros coinciding with the sequence of distinct nonzero complex numbers $(\rho_k)_{k\in\mathbb{N}}$ (see Theorems 2–8). This complements the results of papers [3, 5, 10, 11, 13, 14, 20, 21].

1 Preliminaries

An entire function G is said to be of exponential type $\sigma \in [0;+\infty)$ (see [8, p. 4], [9, p. 4262]), if for any $\varepsilon > 0$ there exists a constant $c(\varepsilon)$ such that

$$\big|G(z)\big| \leq c(\varepsilon)\exp\big((\sigma+\varepsilon)|z|\big)$$

for all $z \in \mathbb{C}$.

Denote by PW^2_σ the set of all entire functions of exponential type $\sigma \in (0;+\infty)$ whose narrowing on $\mathbb{R}$ belongs to the space $L^2(\mathbb{R})$, and by $PW^2_{\sigma,+}$ denote the class of even entire functions from PW^2_σ. According to the Paley-Wiener theorem (see [8, p. 69], [9, p. 4263]), the class PW^2_σ coincides with the class of functions G admitting the representation

$$G(z) = \int_{-\sigma}^{\sigma} e^{itz} g(t)\,dt, \qquad g \in L^2(-\sigma;\sigma),$$

and the class $PW^2_{\sigma,+}$ consists of the functions G representable in the form

$$G(z) = \int_0^{\sigma} \cos(tz) g(t)\,dt, \qquad g \in L^2(0;\sigma).$$

Moreover, $\|g\|_{L^2(0;\sigma)} = \sqrt{2/\pi}\|G\|_{L^2(0;+\infty)}$ and

$$g(t) = \frac{2}{\pi}\int_0^{+\infty} G(z)\cos(tz)\,dz.$$

Let $\log^+ x = \max(0;\log x)$ for $x > 0$. Here and subsequently, by $c_1, c_2, \ldots$ we denote arbitrary positive constants. To prove our main results we need the following auxiliary lemmas.

Lemma 1 ([5, p. 6])**.** *Let an entire function Q be defined by the formula*

$$Q(z) = \sqrt{\frac{2}{\pi}}\int_0^1 \Big(-z^2t^2\cos(tz) + 3tz\sin(tz) + 3\cos(tz)\Big)\, q(t)\,dt, \qquad q \in L^2(0;1). \tag{1}$$

Then for all $z = x + iy = re^{i\varphi} \in \mathbb{C}$, we have

$$\big|Q(z)\big| \leq c_1 \frac{e^{|\operatorname{Im} z|}}{\sqrt{1+|\operatorname{Im} z|}}\big(1+|z|\big)^2,$$

and Q is an even entire function of exponential type $\sigma \leq 1$.

Lemma 2 ([9, p. 4263])**.** *Let Q be an entire function of exponential type $\sigma \leq 1$ for which the integral*

$$\int_{-\infty}^{+\infty} \frac{\log^+ \big|Q(x)\big|}{1+x^2}\,dx$$

exists and let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of nonzero roots of the function $Q(z)$. Then

$$\sum_{k\in\mathbb{N}} \left|\operatorname{Im}\frac{1}{\rho_k}\right| < +\infty.$$

2 Main results

Our principal results are the following statements.

Theorem 1. *An entire function Q has the representation*

$$Q(z) = \int_0^1 z^2 \sqrt{tz} J_{-5/2}(tz) t^4 h(t)\, dt \tag{2}$$

with some function $h \in L^2\left((0;1); x^4 dx\right)$ if and only if it is an even entire function of exponential type $\sigma \le 1$ such that

$$Q(0) = 3\sqrt{\frac{2}{\pi}} \int_0^1 t^2 h(t)\, dt, \tag{3}$$

$$\left.\frac{Q'(z)}{z}\right|_{z=0} = \sqrt{\frac{2}{\pi}} \int_0^1 t^4 h(t)\, dt, \tag{4}$$

and the function $z^{-1}\left(z^{-1}Q'(z)\right)'$ belongs to the space $PW^2_{1,+}$. If these conditions are fulfilled, then

$$h(t) = \sqrt{\frac{2}{\pi}} \frac{1}{t^6} \int_0^{+\infty} \frac{1}{z} \left(\frac{Q'(z)}{z}\right)' \cos(tz)\, dz.$$

Proof. Necessity. Let Q has the representation (2) with some function $h \in L^2\left((0;1); x^4 dx\right)$. Since

$$z^2 \sqrt{tz} J_{-5/2}(tz) = \sqrt{\frac{2}{\pi}} \frac{-z^2 t^2 \cos(tz) + 3tz \sin(tz) + 3\cos(tz)}{t^2},$$

we have

$$Q(z) = \sqrt{\frac{2}{\pi}} \int_0^1 \left(-z^2 t^2 \cos(tz) + 3tz \sin(tz) + 3\cos(tz)\right) t^2 h(t)\, dt, \quad Q(0) = 3\sqrt{\frac{2}{\pi}} \int_0^1 t^2 h(t)\, dt.$$

Therefore, by Lemma 1, the function Q is an even entire function of exponential type $\sigma \le 1$, and

$$Q'(z) = \sqrt{\frac{2}{\pi}} \int_0^1 \left(z^2 t \sin(tz) + z\cos(tz)\right) t^4 h(t)\, dt,$$

$$\frac{Q'(z)}{z} = \sqrt{\frac{2}{\pi}} \int_0^1 \left(tz \sin(tz) + \cos(tz)\right) t^4 h(t)\, dt, \qquad \left.\frac{Q'(z)}{z}\right|_{z=0} = \sqrt{\frac{2}{\pi}} \int_0^1 t^4 h(t)\, dt,$$

$$\left(\frac{Q'(z)}{z}\right)' = \sqrt{\frac{2}{\pi}} \int_0^1 z \cos(tz) t^6 h(t)\, dt, \qquad \frac{1}{z}\left(\frac{Q'(z)}{z}\right)' = \sqrt{\frac{2}{\pi}} \int_0^1 \cos(tz) t^4 q(t)\, dt,$$

where $q(t) := t^2 h(t)$. Since $h \in L^2\left((0;1); x^4 dx\right)$, we have $q \in L^2(0;1)$, and in accordance with the Paley-Wiener theorem, the function $z^{-1}\left(z^{-1}Q'(z)\right)'$ belongs to the space $PW^2_{1,+}$.

Sufficiency. If all the conditions of the theorem hold, then from the formula for the inverse Fourier cosine transformation it follows that the function

$$q(t) = \sqrt{\frac{2}{\pi}} \frac{1}{t^4} \int_0^{+\infty} \frac{1}{z} \left(\frac{Q'(z)}{z}\right)' \cos(tz)\, dz$$

belongs to the space $L^2(0;1)$, and

$$\left(\frac{Q'(z)}{z}\right)' = \sqrt{\frac{2}{\pi}} \int_0^1 z \cos(tz) t^4 q(t)\, dt.$$

Using Fubini's theorem, we get

$$\frac{Q'(z)}{z} - \frac{Q'(z)}{z}\bigg|_{z=0} = \sqrt{\frac{2}{\pi}} \int_0^1 t^4 q(t)\,dt \int_0^z w\cos(tw)\,dw$$
$$= \sqrt{\frac{2}{\pi}} \int_0^1 \big(tz\sin(tz) + \cos(tz) - 1\big) t^2 q(t)\,dt$$
$$= \sqrt{\frac{2}{\pi}} \int_0^1 \big(tz\sin(tz) + \cos(tz) - 1\big) t^4 h(t)\,dt,$$

where $h(t) = t^{-2}q(t) \in L^2\big((0;1);x^4dx\big)$. Further, using (4), we obtain

$$Q'(z) = \sqrt{\frac{2}{\pi}} \int_0^1 \Big(tz^2\sin(tz) + z\cos(tz)\Big)\, t^4 h(t)\,dt.$$

Furthermore, applying Fubini's theorem, we get

$$Q(z) - Q(0) = \sqrt{\frac{2}{\pi}} \int_0^1 t^4 h(t)\,dt \int_0^z \Big(w\cos(tw) + tw^2\sin(tw)\Big)\,dw$$
$$= \sqrt{\frac{2}{\pi}} \int_0^1 \Big(-z^2t^2\cos(tz) + 3tz\sin(tz) + 3\cos(tz) - 3\Big)\, t^2 h(t)\,dt.$$

Hence, taking into account (3), we have

$$Q(z) = \sqrt{\frac{2}{\pi}} \int_0^1 \big(-z^2t^2\cos(tz) + 3tz\sin(tz) + 3\cos(tz)\big) t^2 h(t)\,dt = \int_0^1 z^2\sqrt{tz}J_{-5/2}(tz) t^4 h(t)\,dt.$$

Thus, the theorem is proved. □

Let $\widetilde{E}_{4,+}$ be the class of entire functions Q that can be presented in the form (2) with some function $h \in L^2\left((0;1);x^4dx\right)$, and let $E_{4,+}$ be the class of even entire functions Q of exponential type $\sigma \le 1$ such that conditions (3), (4) are fulfilled with $h \in L^2\left((0;1);x^4dx\right)$ and the function $z^{-1}\left(z^{-1}Q'(z)\right)'$ belongs to the space $PW^2_{1,+}$.

Corollary 1. $\widetilde{E}_{4,+} = E_{4,+}$.

Corollary 2. *The class $E_{4,+}$ coincides with a set of entire functions Q representing in the form* (1).

Remark 1. *In [5], the class $E_{4,+}$ was described in terms of the existence of solutions of some differential equations. Also in [5], examples of entire functions $Q \in E_{4,+}$ are given.*

Theorem 2. *Let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of nonzero complex numbers such that $\rho_k^2 \ne \rho_n^2$ for $k \ne n$. For a system $\{\psi_k : k \in \mathbb{N}\}$ to be incomplete in the space $L^2\left((0;1);x^4dx\right)$ it is necessary and sufficient that a sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$, where $\rho_{-k} := -\rho_k$, $k \in \mathbb{N}$, is a subsequence of zeros of some nonzero entire function $Q \in E_{4,+}$.*

Proof. According to Hahn-Banach theorem (see, e.g., [8, p. 131], [9, p. 4258]), the system $\{\psi_k : k \in \mathbb{N}\}$ is incomplete in $L^2\left((0;1);x^4dx\right)$ if and only if there exists a nonzero function $h \in L^2\left((0;1);x^4dx\right)$ such that

$$\int_0^1 \rho_k^2\sqrt{x\rho_k}J_{-5/2}(x\rho_k)\, x^4 h(x)\,dx = 0$$

for all $k \in \mathbb{N}$. Hence, taking into account Theorem 1, we obtain the required proposition. Theorem 2 is proved. □

Theorem 3. *Let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers such that $|\mathrm{Im}\,\rho_k| \geq \delta\,|\rho_k|$ for all $k \in \mathbb{N}$ and some $\delta > 0$. If a system $\{\psi_k : k \in \mathbb{N}\}$ is complete in $L^2\left((0;1);x^4dx\right)$, then*

$$\sum_{k=1}^{\infty}\frac{1}{|\rho_k|} = +\infty. \tag{5}$$

Proof. Suppose, to the contrary, that the system $\{\psi_k : k \in \mathbb{N}\}$ is not complete in the space $L^2\left((0;1);x^4dx\right)$. Then, by Theorem 2, there exists a nonzero entire function $Q \in E_{4,+}$ for which the sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$ is a subsequence of zeros. By virtue of Corollary 2, the function Q is of the kind (1). Due to Lemma 1, we have $\left|Q(x)\right| \leq c_1\left(1+|x|\right)^2$ for all $x \in \mathbb{R}$. This implies

$$\int_{-\infty}^{+\infty}\frac{\log^+\left|Q(x)\right|}{1+x^2}\,dx < +\infty.$$

Therefore, by Lemma 2, we get

$$\sum_{k\in\mathbb{N}}\left|\mathrm{Im}\,\frac{1}{\rho_k}\right| < +\infty.$$

Since $|\mathrm{Im}\,\rho_k| \geq \delta\,|\rho_k|$, $\delta > 0$, for all $k \in \mathbb{N}$, and

$$\left|\mathrm{Im}\,\frac{1}{\rho_k}\right| = \frac{|\mathrm{Im}\,\rho_k|}{|\rho_k|^2} \geq \frac{\delta}{|\rho_k|},$$

we have

$$\sum_{k=1}^{\infty}\frac{1}{|\rho_k|} < +\infty.$$

This contradicts condition (5). Thus, the theorem is proved. □

Theorem 4. *Let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers such that $\rho_k^2 \neq \rho_m^2$ for $k \neq m$. Let a sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$, where $\rho_{-k} := -\rho_k$, be a sequence of zeros of some even entire function G of exponential type $\sigma \leq 1$ for which on the rays $\{z : \arg z = \varphi_j\}$, $j \in \{1;2;3;4\}$, $\varphi_1 \in [0;\pi/2)$, $\varphi_2 \in [\pi/2;\pi)$, $\varphi_3 \in (\pi;3\pi/2]$, $\varphi_4 \in (3\pi/2;2\pi)$, we have*

$$\left|G(z)\right| \geq c_2\left(1+|z|\right)^2 e^{|\,\mathrm{Im}\,z|}.$$

Then the system $\{\psi_k : k \in \mathbb{N}\}$ is complete in $L^2\left((0;1);x^4dx\right)$.

Proof. Assume the converse. Then, according to Theorem 2, there exists a nonzero even entire function $Q \in E_{4,+}$ for which the sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$ is a subsequence of zeros.

Let $V(z) = Q(z)/G(z)$. Then V is an even entire function of order $\tau \leq 1$, for which by Corollary 2 and Lemma 1, we obtain

$$\left|V(z)\right| \leq c_3\frac{1}{\sqrt{1+|\,\mathrm{Im}\,z|}}, \quad \arg z = \varphi_j, \quad j \in \{1;2;3;4\}.$$

Therefore, according to the Phragmén-Lindelöf theorem (see [8, p. 38], [9, p. 4263]), we get $V(z) \equiv 0$. Hence $Q(z) \equiv 0$. This contradiction proves the theorem. □

Corollary 3. *Let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of zeros of the function $J_{-5/2}$. Then the system $\{\psi_k : k \in \mathbb{N}\}$ is complete in $L^2\left((0;1); x^4 dx\right)$.*

Proof. Indeed, a sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$, where $\rho_{-k} = -\rho_k$, is a sequence of zeros of an entire function $G(z) = -z^2\cos z + 3z\sin z + 3\cos z$, and this function satisfies the conditions of Theorem 4. Therefore, a system $\{\psi_k : k \in \mathbb{N}\}$ is complete in $L^2\left((0;1); x^4 dx\right)$. Corollary 3 is proved. □

Theorem 5. *Let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers such that $\rho_k^2 \neq \rho_m^2$ for $k \neq m$. Let a sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$, where $\rho_{-k} := -\rho_k$, be a sequence of zeros of some even entire function $G \notin E_{4,+}$ of exponential type $\sigma \leq 1$ for which on the rays $\{z : \arg z = \varphi_j\}$, $j \in \{1;2;3;4\}$, $\varphi_1 \in [0;\pi/2)$, $\varphi_2 \in [\pi/2;\pi)$, $\varphi_3 \in (\pi;3\pi/2]$, $\varphi_4 \in (3\pi/2;2\pi)$, the inequality*

$$\left|G(z)\right| \geq c_4\left(1+|z|\right)^{-\alpha} e^{|\operatorname{Im} z|}$$

holds with $\alpha < 1/2$. Then the system $\left\{\psi_k : k \in \mathbb{N}\right\}$ is complete in $L^2\left((0;1); x^4 dx\right)$.

Proof. Assume the converse. Then, according to Theorem 2, there exists a nonzero even entire function $Q \in E_{4,+}$ for which the sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$ is a subsequence of zeros.

Let $V(z) = Q(z)/G(z)$. Then V is an even entire function of order $\tau \leq 1$, for which by Corollary 2 and Lemma 1, we get

$$\left|V(z)\right| \leq c_5 \frac{\left(1+|z|\right)^{\alpha+2}}{\sqrt{1+|\operatorname{Im} z|}}, \quad \arg z = \varphi_j, \quad j \in \{1;2;3;4\}.$$

Since $\alpha + 2 < 5/2$, according to the Phragmén-Lindelöf theorem, the function V is a polynomial of degree $\zeta < 2$. However, V is an even entire function, and therefore the function V is a constant. Hence, $Q(z) = c_6 G(z)$ and $Q \notin E_{4,+}$. Thus, we have a contradiction and the proof of the theorem is completed. □

Theorem 6. *Let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers such that $\rho_k^2 \neq \rho_m^2$ for $k \neq m$. Let a sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$, where $\rho_{-k} := -\rho_k$, be a sequence of zeros of some even entire function $F \notin E_{4,+}$ of exponential type $\sigma \leq 1$ such that*

$$\left|F(x+i\eta)\right| \geq \delta|x|^{-\alpha}, \quad \delta > 0, \quad |x| > 1, \tag{6}$$

for some $\alpha < 0$ and $\eta \in \mathbb{R}$. Then the system $\{\psi_k : k \in \mathbb{N}\}$ is complete in $L^2\left((0;1); x^4 dx\right)$.

Proof. Let $F \notin E_{4,+}$ and the inequality (6) is true. Suppose, to the contrary, that the system $\{\psi_k : k \in \mathbb{N}\}$ is not complete in $L^2\left((0;1); x^4 dx\right)$. Then, by Theorem 2, there exists a nonzero even entire function $Q \in E_{4,+}$ which vanishes at the points ρ_k. However, the sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$ is a sequence of zeros of an even entire function $F(z) \notin E_{4,+}$ of exponential type $\sigma \leq 1$. Therefore, $T(z) = Q(z)/F(z)$ is an even entire function of order $\tau \leq 1$. Since $Q \in E_{4,+}$, taking into account Corollary 2 and Lemma 1, we obtain

$$\left|Q(x+i\eta)\right| \leq c_7 \frac{e^{|\eta|}}{\sqrt{1+|\eta|}} \left(1+\sqrt{x^2+\eta^2}\right)^2, \quad x \in \mathbb{R}.$$

Using (6), we get

$$\left|T(x+i\eta)\right| \leq c_8\left(1+|x|\right)^{2+\alpha}, \quad x \in \mathbb{R}.$$

In view of this, we have that $T(z)$ is a polynomial of degree $\zeta < 2$. Further, since T is an even entire function, then $T(z) = c_9$. Furthermore, $F(z) = c_{10}Q(z)$ and $F(z) \in E_{4,+}$. This contradiction concludes the proof of the theorem. □

Let $n(t)$ be the number of points of the sequence $(\rho_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ satisfying the inequality $|\rho_k| \le t$, i.e. $n(t) := \sum\limits_{|\rho_k|\le t} 1$, and let

$$N(r) := \int_0^r \frac{n(t)}{t}\,dt, \quad r > 0.$$

Theorem 7. *Let $(\rho_k)_{k\in\mathbb{N}}$ be an arbitrary sequence of distinct nonzero complex numbers. If*

$$\limsup_{r\to+\infty} \Big(N(r) - \frac{2r}{\pi} + \frac{1}{2}\log r - 2\log(1+r) \Big) = +\infty,$$

then the system $\{\psi_k : k \in \mathbb{N}\}$ is complete in $L^2\left((0;1); x^4 dx\right)$.

Proof. It suffices to assume the incompleteness of the system $\{\psi_k : k \in \mathbb{N}\}$ and prove that

$$\limsup_{r\to+\infty} \Big(N(r) - \frac{2r}{\pi} + \frac{1}{2}\log r - 2\log(1+r) \Big) < +\infty. \tag{7}$$

By virtue of Theorem 2, there exists a nonzero even entire function $Q \in E_{4,+}$ of exponential type $\sigma \le 1$ for which the sequence $(\rho_k)_{k\in\mathbb{N}}$ is a subsequence of zeros. We may consider that $Q(0) = 1$. Then, consecutively applying the Jensen formula (see [8, p. 10], [9, p. 4316]), Corollary 2 and Lemma 1, we obtain

$$\begin{aligned} N(r) &\le \frac{1}{2\pi}\int_0^{2\pi} \log\left|Q\left(re^{i\varphi}\right)\right| d\varphi \\ &\le c_{11} + \frac{1}{2\pi}\int_0^{2\pi} \Big(r|\sin\varphi| - \frac{1}{2}\log\left(1 + r|\sin\varphi|\right) + 2\log(1+r) \Big) d\varphi \\ &\le c_{11} + \frac{1}{2\pi}\int_0^{2\pi} \Big(r|\sin\varphi| - \frac{1}{2}\log r - \frac{1}{2}\log|\sin\varphi| + 2\log(1+r) \Big) d\varphi \\ &= \frac{2r}{\pi} - \frac{1}{2}\log r + 2\log(1+r) + c_{12}, \quad r > 0, \end{aligned}$$

whence it follows (7). The theorem is proved. □

Theorem 8. *Let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers. Assume that $|\rho_k| \le \Delta k + \beta + \alpha_k$ for $0 < \Delta < \frac{\pi}{2+\pi}$, $-\Delta < \beta < 1 - \frac{2\Delta}{\pi}(1+\pi)$. Let a sequence $(\alpha_k)_{k\in\mathbb{N}}$ be such that $\alpha_k \ge 0$, $\alpha_k = O(1)$ as $k \to +\infty$ and*

$$\sum_{k=1}^{\infty} |\alpha_{k+1} - \alpha_k| < +\infty, \qquad \sum_{k=1}^{\infty} \frac{\alpha_k}{k} < +\infty.$$

Then the system $\{\psi_k : k \in \mathbb{N}\}$ is complete in $L^2\left((0;1); x^4 dx\right)$.

Proof. Let $\mu_k = \Delta k + \beta + \alpha_k$, $k \in \mathbb{N}$, and

$$n_1(t) = \sum_{\mu_k \le t} 1, \qquad N_1(r) = \int_0^r \frac{n_1(t)}{t}\,dt, \quad r > 0.$$

Then $n(t) \ge n_1(t)$, $N(r) \ge N_1(r)$ and $n_1(t) = m$ for $\Delta m + \beta + \alpha_m \le t < \Delta(m+1) + \beta + \alpha_{m+1}$ $\left(n_1(t) = 0 \text{ on } (0;\mu_1)\right)$. Let $r \in [\mu_s; \mu_{s+1})$. Then $s = \frac{r}{\Delta} + O(1)$ as $r \to +\infty$.

Therefore, under the assumptions of the theorem, by analogy with [7, p. 894] (see also [6, p. 9]), we obtain

$$N_1(r) \geq \sum_{k=1}^{s-1} k \log \frac{\Delta(k+1)+\beta}{\Delta k+\beta} + O(1) - \left| \sum_{k=1}^{s-1} k \left(\log \frac{\Delta(k+1)+\beta+\alpha_{k+1}}{\Delta k+\beta+\alpha_k} - \log \frac{\Delta(k+1)+\beta}{\Delta k+\beta} \right) \right|$$
$$\geq \frac{r}{\Delta} - \left(\frac{1}{2}+\frac{\beta}{\Delta}\right) \log r - c_{13} \sum_{k=1}^{\infty} \left(|\alpha_{k+1}-\alpha_k| + \frac{\alpha_k}{k} \right) + O(1) \geq \frac{r}{\Delta} - \left(\frac{1}{2}+\frac{\beta}{\Delta}\right) \log r + O(1),$$

as $r \to +\infty$. In view of this, for $0 < \Delta < \frac{\pi}{2+\pi}$ and $-\Delta < \beta < 1 - \frac{2\Delta}{\pi}(1+\pi)$, we get

$$\limsup_{r\to+\infty} \Big(N(r) - \frac{2r}{\pi} + \frac{1}{2}\log r - 2\log(1+r) \Big)$$
$$\geq \limsup_{r\to+\infty} \Big(N_1(r) - \frac{2r}{\pi} + \frac{1}{2}\log r - 2\log(1+r) \Big)$$
$$\geq \limsup_{r\to+\infty} \left(\frac{r}{\Delta} - \left(\frac{1}{2}+\frac{\beta}{\Delta}\right)\log r - \frac{2r}{\pi} + \frac{1}{2}\log r - 2\log(1+r) + O(1) \right)$$
$$\geq \limsup_{r\to+\infty} \left(r\left(\frac{1}{\Delta} - \frac{2}{\pi}\right) - \left(\frac{\beta}{\Delta}+2\right)\log(1+r) + O(1) \right)$$
$$\geq \limsup_{r\to+\infty} \left(r\left(\frac{1}{\Delta} - \frac{2}{\pi} - \frac{\beta}{\Delta} - 2\right) + O(1) \right) = +\infty.$$

Finally, according to Theorem 7, we obtain the required proposition. The proof of theorem is completed. ☐

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Хаць Р.В. *Повнота систем функцій Бесселя індексу* $-5/2$ // Карпатські матем. публ. — 2024. — Т.16, №1. — C. 93–102.

Нехай $L^2((0;1);x^4dx)$ — ваговий простір Лебега всіх вимірних функцій $f:(0;1)\to\mathbb{C}$, для яких $\int_0^1 t^4|f(t)|^2\,dt<+\infty$, $J_{-5/2}$ — функція Бесселя першого роду індексу $-5/2$ і $(\rho_k)_{k\in\mathbb{N}}$ — послідовність різних відмінних від нуля комплексних чисел. Знайдено необхідні та достатні умови повноти системи $\{\rho_k^2\sqrt{x\rho_k}J_{-5/2}(x\rho_k):k\in\mathbb{N}\}$ у просторі $L^2((0;1);x^4dx)$ в термінах цілої функції, множина нулів якої співпадає з послідовністю $(\rho_k)_{k\in\mathbb{N}}$. При цьому, досліджено інтегральне зображення деякого класу $E_{4,+}$ парних цілих функцій експоненційного типу $\sigma\le 1$. Це доповнює аналогічні результати Б. Винницького, В. Дільного, О. Шавали та автора статті про апроксимаційні властивості систем функцій Бесселя з від'ємним півцілим індексом, меншим за -1.

Ключові слова і фрази: функція Бесселя, теорема Пелі-Вінера, теорема Фрагмена-Ліндельофа, теорема Фубіні, теорема Гурвіца, теорема Гана-Банаха, формула Єнсена, ціла функція експоненційного типу, повна система.