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Application of the method of averaging to boundary value problems for differential equations with non-fixed moments of impulse

Stanzhytskyi O.M.[1,✉], Uteshova R.E.[2], Mukash M.[3], Mogylova V.V.[4]

The method of averaging is applied to study the existence of solutions of boundary value problems for systems of differential equations with non-fixed moments of impulse action. It is shown that if an averaged boundary value problem has a solution, then the original problem is solvable as well. Here the averaged problem for the impulsive system is a simpler problem of ordinary differential equations.

Key words and phrases: small parameter, averaging method, fixed point, impulse action, boundary value problem.

[1] Taras Shevchenko National University of Kyiv, 64/13 Volodymyrska str., 01601, Kyiv, Ukraine
[2] International Information Technology University, 34/1 Manas str., 050040, Almaty, Kazakhstan
[3] K. Zhubanov Aktobe Regional University, 34 A. Moldagulova ave., 030000, Aktobe, Kazakhstan
[4] National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", 37 Peremohy ave., 03056, Kyiv, Ukraine
✉ Corresponding author
E-mail: ostanzh@gmail.com (Stanzhytskyi O.M.), ruteshova1@gmail.com (Uteshova R.E.), mukashma1983@gmail.com (Mukash M.), mogylova.viktoria@gmail.com (Mogylova V.V.)

Introduction

We consider the following boundary value problem for a system of differential equations with impulse action at non-fixed moments of time and a small parameter:

$$\begin{aligned} \dot{x} = \varepsilon X(t,x), \quad t \neq t_i(x), \\ \Delta x|_{t=t_i(x)} = \varepsilon I_i(x), \\ F(x(0), x(T/\varepsilon)) = 0. \end{aligned} \tag{1}$$

Here $\varepsilon > 0$ is a small parameter, $T > 0$ is fixed, $t_i(x) < t_{i+1}(x)$, $i = 1, 2, \dots$, are the moments of impulse action, X, I_i, and F are d-dimensional vector functions.

Definition 1. *A function $x(t) : [0, T/\varepsilon] \to \mathbb{R}^d$ is a solution of system* (1) *if the following conditions are satisfied:*

(i) the set $A = \{t \in (0, T/\varepsilon], t = t_i(x(t))$ for some $i\}$ of the points of impulse action is finite (possibly empty);

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(ii) $x(t)$ *is continuously differentiable for all* $t \in (0, T/\varepsilon] \setminus A$*;*

(iii) $x(t)$ *satisfies the first equation in* (1) *for all* $t \in (0, T/\varepsilon] \setminus A$*;*

(iv) $x(t)$ *satisfies the equality* $\triangle x = x(t+0) - x(t) = \varepsilon I_i(x(t))$ *for all* $t \in A$*.*

In addition, if the function $x(t)$ *satisfies the condition* $F(x(0), x(T/\varepsilon)) = 0$*, then it is a solution of the boundary value problem* (1)*.*

We assume that $x(t)$ is left continuous, so that $x(t) = x(t-0)$ for all $t \in A$.
Assuming that there exist the limits

$$X_0(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T X(t,x)\,dt, \qquad I_0(x) = \lim_{T\to\infty} \frac{1}{T} \sum_{0<t_i(x)<T} I_i(x), \tag{2}$$

we put the problem (1) in correspondence to the averaged boundary value problem

$$\begin{aligned} \dot{y} &= \varepsilon[X_0(y) + I_0(y)], \\ F(y(0), &y(T/\varepsilon)) = 0, \end{aligned} \tag{3}$$

or, on the slow time scale $\tau = \varepsilon t$,

$$\frac{dy}{d\tau} = X_0(y) + I_0(y), \qquad F(y(0), y(T)) = 0. \tag{4}$$

The aim of this paper is to prove that if the averaged boundary value problem (3) has a solution, then, for small values of the parameter ε, the original boundary value problem (1) also has a solution that belongs to a small neighborhood of the solution of the averaged problem. The exact statement of the problem and the formulation of the main result are provided in Section 1.

Impulsive systems of differential equations serve as mathematical models of objects that, in the course of their evolution, are exposed to the action of short-term forces. A fairly complete theory of such systems is presented in the monograph [17]. In our paper, we will use the notation and some facts from this monograph. The study of real-life problems with state-dependent impulse effects can be found, for example, in [2, 6, 9].

Much research has been done on non-fixed impulsive initial value problems. For these problems, the existence, stability and other asymptotic properties of solutions are studied in [1, 4] and many other papers. However, in regard to boundary value problems for equations with impulse action, the majority of results concern jumps only at fixed times. This is due to the fact that non-fixed impulses significantly change properties of boundary value problems, which is explained in detail in [12].

To our knowledge, the first results on boundary value problems for non-fixed impulsive systems were obtained in [16] for a periodic case. In order to study periodic solutions of weakly nonlinear impulsive systems with non-fixed moments of time, the authors proposed a method that reduces this problem to a family of similar problems with fixed moments of impulse action. This technique was successfully applied to the study of almost periodic solutions in [3,5]. However, this method assumes that the original system has a roughly linear part and a small nonlinearity, which makes it possible to search for solutions in the integral representation using Green's function. We do not assume such a structure in system (1).

Another approach to the study of this kind of boundary value problems was proposed in [11, 13, 14]. It is based on the ideas of the Samoilenko numerical-analytical method. The authors not only prove the existence of solutions to boundary value problems, but also develop constructive approximation schemes for their search. However, it should be said that the authors seek a solution of a boundary value problem with a predetermined number of impulse actions. This number is included in the conditions of the theorems, which makes it much more difficult to verify these conditions if the number of impulses increases.

In the present paper, we use the averaging method to solve the boundary value problem (1). This method is one of the most widespread and effective methods for the analysis of nonlinear dynamical systems. The averaging method, proposed by Krylov and Bogolyubov originally for ordinary differential equations, was later developed and applied to various problems. For example, in [20] it was applied to the study of the existence of bounded on the entire axis solutions of non-autonomous systems of differential equations. The method was employed to the study of optimal control problems [7, 8, 10, 23, 25] and invariant sets of stochastic systems [21, 22].

The averaging method was successfully applied to boundary value problems for systems of differential equations in [19]. It made it possible to reduce the solution of a boundary value problem for a non-autonomous multifrequency system to the study of a similar problem for an autonomous averaged system. In [24,26], this method was applied to boundary value problems for systems of integro-differential equations.

The outline of the proof of our main result is as follows.

Step I. We first consider the boundary value problem for the system with impulse effect at fixed moments t_i on $[0, T/\varepsilon]$:

$$\begin{aligned} \dot{x} &= \varepsilon X(t,x), \quad t \neq t_i, \\ \Delta x|_{t=t_i} &= \varepsilon I_i(x(t_i)), \\ F(x(0), x(T/\varepsilon)) &= 0. \end{aligned} \tag{5}$$

By using the Samoilenko theorem on averaging of impulsive systems [15] and the idea proposed in [19], we prove the existence and uniqueness of a solution of the boundary value problem (5).

Step 2. Let us fix p points $y_1, y_2, \dots, y_p$ in some neighborhood of a solution of the averaged problem (4) and consider the following boundary value problem:

$$\begin{aligned} \dot{x} &= \varepsilon X(t,x), \quad t \neq t_i(y_i), \\ \Delta x|_{t=t_i(y_i)} &= \varepsilon I_i(x(t_i(y_i))), \\ F(x(0), x(T/\varepsilon)) &= 0. \end{aligned}$$

From Step 1, we conclude that this problem, for a sufficiently small ε, has a unique solution $x(t, y_1, \dots, y_p)$. If we choose $y_1, \dots, y_p$, so that

$$y_i = x(t_i(y_i), y_1, \dots, y_p), \quad i = \overline{1,p}, \tag{6}$$

then the function $x(t, y_1, \dots, y_p)$ is the desired solution of problem (1). To prove the existence of $y_1, \dots, y_p$, as a solution of system (6), we use a fixed-point theorem.

Note that the idea suggested in Step 2 was previously used in [3, 5, 16]. However, as mentioned above, the presence of the rough linear part in the system enabled the authors to make use of the integral representation via Green's functions to prove the existence of a solution of system (6). In our case, such a representation cannot be written, therefore the solvability of (6) is proved by a different method. This proof is a distinguishing feature of our paper.

The remainder of this paper is structured as follows. In Section 1, we present the statement of the problem and formulate the main results. Section 2 provides the auxiliary results regarding the dependence of impulsive systems on initial data and solving a boundary value problem for systems with fixed moments of impulse action. The main result is proved in Section 3. The last section provides some illustrative examples.

1 Problem statement and the main result

The following notation will be used: $|\cdot|$ is the norm in $\mathbb{R}^d$, and $\|\cdot\|$ is the norm of a matrix that is consistent with that of a vector. We set $U_a = \{x \in \mathbb{R}^d : |x| \le a\}$.

We consider problem (1) under the assumption that the following conditions are satisfied.

1.1. The functions $X(t,x)$ and $I_i(x)$ are continuous in a set $Q = \{t \ge 0, x \in U_a\}$, bounded by a constant $M > 0$ and, with respect to x, satisfy the Lipschitz condition with a constant $L > 0$.

1.2. There exist uniform (in $x \in U_\rho$) limits (2).

1.3. The averaged boundary value problem (4) has a solution $y = y(\tau) = y(\varepsilon, \tau)$ that belongs to U_a with some ρ-neighborhood. In this neighborhood, the function $X(t,x)$ has partial derivatives $\frac{\partial X(t,x)}{\partial x}$ that are unifromly continuous with respect to x, and the functions $I_i(x)$ have partial derivatives $\frac{\partial I_i(x)}{\partial x}$ that are uniformly continuous with respect to i. The functions $X_0(x)$, $I_0(x)$, and $F(x,y)$, in the indicated ρ-neighborhood, have continuous partial derivatives $\frac{\partial X_0(x)}{\partial x}$, $\frac{\partial I_0(x)}{\partial x}$, $\frac{\partial F(x,y)}{\partial x}$, $\frac{\partial F(x,y)}{\partial y}$, and

$$\det \frac{\partial F_0(x_0)}{\partial x_0} \ne 0, \tag{7}$$

where $x_0 = y(0)$, $F_0(x_0) = F(x_0, y(T, x_0))$.

1.4. There exist uniform (in $x \in U_a$) limits

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\partial X(t,x)}{\partial x}\, dt = \frac{\partial X_0(x)}{\partial x}, \qquad \lim_{T\to\infty} \frac{1}{T} \sum_{0<t_i(x)<T} \frac{\partial I_i(x)}{\partial x} = \frac{\partial I_0(x)}{\partial x}.$$

1.5. The moments $\{t_i(x)\}$ of impulse action are continuous functions in U_a, and the surfaces $t = t_i(x)$ satisfy the condition of separability, i.e.

$$\min_{x\in U_a} t_{i+1}(x) > \max_{x\in U_a} t_i(x), \quad i = 1, 2, \ldots. \tag{8}$$

We assume that there exists a constant $C > 0$ such that, for all $t > 0$ and $x \in U_a$,

$$i(t,x) \le Ct, \tag{9}$$

where $i(t,x)$ is the number of impulses on $(0,t)$.

It is also supposed that the solutions of system (1) cross each surface $t = t_i(x)$ at most once.

Remark 1. *Sufficient conditions for such behavior of solutions are well studied (see, for example, [5, 17, 18]).*

We need to impose an additional condition on the relative position of the curve $y = y(\varepsilon t)$ and the surfaces $t = t_i(x)$ for $t = \frac{T}{\varepsilon}$. For every ε, due to Condition 1.5, there are three possible relative positions of the plane $t = \frac{T}{\varepsilon}$ and the surfaces $t = t_i(x)$:

1) the plane $t = \frac{T}{\varepsilon}$ does not intersect any surface $t = t_i(x)$;

2) for some i, $t_i(x) \equiv \frac{T}{\varepsilon}$, $x \in U_a$;

3) for some i, the plane $t = \frac{T}{\varepsilon}$ intersects a surface $t = t_i(x)$.

It follows from (8) that the plane cannot intersect more than one of these surfaces. Let us denote $N_i(\varepsilon) = \{x \in U_a : \frac{T}{\varepsilon} = t_i(x)\}$.

Condition A. There exist $\mu > 0$ and $\nu > 0$ such that, if $N_i(\varepsilon) \neq \varnothing$ and $N_i(\varepsilon) \neq U_a$ for some $\varepsilon < \nu$, then

$$\rho(y(T), N_i(\varepsilon)) > \mu.$$

The main result of this paper is provided in the following statement.

Theorem 1. *Let Conditions 1.1–1.5 and Condition A hold. Then there exist $\varepsilon_0 > 0$ and $0 < \sigma_0 < \rho$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the boundary value problem* (1) *has a solution $x(t, \varepsilon)$ in the σ_0-neighborhood of $y(\varepsilon t)$, i.e. $|x(t, \varepsilon) - y(\varepsilon t)| < \sigma_0$, $t \in [0, T/\varepsilon]$, and*

$$\sup_{t \in [0, T/\varepsilon]} |x(t, \varepsilon) - y(\varepsilon t)| \to 0, \quad \varepsilon \to 0. \tag{10}$$

2 Auxiliary statements

In this section, we present a number of auxiliary results regarding the properties of solutions of systems with fixed moments of impulse action.

2.1 Continuous dependence on initial conditions

Let us consider the system of differential equations

$$\begin{aligned} \dot{x} &= X(t, x), \quad t \neq t_i(x), \\ \Delta x|_{t=t_i(x)} &= I_i(x), \end{aligned} \tag{11}$$

for $t \in [0, T]$ and $x \in U_a$. We will assume that system (11) satisfies Conditions 1.1 and 1.5.

By condition (9), there are no more than CT moments of impulse action on the interval $(0, T)$. Let us take $y_1, \ldots, y_p \in U_a$ and generate p moments of impulse action, namely $\{\tau_i(y_i)\}_1^p$. There is no loss of generality in assuming that $\tau_p(y_p) \leq T$.

Let $x(t, y)$, where $y = (y_1, \ldots, y_p)$, be a solution of the impulsive system with fixed moments

$$\begin{aligned} \dot{x} &= X(t, x), \quad t \neq t_i(y_i), \\ \Delta x|_{t=t_i(y_i)} &= I_i(x(t_i)), \end{aligned}$$

subject to the initial condition $x(0, y) = x(y)$. Let $x(t, z)$ be a solution of an analogous system, which is generated by using the set of impulse moments for a set $z = (z_1, \ldots, z_p)$, where $z_i \in U_a$, with the initial condition $x(0,z) = x(z)$. The continuity of $t_i(x)$ implies that $t_i(z_i) \to t_i(y_i)$ as $z \to y$. Hence, for z sufficiently close to y, the number of impulses $t_i(z_i)$ on $[0,T]$ is no less than $p-1$ and no greater than p.

Let $\underline{t_i} = \min\{t_i(y_i), t_i(z_i)\}$, $\overline{t_i} = \max\{t_i(y_i), t_i(z_i)\}$.

Lemma 1 (Continuous dependence). *Under Conditions 1.1 and 1.5, if $z \to y$ and $x(z) \to x(y)$, then*

$$\sup_{\substack{t \in (\overline{t_i}, \underline{t_{i+1}}] \\ i=\overline{1,p-1}}} |x(t,z) - x(t,y)| \to 0. \tag{12}$$

Proof. Without loss of generality, we can assume that $t_i(y_i) \le t_i(z_i)$, $i = \overline{1,p}$. It is obvious that the following inequality holds on $[0, t_1(y_1)]$:

$$|x(t,y) - x(t,z)| \le |x(y) - x(z)| e^{LT}. \tag{13}$$

Let us now estimate the difference between the solutions on $(t_1(z_1), t_2(y_2)]$. We have

$$x(t,y) = x(t_1(z_1), y) + \int_{t_1(z_1)}^{t} X(s, x(s,y))\, ds,$$

$$x(t,z) = x(t_1(z_1), z) + I_1(x(t_1(z_1)), z) + \int_{t_1(z_1)}^{t} X(s, x(s,z))\, ds.$$

In addition,

$$x(t_1(z_1), y) = x(t_1(y_1), y) + I_1(x(t_1(y_1)), y) + \int_{t_1(y_1)}^{t_1(z_1)} X(t, x(t,y))\, dt.$$

Let us estimate the difference $x(t_1(z_1), z) - x(t_1(y_1), y)$. We have

$$|x(t_1(z_1), z) - x(t_1(y_1), y)| \le |x(t_1(z_1), z) - x(t_1(y_1), z)| + |x(t_1(y_1), z) - x(t_1(y_1), y)|.$$

But

$$x(t_1(z_1), z) = x(t_1(y_1), z) + \int_{t_1(y_1)}^{t_1(z_1)} X(t, x(t,z))\, dt,$$

which, due to Condition 1.1, implies

$$|x(t_1(z_1), z) - x(t_1(y_1), z)| \le M |t_1(z_1) - t_1(y_1)|. \tag{14}$$

Then from (13) and (14) we get

$$|x(t_1(z_1), z) - x(t_1(y_1), y)| \to 0, \quad z \to y. \tag{15}$$

So, we have

$$\begin{aligned} |x(t,y) - x(t,z)| \le{}& |x(t_1(z_1), z) - x(t_1(y_1), y)| + |I_1(x(t_1(y_1), y)) - I_1(x(t_1(z_1), z))| \\ &+ \Big| \int_{t_1(y_1)}^{t_1(z_1)} X(t, x(t,y))\, dt \Big| + \int_{t_1(z_1)}^{t} \big| X(s, x(s,y)) - X(s, x(s,z)) \big|\, ds \\ \le{}& |X(t_1(z_1), z) - X(t_1(y_1), y)| + L|X(t_1(y_1), y) - X(t_1(z_1), z)| \\ &+ M|t_1(z_1) - t_1(y_1)| + L \int_{t_1(z)}^{t} \big| X(s,y) - X(s,z) \big|\, ds. \end{aligned}$$

Therefore, using Gronwall's inequality and taking into account (15) and the continuity of $t_1(x)$, we obtain (12) on the interval $(t_1(z_1), t_2(y_2))$. On the subsequent intervals, the proof is obtained in a similar way. Now, taking into account the finite number of the intervals $(\overline{t_i}, \underline{t_{i+1}}]$ on $[0,T]$, we complete the proof of Lemma 1. □

Let us now study the continuous differentiability of a solution with respect to initial data for the following system with fixed moments of impulse action:

$$\begin{aligned} \dot{x} &= X(t,x), \quad t \neq t_i, \\ \Delta x|_{t=t_i} &= I_i(x). \end{aligned} \tag{16}$$

Here $t \in [0,T]$, $x \in U_a$, and $t_i < t_{i+1}$ are the moments of impulse on $[0,T]$. We will assume that the number of such moments is finite on $[0,T]$.

Let $x(t,x_0)$ be a solution of system (16) on $[0,T]$ subject to the initial condition $x(0,x_0) = x_0$.

Lemma 2 (Continuous differentiability with respect to initial data)**.** *Let system* (16) *satisfy Conditions 1.1 and 1.5, and the functions $X(t,x)$ and $I_i(x)$ are continuously differentiable with respect to x for $t \in [0,T]$, $x \in U_a$. Then the solution $x(t,x_0)$ is continuously differentiable with respect to x_0 and the function $z(t) = \frac{\partial X(t,x_0)}{\partial x_0}$ satisfies the linear variational equation with impulse action*

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial X(t,x(t,x_0))}{\partial x} z, \quad t \neq t_i, \\ \Delta z|_{t=t_i} &= \frac{\partial I_i(x(t,x_0))}{\partial x} z(t_i). \end{aligned}$$

The proof of Lemma 2 is standard and carried out similarly to the case of ordinary differential equations; see also [17].

2.2 Averaging of the variational equation

We consider the system of differential equations with impulse action at fixed moments of time t_i:

$$\begin{aligned} \dot{x} &= \varepsilon X(t,x), \quad t \neq t_i, \\ \Delta x|_{t=t_i} &= \varepsilon I_i(x(t_i)), \end{aligned} \tag{17}$$

here $i = 1,2,\dots$; $\varepsilon > 0$ is a small parameter. Let the following conditions be met.

2.1. The functions $X(t,x)$ and $I_i(x)$ satisfy Conditions 1.1 for $t \geq 0$ and $x \in D$; D is a domain in $\mathbb{R}^d$.

2.2. There exist uniform (in $x \in D$) limits

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X(t,x)\,dt = X_0(x), \qquad \lim_{t\to\infty} \frac{1}{T} \sum_{0<t_i<T} I_i(x) = I_0(x).$$

2.3. There exists a constant $C > 0$ such that

$$i(t) \leq Ct, \tag{18}$$

where $i(t)$ is the number of impulses on $(0,t)$.

2.4. The averaged system

$$\dot{y} = \varepsilon[X_0(y) + I_0(y)] \tag{19}$$

has a solution $y = y(\varepsilon t, x_0)$, $y(0,x_0) = x_0 \in D$, that, for $\varepsilon = 1$ and $t \in [0,T]$, belongs to D together with some ρ-neighborhood.

The next proposition follows directly from the classical Samoilenko theorem on the averaging in impulsive systems [15].

Proposition 1. *Let Conditions 2.1–2.4 hold. Then, for all $\eta > 0$, there exists $\varepsilon_0(\eta) > 0$ such that for all $\varepsilon < \varepsilon_0$ the system of equations* (17) *has a unique solution $x(t, x_0)$ $(x(0, x_0) = x_0)$, defined for $t \in [0, T/\varepsilon]$ and satisfying the inequality*

$$|x(t, x_0) - y(\varepsilon t, x_0)| \leq \eta, \quad t \in [0, T/\varepsilon]. \tag{20}$$

Remark 2. *Condition 2.2 implies the existence of a continuous function $\varphi(t)$ that monotonically tends to zero as $t \to \infty$ such that*

$$\Big| \int_0^t X(s, x)\, ds - X_0(x)t \Big| + \Big| \sum_{0<t_i(x)<t} I_i(x) - I_0(x)t \Big| \leq \varphi(t)t. \tag{21}$$

Further, it follows from the proof of the above-mentioned Samoilenko theorem that ε_0 in Proposition 1 does not depend on the position of the points of impulse action $\{t_i\}$ on $[0, T/\varepsilon]$, but depends only on the constant C from the majorant estimate (18) *and the function $\varphi(t)$ from* (21).

Remark 3. *The solution $x(t, x_0)$, obviously, depends on ε. However, for convenience, we will omit this dependence in the notation.*

Suppose that, in addition to 2.1–2.4, the following condition is met.

2.5. The functions $X(t, x)$ and $I_i(x)$ are continuously differentiable for $t \geq 0$, $x \in D$, and there exist uniform (in $x \in D$) limits

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\partial X(t, x)}{\partial x}\, dt = \frac{\partial X_0(x)}{\partial x}, \qquad \lim_{T\to\infty} \frac{1}{T} \sum_{0<t_i<T} \frac{\partial I_i(x)}{\partial x} = \frac{\partial I_0(x)}{\partial x}. \tag{22}$$

Theorem 2. *(On the averaging of variational equations). Let Conditions 2.1–2.5 hold, and let $\frac{\partial X(t,x)}{\partial x}$ and $\frac{\partial I_i(x)}{\partial x}$ be Lipschitz functions with respect to x with a constant L in the domain $t \geq 0$, $x \in D$. Then, for any $\eta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\eta) > 0$ such that, for $\varepsilon \in (0, \varepsilon_0]$, the derivatives with respect to initial data of solutions of the exact equation* (17) *and the averaged equation* (19) *satisfy the following inequality:*

$$\left\| \frac{\partial x(t, z)}{\partial z}\Big|_{z=x_0} - \frac{\partial y(\varepsilon t, z)}{\partial z}\Big|_{z=x_0} \right\| \leq \eta, \quad t \in [0, T/\varepsilon]. \tag{23}$$

Proof. Let us denote $z(t) = \frac{\partial x(t,z)}{\partial z}\Big|_{z=x_0}$ and $z_1(t) = \frac{\partial y(\varepsilon t,z)}{\partial z}\Big|_{z=x_0}$. According to Lemma 2, $z(t)$ satisfies the linear variational matrix equation

$$\dot{z} = \varepsilon \frac{\partial X(t, x(t, x_0))}{\partial x} z, \quad t \neq t_i,$$
$$\Delta z\big|_{t=t_i} = \varepsilon \frac{\partial I_i(x(t_i, x_0))}{\partial x} z(t_i),$$

and $z_1(t)$ satisfies, respectively, the equation

$$\dot{z}_1 = \varepsilon \frac{\partial (X_0(y(\varepsilon t, x_0) + I_0(y(\varepsilon t, x_0)))}{\partial x} z_1. \tag{24}$$

It follows from Proposition 1 that there exists $\varepsilon_1 > 0$ such that, for $\varepsilon \in (0, \varepsilon_1]$, the solution $x(t, x_0)$ of the exact system belongs to the domain D for $t \in [0, T/\varepsilon]$. Therefore, for $x(t, x_0)$ the following estimate $|x(t, x_0)| \leq |x_0| + TM$, $t \in [0, T/\varepsilon]$, holds.

A similar estimate is valid for the averaged problem. The solution $z(t)$ admits the integro-summary representation

$$z(t) = z(0) + \varepsilon \int_0^t \frac{\partial X(s, x(s, x_0))}{\partial x} z(s)\, ds + \varepsilon \sum_{0<t_i<t} \frac{\partial I_i(x(t_i), x_0)}{\partial x} z(t_i),$$

and this solution, obviously, exists on the entire interval $[0, T/\varepsilon]$. The functions X and I_i are Lipschitz, hence, their partial derivatives $\frac{\partial X}{\partial x}$ and $\frac{\partial I_i}{\partial x}$ are bounded by the constant L for $t \in [0, T/\varepsilon]$. So, we get

$$\|z(t)\| \leq \|z(0)\| + \varepsilon \int_0^t L\|z(s)\|\, ds + \varepsilon \sum_{0<t_i<t} L\|z(t_i)\|.$$

Hence, due to an analogue of Gronwall's lemma [17], we obtain

$$\|z(t)\| \leq \|z(0)\|(1+\varepsilon L)^{i(t)} e^{\varepsilon L t}(1+\varepsilon L)^{C\frac{T}{\varepsilon}} \leq \|z(0)\| e^{LT + \frac{CT}{L}}. \tag{25}$$

Using the Gronwall lemma for the solution of system (24), we get $\|z_1(t)\| \leq \|z_1(0)\| e^{2LT}$.

Let us now fix $\eta > 0$ and estimate the difference $z(t) - z_1(t)$ on $[0, T/\varepsilon]$. Let $x(t, x_0) = x(t)$ and $y(\varepsilon t, x_0) = y(t)$. Then

$$\begin{aligned}
\|z(t) - z_1(t)\| &= \Big\|\varepsilon \int_0^t \tfrac{\partial X(s,x(s))}{\partial x} z(s)\, ds + \varepsilon \sum_{0<t_i<t} \tfrac{\partial I_i(x(t_i))}{\partial x} z(t_i) \\
&\quad - \varepsilon \int_0^t \tfrac{\partial X_0(y(s))}{\partial x} z_1(s)\, ds - \varepsilon \int_0^t \tfrac{\partial I_0(y(s))}{\partial x} z_1(s)\, ds\Big\| \\
&\leq \varepsilon \Big\| \int_0^t [\tfrac{\partial X(s,x(s))}{\partial x} - \tfrac{\partial X(s,y(s))}{\partial x}] z(s)\, ds\Big\| + \varepsilon \Big\| \sum_{0<t_i<t} [\tfrac{\partial I_i(x(t_i))}{\partial x} - \tfrac{\partial I_i(y(t_i))}{\partial x}] z(t_i)\Big\| \\
&\quad + \varepsilon \Big\| \int_0^t \tfrac{\partial X(s,y(s))}{\partial x} [z(s) - z_1(s)]\, ds\Big\| + \varepsilon \Big\| \int_0^t [\tfrac{\partial X(s,y(s))}{\partial x} - \tfrac{\partial X_0(y(s))}{\partial x}] z_1(s)\, ds\Big\| \\
&\quad + \varepsilon \Big\| \sum_{0<t_i<t} \tfrac{\partial I_i(y(t_i))}{\partial x} [z(t_i) - z_1(t_i)]\Big\| + \varepsilon \Big\| \sum_{0<t_i<t} \tfrac{\partial I_i(y(t_i))}{\partial x} z_1(t_i) - \int_0^t \tfrac{\partial I_0(y(s))}{\partial x} z_1(s)\, ds\Big\| \\
&= J_1 + J_2 + J_3 + J_4 + J_5 + J_6.
\end{aligned} \tag{26}$$

Let us estimate each term of the sum in (26). Taking into account that $\frac{\partial X}{\partial x}$ is Lipschitz and (25), we obtain that the first term in (26) admits the estimate

$$J_1 \leq \varepsilon \int_0^t L|x(s) - y(s)|\, ds \|z(0)\| e^{LT + \frac{CT}{L}}.$$

By Proposition 1, we can choose ε so that $|x(s) - y(s)|$ is arbitrarily small. Hence there exists $\varepsilon_2 \leq \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon_2]$ we have the estimate

$$J_1 \leq \frac{\eta}{b}, \tag{27}$$

here $b > 0$ is a constant that will be defined below.

The second term J_2 is estimated in a similar way:

$$J_2 \le \varepsilon \sum_{0<t_i<t} L|x(t_i) - y(t_i)| \, \|z(0)\| e^{LT+\frac{CT}{L}} \le \frac{\eta}{b}. \tag{28}$$

The term J_3, obviously, admits the estimate

$$J_3 \le \varepsilon L \int_0^t \|z(s) - z_1(s)\| \, ds. \tag{29}$$

To estimate the fourth term J_4, we perform the integration over the entire interval $[0, T/\varepsilon]$, assuming that the integrand is equal to zero to the right of t. Let us divide $[0, T/\varepsilon]$ into n equal parts by points $\{\tau_k\}_1^n$. Then J_4 takes the form

$$\begin{aligned} J_4 \le \varepsilon \Big\| \sum_{k=0}^{n-1} \Big[\int_{\tau_k}^{\tau_{k+1}} \Big(\frac{\partial X(s, y(s))}{\partial x} z_1(s) - \frac{\partial X(s, y(\tau_k))}{\partial x} \Big) z_1(\tau_k) \, ds \\ + \int_{\tau_k}^{\tau_{k+1}} \Big(\frac{\partial X_0(y(\tau_k))}{\partial x} z_1(\tau_k) - \frac{\partial X_0(y(s))}{\partial x} z_1(s) \Big) \, ds \Big] \\ + \varepsilon \sum_{k=0}^{n-1} \int_{\tau_k}^{\tau_{k+1}} \Big(\frac{\partial X(s, y(\tau_k))}{\partial x} - \frac{\partial X_0(y(\tau_k))}{\partial x} \Big) \, ds z_1(\tau_k) \Big\| = \|J_{41} + J_{42} + J_{43}\|. \end{aligned} \tag{30}$$

For J_{41} we obtain the estimate

$$\begin{aligned} \|J_{41}\| &= \varepsilon \sum_{k=0}^{n-1} \Big[\int_{\tau_k}^{\tau_{k+1}} \Big\| \Big(\frac{\partial X(s, y(s))}{\partial x} - \frac{\partial X(s, y(\tau_k))}{\partial x} \Big) z_1(s) + \frac{\partial X(s, y(\tau_k))}{\partial x} (z_1(s) - z_1(\tau_k)) \Big\| \, ds \Big] \\ &\le \varepsilon \sum_{k=0}^{n-1} \int_{\tau_k}^{\tau_{k+1}} \big(L|y(s) - y(\tau_k)| \, \|z_1(s)\| + L\|z_1(s) - z_1(\tau_k)\| \big) \, ds. \end{aligned}$$

But $y(t) = y(\tau_k) + \varepsilon \int_{\tau_k}^t [X_0(y(s)) + I_0(y(s))] \, ds$. So, we have $|y(t) - y(\tau_k)| \le \varepsilon 2M \frac{T}{\varepsilon n} = \frac{2MT}{n}$, $t \in [\tau_k, \tau_{k+1}]$, and, similarly,

$$\|z_1(s) - z_1(\tau_k)\| \le \varepsilon \int_{\tau_k}^s \Big\| \frac{\partial (X_0(y(t)) + I_0(y(t)))}{\partial x} z_1(t) \Big\| \, dt \le 2\tfrac{T}{n} L \|z_1(0)\| e^{2LT}.$$

Therefore,

$$\begin{aligned} \|J_{41}\| &\le \varepsilon \sum_{k=0}^{n-1} \big[\tfrac{LMT}{n} \tfrac{T}{\varepsilon n} \|z_1(0)\| e^{2LT} + 2L^2 \tfrac{T}{n} \|z_1(0)\| e^{2LT} \tfrac{t}{\varepsilon n} \big] \\ &= \tfrac{1}{n} \big(2LMT^2 \|z_1(0)\| e^{2LT} + 2L^2 T^2 \|z_1(0)\| e^{2LT} \big), \end{aligned} \tag{31}$$

$$\begin{aligned} \|J_{42}\| &\le \varepsilon \sum_{k=0}^{n-1} \int_{\tau_k}^{\tau_{k+1}} \Big\| \Big(\frac{\partial X_0(y(\tau_k))}{\partial x} - \frac{\partial X_0(y(s))}{\partial x} \Big) z_1(\tau_k) + \frac{\partial X_0(y(s))}{\partial x} (z_1(\tau_k) - z_1(s)) \Big\| \, ds \\ &\le \varepsilon \sum_{k=0}^{n-1} \int_{\tau_k}^{\tau_{k+1}} (L|y(\tau_k) - y(s)| \, \|z_1(\tau_k)\| + L\|z_1(\tau_k) - z_1(s)\|) \, ds \\ &\le \tfrac{2LMT^2}{n} \|z_1(0)\| e^{2LT} + \tfrac{2L^2T^2}{n} \|z_1(0)\| e^{2LT}. \end{aligned} \tag{32}$$

Let us now estimate the term J_{43}:

$$\|J_{43}\| \leq \|z_1(0)\|e^{2LT}\Big(\varepsilon\sum_{k=0}^{n-1}\Big\|\int_0^{\tau_{k+1}}\Big(\frac{\partial X(s,y(\tau_k))}{\partial x}-\frac{\partial X_0(y(\tau_k))}{\partial x}\Big)\,ds\Big\| \\ +\varepsilon\sum_{k=0}^{n-1}\Big\|\int_0^{\tau_k}\Big(\frac{\partial X(s,y(\tau_k))}{\partial x}-\frac{\partial X_0(y(\tau_k))}{\partial x}\Big)\,ds\Big\|\Big). \tag{33}$$

We notice that, due to the first condition in (22), we can specify a continuous function $\psi(t)$, approaching zero monotonically as $t\to\infty$, such that

$$\Big\|\int_0^t\Big(\frac{\partial X(s,x)}{\partial x}-\frac{\partial X_0(x)}{\partial x}\Big)\,ds\Big\| \leq t\psi(t), \tag{34}$$

uniformly in $x\in D$.

So, for (33), we get

$$\|z_1(0)\|e^{2LT}\Big(\varepsilon\sum_{k=0}^{n-1}\tau_{k+1}\psi(\tau_{k+1})+\sum_{k=0}^{n-1}\tau_k\psi(\tau_k)\Big) \leq \|z_1(0)\|e^{2LT}2nT\psi(\tfrac{T}{n\varepsilon}), \tag{35}$$

if t belongs to any segment $[\tau_k,\tau_{k+1}]$, except for the first one.

But if $t\in[0,\frac{T}{\varepsilon n}]$, then, by Dini's theorem, we have

$$\|J_{43}\| \leq \|z_1(0)\|e^{2LT}\varepsilon t\psi(t) = \|z_1(0)\|e^{2LT}\tau t\psi(\tau/\varepsilon) \leq \|z_1(0)\|e^{2LT}\sup_{\tau\in[0,T]}(\tau\psi(\tau/\varepsilon)) \to 0 \tag{36}$$

as $\varepsilon\to 0$. Therefore, by virtue of (30)–(36) we have the next estimate

$$\|J_4\| \leq \frac{1}{n}\big(4LMT^2||z_1(0)||e^{2LT}+4L^2T^2||z_1(0)||e^{2LT}\big) \\ +||z_1(0)||e^{2LT}2nT\psi\Big(\frac{T}{\varepsilon n}\Big)+||z_1(0)||e^{2LT}\sup_{\tau\in[0,T]}\Big(\tau\psi\Big(\frac{\tau}{\varepsilon}\Big)\Big). \tag{37}$$

The term J_5 is estimated in the same way as J_3:

$$||J_5|| \leq \varepsilon L\sum_{0<t_i<t}\|z(t_i)-z_1(t_i)\|. \tag{38}$$

Let us obtain the estimate for J_6 in (26). We have

$$\|J_6\| \leq \varepsilon\sum_{k=0}^{n-1}\Bigg[\sum_{\substack{\tau_k\leq t_i<\tau_{k+1}\\ t_i<t}}\frac{\partial I_i(y(t_i))}{\partial x}z_1(t_i)-\int_{\tau_k}^{\tau_{k+1}}\frac{\partial I_0(y(s))}{\partial x}z_1(s)\,ds\Bigg].$$

Let us now estimate each term in the last sum:

$$\varepsilon\Bigg(\sum_{\substack{\tau_k\leq t_i<\tau_{k+1}\\ t_i<t}}\frac{\partial I_i(y(t_i))}{\partial x}z_1(t_i)-\int_{\tau_k}^{\tau_{k+1}}\frac{\partial I_0(y(s))}{\partial x}z_1(s)\,ds\Bigg) \\ =\varepsilon\Bigg[\sum_{\substack{\tau_k\leq t_i<\tau_{k+1}\\ t_i<t}}\frac{\partial I_i(y(t_i))}{\partial x}z_1(t_i)-\frac{\partial I_0(y(\tau_k))}{\partial x}z_1(\tau_k)(\tau_{k+1}-\tau_k) \\ +\frac{\partial I_0(y(\tau_k))}{\partial x}z_1(\tau_k)(\tau_{k+1}-\tau_k)-\int_{\tau_k}^{\tau_{k+1}}\frac{\partial I_0(y(s))}{\partial x}z_1(s)\,ds\Bigg]. \tag{39}$$

But

$$
\begin{aligned}
&\Big\| \frac{\partial I_0(y(\tau_k))}{\partial x} z_1(\tau_k)(\tau_{k+1}-\tau_k) - \int_{\tau_k}^{\tau_{k+1}} \frac{\partial I_0(y(s))}{\partial x} z_1(s)\,ds \Big\| \\
&\qquad = \Big\| \int_{\tau_k}^{\tau_{k+1}} \Big[\frac{\partial I_0(y(\tau_k))}{\partial x} z_1(\tau_k) - \frac{\partial I_0(y(s))}{\partial x} z_1(s) \Big]\,ds \Big\| \\
&\qquad = \Big\| \int_{\tau_k}^{\tau_{k+1}} \Big[\Big(\frac{\partial I_0(y(\tau_k))}{\partial x} - \frac{\partial I_0(y(s))}{\partial x} \Big) z_1(\tau_k) + \frac{\partial I_0(y(s))}{\partial x} [z_1(\tau_k) - z_1(s)] \Big]\,ds \Big\| \\
&\qquad \le L \int_{\tau_k}^{\tau_{k+1}} \big(|y(\tau_k) - y(s)|\, \|z_1(\tau_k)\| + L\|z_1(\tau_k) - z_1(s)\| \big)\,ds \\
&\qquad \le \int_{\tau_k}^{\tau_{k+1}} \big(L\tfrac{2MT}{n} \|z_1(0)\| e^{2LT} + \tfrac{L^2 2T}{n} \|z_1(0)\| e^{2LT} \big)\,ds \\
&\qquad \le \big(\tfrac{2MLT}{n} \|z_1(0)\| e^{2LT} + \tfrac{2L^2T}{n} \|z_1(0)\| e^{2LT} \big) \tfrac{T}{n\varepsilon}.
\end{aligned} \tag{40}
$$

Let us estimate the first term in (39):

$$
\begin{aligned}
&\sum_{\substack{\tau_k \le t_i < \tau_{k+1} \\ t_i < t}} \frac{\partial I_i(y(t_i))}{\partial x} z_1(t_i) - \frac{\partial I_0(y(\tau_k))}{\partial x} z_1(\tau_k)(\tau_{k+1}-\tau_k) = \sum_{\substack{\tau_k \le t_i < \tau_{k+1} \\ t_i < t}} \frac{\partial I_i(y(t_i))}{\partial x} z_1(t_i) \\
&- \sum_{\substack{\tau_k \le t_i < \tau_{k+1} \\ t_i < t}} \frac{\partial I_i(y(\tau_k))}{\partial x} z_1(\tau_k) + \sum_{\substack{\tau_k \le t_i < \tau_{k+1} \\ t_i < t}} \frac{\partial I_i(y(\tau_k))}{\partial x} z_1(\tau_k) - \frac{\partial I_0(y(\tau_k))}{\partial x} z_1(\tau_k)(\tau_{k+1}-\tau_k).
\end{aligned} \tag{41}
$$

We now estimate the first difference in (41), representing it in the form

$$
\begin{aligned}
&\sum_{\substack{\tau_k \le t_i < \tau_{k+1} \\ t_i < t}} \Big\| \Big(\frac{\partial I_i(y(t_i))}{\partial x} - \frac{\partial I_i(y(\tau_k))}{\partial x} \Big) z_1(t_i) + \frac{\partial I_i(y(\tau_k))}{\partial x} (z_1(t_i) - z_1(\tau_k)) \Big\| \\
&\qquad \le \sum_{\substack{\tau_k \le t_i < \tau_{k+1} \\ t_i < t}} (L|y(t_i) - y(\tau_k)| \|z_1(t_i)\| + L\|z_1(t_i) - z_1(\tau_k)\| \\
&\qquad \le \sum_{\substack{\tau_k \le t_i < \tau_{k+1} \\ t_i < t}} \big(L\tfrac{2MT}{n} \|z_1(0)\| e^{2LT} + \tfrac{2LT}{n} \|z_1(0)\| e^{2LT} \big).
\end{aligned} \tag{42}
$$

Let us get the estimate for the second difference in (41):

$$
\begin{aligned}
&\Big\| \sum_{\substack{\tau_k \le t_i < \tau_{k+1} \\ t_i < t}} \Big[\frac{\partial I_i(y(\tau_k))}{\partial x} z_1(\tau_k) - \int_{\tau_k}^{\tau_{k+1}} \frac{\partial I_0(y(\tau_k))}{\partial x} z_1(\tau_k)\,ds \Big] \Big\| \\
&\qquad \le \varepsilon \Big(\Big\| \sum_{\substack{0 < t_i \le \tau_k \\ t_i < t}} \frac{\partial I_i(y(\tau_k))}{\partial x} - \int_0^{\tau_k} \frac{\partial I_0(y(\tau_k))}{\partial x}\,ds \Big\| \\
&\qquad\qquad + \Big\| \sum_{\substack{0 < t_i < \tau_{k+1} \\ t_i < t}} \frac{\partial I_i(y(\tau_k))}{\partial x} - \int_0^{\tau_{k+1}} \frac{\partial I_0(y(\tau_k))}{\partial x}\,ds \Big\| \Big) \|z_1(0)\| e^{2LT}.
\end{aligned} \tag{43}
$$

Due to the second condition in (22), the expressions in (43) admit the estimates analogous to (35). Namely, the first expression is no greater than $\tau_k \psi(\tau_k)$ and the second one is no greater than $\tau_{k+1}\psi(\tau_{k+1})$. Thus, due to (42) and (43), we get that (41) admits the estimate

$$
\Big(\sum_{\substack{\tau_k \le t_i < \tau_{k+1} \\ t_i < t}} \big(\tfrac{L2MT}{n} + \tfrac{2LM}{n} \big) + \tau_k \psi(\tau_k) + \tau_{k+1}\psi(\tau_{k+1}) \Big) \|z_1(0)\| e^{2LT}. \tag{44}
$$

From (40) and (44) we obtain that (39) is estimated by the following expression:

$$\Big[\big(\tfrac{2MLT}{n}+2L^2Tn\big)\tfrac{T}{n}+\sum_{\substack{\tau_k\le t_i<\tau_{k+1}\\ t_i<t}}\big(\tfrac{2LMT}{n}+\tfrac{2LM}{n}\big)+\tau_k\psi(\tau_k)+\tau_{k+1}\psi(\tau_{k+1})\Big]\|z_1(0)\|e^{2LT}.$$

Then, by (43) and (39), and taking into account (18), we get the following estimate for J_6:

$$\begin{aligned}\|J_6\| &\le \tfrac{2MLT+2L^2T}{n}T+\tfrac{2LMT+2LM}{n}\varepsilon\sum_{0<t_i<\frac{T}{\varepsilon}}1+\varepsilon\sum_{k=0}^{n-1}[\tau_k\psi(\tau_k)+\tau_{k+1}\psi(\tau_{k+1})]\\ &\le \tfrac{2MLT+2L^2T}{n}T+\tfrac{2LMT+2LM}{n}CT+2\varepsilon\sum_{k=0}^{n-1}\tfrac{T}{\varepsilon}\psi\big(\tfrac{T}{\varepsilon n}\big)\\ &=\tfrac{1}{n}[2MLT^2+2L^2T^2+(2LMT+2LM)CT]+2nT\psi\big(\tfrac{T}{\varepsilon n}\big).\end{aligned}\tag{45}$$

Let us choose n sufficiently large so that the first terms in (37) and (45) are less than η/b and fix this n. Then, by choosing $\varepsilon_3\le\varepsilon_2$ sufficiently small, for fixed n and all $\varepsilon\in(0,\varepsilon_3]$, we make the remaining terms in (37) and (45) less than η/b.

So, from (27)–(29), (37), (38), and (45) we obtain

$$\|z(t)-z_1(t)\|\le\tfrac{\eta}{b}+\tfrac{\eta}{b}+\varepsilon L\int_0^t\|z(s)-z_1(s)\|\,ds+\tfrac{\eta}{b}+\varepsilon L\sum_{0<t_i<t}\|z(t_i)-z_1(t_i)\|+\tfrac{\eta}{b}.$$

Now, by using the analogue of Gronwall's lemma [17] and making an appropriate choice of b, we finish the proof of Theorem 2. □

Remark 4. *It follows from the proof of Theorem 2 that, similarly to Remark 2, the estimates obtained there do not depend on the position of the points of impulse action $\{t_i\}$ on $[0,T/\varepsilon]$, but depend of constant C from* (18)*, and function $\psi(t)$ from* (34). *Hence ε_0 does not depend on the position of $\{t_i\}$ as well.*

Remark 5. *It can be easily shown, by making slight changes in the proof, that the statement of Theorem 2 remains true if we replace the Lipschitz conditions for $\frac{\partial X(t,x)}{\partial x}$ and $\frac{\partial I_i(x)}{\partial x}$, $i=1,2,\dots$, with the condition of their uniform, in $t\ge 0$, continuity with respect to x in the ρ-neighborhood of the averaged solution $y(\tau)$. We omitted the proof to avoid cumbersome calculations.*

2.3 The boundary value problem for fixed moments of impulse action

We consider the boundary value problem for system (17) subject to the boundary condition

$$F(x(0),x(T/\varepsilon))=0.\tag{46}$$

Theorem 3. *Suppose that Conditions 2.1–2.3 hold and, in addition, the following conditions are satisfied.*

2.6. *The averaged boundary value problem*

$$\dot{y} = \varepsilon[X_0(y) + I_0(y)],$$
$$F(y(0), y(T/\varepsilon) = 0,$$

has a solution $y = y(\varepsilon t) = y(\tau)$ that belongs to D together with some ρ-neighborhood, in which the function $X(t,x)$ has partial derivatives $\frac{\partial X(t,x)}{\partial x}$, continuous with respect to x in the ρ-neighborhood of $y(\tau)$, and the functions $I_i(x)$ have partial derivatives $\frac{\partial I_i(x)}{\partial x}$, uniformly continuous with respect to $i \in \mathbb{N}$. The functions $X_0(x)$, $I_0(x)$, and $F(x,y)$ have continuous partial derivatives $\frac{\partial X_0(x)}{\partial x}$, $\frac{\partial I_0(x)}{\partial x}$, $\frac{\partial F(x,y)}{\partial x}$, $\frac{\partial F(x,y)}{\partial y}$ in the indicated ρ-neighborhood, and

$$\det \frac{\partial F_0(x)}{\partial x_0} \neq 0,$$

where $x_0 = y(0), F_0(x_0) = F(x_0, y(T, x_0))$.

2.7. *The following limits exist for x in the ρ-neighborhood of $y(\tau)$:*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\partial X(t,x)}{\partial x}\, dt = \frac{\partial X_0(x)}{\partial x}, \qquad \lim_{T\to\infty} \frac{1}{T} \sum_{0<t_i<T} \frac{\partial I_i(x)}{\partial x} = \frac{\partial I_0(x)}{\partial x}.$$

Then there exists $\varepsilon_0 > 0$ and $\sigma_0 < \rho$ such that for $\varepsilon \in (0, \varepsilon_0)$ the boundary value problem (17), (46) *has a unique solution $x(t,\varepsilon)$ that belongs to the σ_0-neighborhood of $y(\varepsilon t)$, i.e.*

$$|x(t,\varepsilon) - y(\varepsilon t)| < \sigma_0, \quad t \in [0, T/\varepsilon], \quad \varepsilon \in (0, \varepsilon_0), \tag{47}$$

and

$$\sup_{t\in[0,T/\varepsilon]} |x(t,\varepsilon) - y(\varepsilon t)| \to 0, \quad \varepsilon \to 0. \tag{48}$$

Proof. Let $x_0 = y(0)$ be an initial value of the solution. We will seek the solution of (17), (46) of the form

$$x(t,\varepsilon) = x(t, x_0 + \bar{x}, \varepsilon), \tag{49}$$

where $\bar{x}$ is chosen from a neighborhood of zero. Let us consider the solution $y(\tau, x_0 + \bar{x})$ of the averaged problem. For the difference between $y(\tau)$ and $y(\tau, x_0 + \bar{x})$, by Gronwall's inequality, the following inequality

$$|y(\tau) - y(\tau, x_0 + \bar{x})| \leq |\bar{x}| e^{LT}, \tag{50}$$

holds until $y(\tau, x_0 + \bar{x})$ reaches the boundary of the domain D. So, if $|\bar{x}| \leq \frac{\rho}{2} e^{-LT}$, then the solution $y(\tau, x_0 + \bar{x})$ exists for $\tau \in [0, T]$ and belongs to the $\frac{\rho}{2}$-neighborhood of $y(\tau)$.

We determine the unknown parameter $\bar{x}$ in (49) from the equation

$$F\big(x_0 + \bar{x}, x(T/\varepsilon, x_0 + \bar{x}, \varepsilon)\big) = 0. \tag{51}$$

Note that, in view of Proposition 1, the solution $x(t, x_0 + \bar{x}, \varepsilon)$ of the exact system, for sufficiently small $\varepsilon > 0$, exists on the interval $[0, T/\varepsilon]$, and for any $\eta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\eta) > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, the next estimate holds

$$|x(t, x_0 + \bar{x}, \varepsilon) - y(\varepsilon t, x_0 + \bar{x})| < \eta(\varepsilon) \to 0, \quad \varepsilon \to 0. \tag{52}$$

Therefore, for $\varepsilon \in (0, \varepsilon_0)$, if ε_0 is sufficiently small, then the map in (51), as a map in $\bar{x}$, is well-defined in a ball $B_r(0)$, where $r = \frac{\rho}{2}e^{-LT}$. It should be also noted that, since $y(\tau)$ is a bounded function for $\tau \in [0, T]$, then, due to (50) and (52), $x(t, x_0 + \bar{x}, \varepsilon)$ belongs to a bounded domain for $t \in [0, T/\varepsilon]$, $\varepsilon \in (0, \varepsilon_0)$.

It follows from Condition 2.6 that system (17) satisfies the conditions of Lemma 2. Hence the map $F\big(x_0 + \bar{x}, x(T/\varepsilon, x_0 + \bar{x}, \varepsilon)\big)$, for $\bar{x} \in B_r(0)$, has uniformly continuous partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ and is continuously differentiable with respect to $\bar{x}$. Therefore, there exists a constant $N(\tau) > 0$ such that $\|\frac{\partial F}{\partial x}\| \le N(\tau)$ and $\|\frac{\partial F}{\partial y}\| \le N(\tau)$, for $\bar{x} \in B_r(0)$.

Next, we have

$$
\begin{aligned}
F\big(x_0 + \bar{x}, x(T/\varepsilon, x_0 + \bar{x}, \varepsilon)\big) - F(x_0 + \bar{x}, y(T, x_0)) &= F\big(x_0 + \bar{x}, x(T/\varepsilon, x_0 + \bar{x}, \varepsilon)\big) \\
&-F(x_0 + \bar{x}, y(T, x_0 + \bar{x}) + F(x_0 + \bar{x}, y(T, x_0 + \bar{x}) - F(x_0 + \bar{x}, y(T, x_0)).
\end{aligned}
$$

Let $R_1(\bar{x}, \varepsilon) = F\big(x_0 + \bar{x}, x(T/\varepsilon, x_0 + \bar{x}, \varepsilon)\big) - F\big(x_0 + \bar{x}, y(T, x_0 + \bar{x})\big)$. By (52), the following estimate holds:

$$
|R_1(\bar{x}, \varepsilon)| \le N(r)|x(\tfrac{T}{\varepsilon}, x_0 + \bar{x}, \varepsilon) - y(T, x_0 + \bar{x})| \le N(r)\eta(\varepsilon) \to 0, \quad \varepsilon \to 0. \tag{53}
$$

We have

$$
\begin{aligned}
&F(x_0 + \bar{x}, y(T, x_0 + \bar{x})) - F(x_0, y(T, x_0)) = \Big(\frac{\partial F(x_0, y(T, x_0))}{\partial x} + \frac{\partial F(x_0, y(T, x_0))}{\partial y}\frac{\partial y(T, x_0)}{\partial x_0}\Big)\bar{x} \\
&+ \int_0^1 \Big(\frac{\partial F(x_0 + s\bar{x}, y(T, x_0 + \bar{x}))}{\partial x} - \frac{\partial F(x_0, y(T, x_0))}{\partial x}\Big)\bar{x}\, ds \\
&+ \int_0^1 \Big(\frac{\partial F(x_0 + \bar{x}, y(T, x_0 + s\bar{x}))}{\partial y}\frac{\partial y(T, x_0 + s\bar{x}))}{\partial z}\Big|_{z=x_0+s\bar{x}} - \frac{\partial F(x_0, y(T, x_0))}{\partial y}\tfrac{\partial y(T,x_0)}{\partial z}\Big|_{z=x_0}\Big)\bar{x}\, ds \\
&= \Big(\frac{\partial F(x_0, y(T, x_0))}{\partial x} + \frac{\partial F(x_0, y(T, x_0))}{\partial y}\frac{\partial y(T, x_0)}{\partial z}\Big|_{z=x_0}\Big)\bar{x} + R_2(\bar{x})\bar{x} + R_3(\bar{x})\bar{x}.
\end{aligned}
\tag{54}
$$

Let us consider each term in (54) separately. The first term, due to the definition of $F_0(x_0)$ in Condition 2.6, can be represented as $\frac{\partial F_0}{\partial x_0}\bar{x}$.

For the second term, in view of the uniform continuity of partial derivatives and (50), we obtain that, for $|\bar{x}| \le \tau$, there is some function $\delta(r)$ such that $\|R_2(\bar{x})\| \le \delta(r) \to 0$, $r \to 0$, where $r \le \frac{\rho}{2}e^{-LT}$.

To estimate the third term, we note that the derivative $\frac{\partial y(T,z)}{\partial z}$ is a continuous function of the parameter z. Therefore, due to the uniform continuity of partial derivatives in the third term, for $|\bar{x}| \le r$, we obtain the estimate $\|R_3(\bar{x})\| \le \delta_1(r) \to 0$, $r \to 0$, with some function $\delta_1(r)$.

Now, we turn to equation (51) and rewrite it as $\bar{x} = -\big(\frac{\partial F_0}{\partial x_0}\big)^{-1}(R_1(\bar{x}, \varepsilon) + (R_2(\bar{x}) + R_3(\bar{x}))\bar{x}$, or

$$
\bar{x} = \big(\tfrac{\partial F_0}{\partial x_0}\big)^{-1}M(\bar{x}, \varepsilon). \tag{55}
$$

The following estimate holds for $M(\bar{x}, \varepsilon)$:

$$
|M(\bar{x}, \varepsilon)| \le N(r)\eta(\varepsilon) + \delta_2(r)\bar{x}, \tag{56}
$$

here $\delta_2(r) = \max\{\delta(r), \delta_1(r)\}$. Note that $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$, and $\delta_2(r) \to 0$ as $r \to 0$.

Let us estimate $\frac{\partial M}{\partial \bar{x}}$. We have

$$\begin{aligned}\frac{\partial R_1(\bar{x},\varepsilon)}{\partial \bar{x}} &= \frac{\partial F(x_0+\bar{x}, x(T/\varepsilon, x_0+\bar{x},\varepsilon))}{\partial x}\Big|_{x=x_0+\bar{x}} \\ &+ \frac{\partial F(x_0+\bar{x}, x(T/\varepsilon, x_0+\bar{x},\varepsilon))}{\partial y}\frac{\partial x\big(T/\varepsilon,x_0+\bar{x},\varepsilon\big)}{\partial z}\Big|_{z=x_0+\bar{x}} \\ &- \frac{\partial F(x_0+\bar{x}, y(\tau, x_0+\bar{x})}{\partial x}\Big|_{x=x_0+\bar{x}} - \frac{\partial F(x_0+\bar{x}, y(\tau, x_0+\bar{x})}{\partial y}\frac{\partial y(\tau,x_0+\bar{x})}{\partial z}\Big|_{z=x_0+\bar{x}}.\end{aligned}$$

We note that, by Theorem 2, whose conditions are met, the difference

$$\frac{\partial x(T/\varepsilon, x_0+\bar{x},\varepsilon)}{\partial z} - \frac{\partial y(\tau, x_0+\bar{x})}{\partial z}$$

can be made arbitrarily small by an appropriate choice of ε. So, from (52) and the uniform continuity of $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ in domain $|\bar{x}| \le r$, we obtain the following estimate for $\frac{\partial R_1}{\partial \bar{x}}$:

$$\Big\|\frac{\partial R_1(\bar{x},\varepsilon)}{\partial \bar{x}}\Big\| \le \delta_3(\varepsilon) \to 0, \quad \varepsilon \to 0, \quad |\bar{x}| \le r, \tag{57}$$

for some function $\delta_3(\varepsilon)$.

Let us now estimate $\frac{\partial R_2}{\partial \bar{x}}$. It follows from (54) that R_2 can be represented as

$$R_2(\bar{x}) = F(x_0+\bar{x}, y(T, x_0+\bar{x})) - F(x_0+\bar{x}, y(T, x_0+\bar{x})) - \frac{\partial F(x_0, y(T, x_0))}{\partial x}\bar{x}.$$

We then obtain

$$\begin{aligned}\Big\|\frac{\partial R_2}{\partial \bar{x}}\Big\| \le \Big\|&\frac{\partial F(x_0+\bar{x}, y(T, x_0+\bar{x}))}{\partial x} - \frac{\partial F(x_0+\bar{x}, y(T, x_0))}{\partial x}\Big\| \\ &+ \Big\|\frac{\partial F(x_0+\bar{x}, y(T, x_0+\bar{x}))}{\partial y}\frac{\partial y(T, x_0+\bar{x})}{\partial z}\Big|_{z=x_0+\bar{x}} \\ &- \frac{\partial F(x_0, y(T, x_0+\bar{x}))}{\partial y}\frac{\partial y(T, x_0+\bar{x})}{\partial z}\Big|_{z=x_0+\bar{x}}\Big\| \le \delta_4(\bar{x}) \to 0, \quad r \to 0.\end{aligned} \tag{58}$$

Similarly, taking into account

$$R_3(\bar{x}) = F(x_0, y(T, x_0+\bar{x})) - \frac{\partial F(x_0, y(T, x_0))}{\partial y}\frac{\partial y(T, x_0)}{\partial z}\Big|_{z=x_0}\bar{x},$$

we get that, in view of the uniform continuity of $\frac{\partial F}{\partial y}$ and $\frac{\partial y(T,z)}{\partial z}$, $\frac{\partial R_3}{\partial \bar{x}}$ admits the estimate

$$\begin{aligned}&\Big\|\frac{\partial R_3(\bar{x})}{\partial \bar{x}}\Big\| \\ &\le \Big\|\frac{\partial F(x_0, y(T, x_0+\bar{x}))}{\partial y}\frac{\partial y(T, x_0+\bar{x})}{\partial z}\Big|_{z=x_0+\bar{x}} - \frac{\partial F(x_0, y(T, x_0))}{\partial y}\frac{\partial y(T, x_0)}{\partial z}\Big|_{z=x_0}\Big\| \\ &\le \delta_5(\bar{x}) \to 0, \quad r \to 0,\end{aligned} \tag{59}$$

Hence, by (57)–(59), we get the estimate

$$\Big\|\frac{\partial M(\bar{x},\varepsilon)}{\partial \bar{x}}\Big\| \le \delta_3(\varepsilon) + \delta_6(r) = \zeta(\varepsilon, r) \to 0, \quad \varepsilon \to 0, \quad r \to 0.$$

Let $C_1 = \left\| \frac{\partial F_0}{\partial x_0} \right\|^{-1}$. Let us choose r and $\varepsilon_1 \leq \varepsilon_0$ so that

$$\delta_r(r) \leq \frac{1}{2} \qquad \text{and} \qquad \eta(\varepsilon) \leq \frac{r}{2C_1 N(r)}, \tag{60}$$

respectively.

Then, if $|\bar{x}| \leq r$, it follows from (56) that

$$C_1 |M(\bar{x}, \varepsilon)| \leq C_1 (N(r)\eta(\varepsilon) + \delta_2(r)|\bar{x}|) \leq \tfrac{r}{2} + \tfrac{r}{2} = r.$$

Thus, if inequalities (60) hold, $\left(\frac{\partial F_0}{\partial x_0}\right)^{-1} M(\bar{x}, \varepsilon)$ maps the ball $B_r(0)$ into itself. Moreover, if we choose ε and r so that, besides (60), the inequality $\zeta(\varepsilon, r) < 1$ holds, then the map (55) is contractive. Therefore, this map has a unique fixed point $\bar{x}^* = \bar{x}^*(\varepsilon, r)$, which is the initial value of the solution of the boundary value problem (17), (46). The estimates (52) and (53) imply the estimate (47) for the same $\sigma_0 > 0$.

Let us now choose r as a function of parameter ε so that $r(\varepsilon) \to 0$ as $\varepsilon \to 0$, and let us choose $\varepsilon_1 \leq \varepsilon_0$ so that the function $\eta(\varepsilon)$ in (53) satisfies the inequality

$$\frac{\eta(\varepsilon)}{r(\varepsilon)} \leq \frac{1}{2C_1 N(r(\varepsilon))}.$$

Note that such a choice is possible since the function $N(r(\varepsilon))$, bounding the partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ in the ball $B_r(0)$, does not increase as $r(\varepsilon)$ decreases. The estimate (48) then follows from (50) and (52). Theorem 3 is proved. □

Remark 6. *In the proof of Theorem 3, we use the estimates of the differences between the solutions of the exact and averaged problems (like* (20)*) and the estimates of the differences between their derivatives with respect to initial data (like* (23)*). These estimates do not depend on the position of the points* $\{t_i\}$ *on* $[0, T/\varepsilon]$*, but depend only on the constant* C *and the functions* $\phi(t)$ *and* $\psi(t)$*.*

3 Proof of the main result

In this section, we present the proof of Theorem 1.

Proof. Let us take an arbitrary set of points $y_1, y_2, \ldots, y_n, \ldots$ in U_a and use them to generate a sequence of moments of impulse action $\tau_1(y_1), \tau_2(y_2), \ldots, \tau_n(y_n), \ldots$. For the chosen set, we construct the system with impulse action at fixed moments of time $\{\tau_i(y_i)\}$, $i = 1, 2, \ldots,$

$$\begin{aligned} \dot{x} &= \varepsilon X(t, x), \quad t \neq t_i(y_i), \\ \Delta x\big|_{t = t_i(y_i)} &= \varepsilon I_i(x), \end{aligned} \tag{61}$$

subject to the boundary condition

$$F(x(0), x(T/\varepsilon)) = 0. \tag{62}$$

Let us choose any $x \in U_a$ and $t > 0$ and fix them. Suppose that the interval $[0, t)$ contains exactly n points $\tau_1(x) < \tau_2(x) < \ldots < \tau_n(x) < t$. By (9), we get $n \leq Ct$. It follows from Condition 1.5 for impulse moments that there exists an increasing sequence of numbers $\{A_i\}_1^\infty$

such that, for any positive integer i, the inequalities $t_i(x) < A_i < t_{i+1}(y)$, $x, y \in U_a$ hold. We then obtain

$$t_1(y_1) < A_1 < t_2(y_2) < A_3 < t_3(y_3) < \ldots < A_{n-2} < t_{n-1}(y_{n-1}) < A_{n-1} < \tau_n(x).$$

Therefore, the number of terms in the sum $\sum\limits_{0<t_i(y_i)<t} I_i(x)$ can be equal to n, or $n-1$, or $n+1$. So, we have three possible cases:

$$\frac{1}{t}\sum_{0<t_i(y_i)<t} I_i(x) = \left[\begin{array}{l} \frac{1}{t}\sum\limits_{0<t_i(x)<t} I_i(x), \\ \frac{1}{t}\sum\limits_{0<t_i(x)<t} I_i(x) - \frac{1}{t}I_n(x), \\ \frac{1}{t}\sum\limits_{0<t_i(x)<t} I_i(x) + \frac{1}{t}I_{n+1}(x). \end{array}\right.$$

In each case, due to Condition 1.2 and the fact that $I_i(x)$ is bounded by a constant M, we have that $\frac{1}{t}\sum\limits_{0<t_i(y_i)<t} I_i(x)$, uniformly in y_i and x, tends to $I_0(x)$ as $t \to \infty$. Moreover,

$$\Big|\frac{1}{t}\sum_{0<t_i(y_i)<t} I_i(x) - I_0(x)\Big| \le \Big|\frac{1}{t}\sum_{0<t_i(x)<t} I_i(x) - I_0(x)\Big| + \frac{M}{t} \le \varphi(t) + \frac{M}{t} = \varphi_1(t),$$

where $\varphi(t)$ is from the inequality (21). Therefore, for any sequence $\{y_i\} \in U_a$ we have that

$$\Big|\sum_{0<t_i(y_i)<t} I_i(x) - I_0(x)t\Big| \le \varphi_1(t)t, \tag{63}$$

and the number of points $t_i(y_i)$ on $(0, t)$ does not exceed $Ct + 1$.

Thus, for any set $\{y_i\} \in U_a$, the boundary value problem (61), (62) satisfy the conditions of Theorem 3. Consequently, for small ε the problem has a unique solution that belongs to some neighborhood of the solution of the averaged problem. Moreover, in view of (63) and Remark 6, the values of ε_0 and σ_0 from Theorem 3 stay the same for any set $\{y_i\}_1^\infty \in U_a$.

Let us now show that the original problem (1), for $\varepsilon < \varepsilon_0$, has a solution that belongs to the σ_0-neighborhood of the solution of the averaged boundary value problem and (10) holds true. Let us choose $\varepsilon < \varepsilon_0$, then fix it and consider the interval $[0, T/\varepsilon]$. Suppose that the number of functions $t_i(x)$, such that $t_i(x) \le T/\varepsilon$, is equal to p. Note that the conditions of the theorem imply that $t_i(x) < T/\varepsilon$ for $i = \overline{1, p-1}$. For our boundary value problem, the behaviour of solutions of system (1) for $t > T/\varepsilon$ does not matter, so we will consider this system on the interval $[0, T/\varepsilon]$. It follows from the definition of the solution of system (1) that the impulse action at the moment $t = T/\varepsilon$ also does not matter. So, instead of the hypersurface $t = t_p(x)$ we will consider the hypersurface $t = \tau(x)$, where $\tau(x)$ is of the form

$$\tau(x) = \begin{cases} t_p(x) & \text{for } x \in U_a \text{ such that } t_p(x) < T/\varepsilon, \\ T/\varepsilon & \text{for } x \in U_a \text{ such that } t_p(x) \ge T/\varepsilon. \end{cases}$$

Reassigning again $\tau(x) := t_p(x)$, we get that $t_i(x) \le T/\varepsilon$ for all $i = \overline{1, p}$ and $x \in U_a$.

Let us consider p points $y_1, \ldots, y_p$ in U_a. Now, $y = (y_1, \ldots, y_p)$ is a vector from the space $\mathbb{R}^{pd}$ with the norm $\|y\|^2 = \sum\limits_{i=1}^{p} \|y_i\|^2$. Obviously, if $y_1, \ldots, y_p \in U_a$ then y belongs to the ball $B_{pa}(0)$ of radius pa centered at the origin in the space $\mathbb{R}^{pd}$.

For the chosen set $y_1, \dots, y_p \in U_a$, we consider the boundary value problem (61), (62). According to Theorem 3, this problem has a unique solution $x^*(t,y) \in U_a$, $t \in [0, T/\varepsilon]$. As follows from Theorem 3 and Remark 6, there exists $\widetilde{\varepsilon} < \nu$ such that for $\varepsilon < \widetilde{\varepsilon}$ the solution $x^*(t,y)$ belongs in $\mu/2$-neighborhood of $y(\varepsilon t)$, and, consequently, $\rho(x^*(\frac{T}{2}, y), N_i(\varepsilon)) > \frac{\mu}{2}$.

Using $x^*(t,y)$, we construct a map $s = s(y) : B_{ap}(0) \to B_{ap}(0)$ in the following way:

$$\begin{aligned} s_i = x^*(t_i(y_i), y), \quad i = \overline{1,p}, \\ s = (s_1, \dots, s_p). \end{aligned} \tag{64}$$

To complete the proof, we need to demonstrate that the constructed map has a fixed point. By Brouwer's theorem, it is sufficient to show the continuity of the map. Let us fix $y \in B_{ap}(0)$ and consider the points $z \in B_{ap}(0)$ sufficiently close to y. The continuity of $t_i(x)$ guarantees that $t_i(y_i)$ and $t_i(z_i)$ are close enough.

The solutions of boundary value problems (61), (62) for $t_i(y_i)$ and $t_i(z_i)$ are of the respective form

$$x^*(t,y) = x^*(y) + \varepsilon \int_0^t X(s, x^*(s,y))\, ds + \varepsilon \sum_{0 < t_i(y_i) < t} I_i(x^*(t_i(y_i)), y),$$

$$x^*(t,z) = x^*(z) + \varepsilon \int_0^t X(s, x^*(s,z))\, ds + \varepsilon \sum_{t_i(z_i)} I_i(x^*(t_i(z_i)), z),$$

where $x^*(y)$ and $x^*(z)$ are the initial values of these solutions, respectively. It follows from Theorem 3 that $x^*(y)$ and $x^*(z)$ are of the form $x^*(y) = x_0 + \bar{x}(y)$ and $x^*(z) = x_0 + \bar{x}(z)$ (see (49)). The functions $\bar{x}(y)$ and $\bar{x}(z)$ can be found by the method of successive approximations, from the following recurrent sequences constructed according to (55):

$$X_n(y) = \left(\frac{\partial F_0}{\partial x_0}\right)^{-1} M(\bar{x}_{n-1}^{(y)}, \varepsilon), \qquad X_n(z) = \left(\frac{\partial F_0}{\partial x_0}\right)^{-1} M(\bar{x}_{n-1}^{(z)}, \varepsilon),$$

where $\bar{x}_0(y) = \bar{x}_0(z) = 0$ and $M(\bar{x}, \varepsilon) = R_1(\bar{x}, \varepsilon) + (R_2(\bar{x}) + R_3(\bar{x}))x$.

Let us show that the functions $\bar{x}_n(y)$ are continuous for all n. In the case $n = 1$, due to the definition of functions $R_1(\bar{x}, \varepsilon)$ and $M(\bar{x}, \varepsilon)$, we have

$$|\bar{x}_1(y) - \bar{x}_1(z)| \le C_1 N_1(r) |x(T/\varepsilon, x_0, y, \varepsilon) - x(T/\varepsilon, x_0, z, \varepsilon)|, \tag{65}$$

where $C_1 = \left\| \frac{\partial F_0}{\partial x_0} \right\|^{-1}$.

Note that, due to Condition A of the theorem, the point $t = T/\varepsilon$ always belongs to the interval appearing in (12). Thus, by Lemma 1, the right hand part of (65) approaches zero as $z \to y$.

If $n = 2$, we have

$$\bar{x}_2(y) = \left(\frac{\partial F_0}{\partial x_0}\right)^{-1} M(\bar{x}_1(y), \varepsilon) \qquad \text{and} \qquad \bar{x}_2(z) = \left(\frac{\partial F_0}{\partial x_0}\right)^{-1} M(\bar{x}_1(z), \varepsilon).$$

Next, due to Lemma 1 and the continuous dependence of solutions of system (3) on initial data, we get

$$\begin{aligned} &|R_1(\bar{x}_1(y), \varepsilon) - R_1(\bar{x}_1(z), \varepsilon)| \\ &\quad = |F(x_0 + \bar{x}_1(y), x(T/\varepsilon, x_0 + \bar{x}_1(y), y, \varepsilon) - F(x_0 + \bar{x}_1(y), y(T, x_0 + \bar{x}_1(y)) \\ &\qquad - F(x_0 + \bar{x}_1(z), x(T/\varepsilon, x_0 + \bar{x}_1(z), z, \varepsilon) + F(x_0 + \bar{x}_1(z), y(T, x_0 + \bar{x}_1(z)) \\ &\quad \le N(r)\big(|\bar{x}_1(y) - \bar{x}_1(z)| + |x(T/\varepsilon, x_0 + \bar{x}_1(y), y, \varepsilon) - x(T/\varepsilon, x_0 + \bar{x}_1(z), z, \varepsilon)| \\ &\qquad + |\bar{x}_1(y) - \bar{x}_1(z)| + |y(T, x_0 + \bar{x}_1(y)) - y(T, x_0 + \bar{x}_1(z))|\big) \to 0, \quad z \to y. \end{aligned}$$

The continuity of $R_2(\bar{x}(y))$ and $R_3(\bar{x}(y))$ follows from the continuity and continuous differentiability of solutions of the averaged problem with respect to initial data.

Thus, for any n, we have

$$|\bar{x}_n(y) - \bar{x}_n(z)| \to 0, \quad z \to y. \tag{66}$$

Since the estimates (56)–(59) are uniform in $\bar{x}(y)$, $|\bar{x}| \le r$, then the sequence $\bar{x}_n(y)$, uniformly in $y_1, \dots, y_p \in U_a$, converges to the initial value $x^*(y)$ of the boundary value problem (61), (62).

Hence, taking into account (66), we obtain the continuity of $x^*(y)$ with respect to $y = (y_1, \dots, y_p)$, $y_i \in U_a$.

Let us now estimate the difference $x^*(t_i(y_i), y) - x^*(t_i(z_i), z)$. Without loss of generality, we can assume that $t_i(y_i) \le t_i(z_i)$ for this particular i. We have

$$|x^*(t_i(y_i), y) - x^*(t_i(z_i), z)| \le |x^*(t_i(y_i), y) - x^*(t_i(y_i), z)| + |x^*(t_i(y_i), z) - x^*(t_i(z_i), z)|.$$

By Lemma 1, the first term approaches zero as $z \to y$. The convergence to zero of the second term is established similarly to (14).

Thus, for any $\varepsilon < \varepsilon_0$ the map (64) is continuous and, consequently, has a fixed point $y^* = (y_1^*, \dots, y_p^*)$. Then the function $x^*(t, y^*)$, obviously, is a solution of the boundary value problem (1). The limit relationship (10) is established similarly to (48) in Theorem 3. This completes the proof of Theorem 1. □

4 Examples

In this section, we provide some examples illustrating Theorem 1.

Example 1. Let us consider the boundary value problem in $\mathbb{R}^d$:

$$\begin{aligned} \dot{x} = \varepsilon x + \varepsilon f(t, x), \quad t \ne t(x), \\ \Delta x\big|_{t=t(x)} = \varepsilon I(x), \\ x(0) = x(\tfrac{T}{\varepsilon}). \end{aligned} \tag{67}$$

Here $t(x) = (\alpha, x) + b$, and the solution undergoes an impulsive perturbation when it reaches the hypersurface $t = (\alpha, x) + b$, $\alpha \in \mathbb{R}^d$, $b \in \mathbb{R}^1$. Assume that $f(t, x)$ is a Lipschitz function with respect to x and is bounded by a constant M. In this example, $F(x, y) = x - y$. We suppose that necessary conditions for the smoothness of $f(t, x)$ and $I(x)$ are satisfied in the domain $U_a = \{x \in \mathbb{R}^d : |x| \le a\}$. Suppose also that the function $f(t, x)$ is periodic in t with period 2π and has the zero mean:

$$\int_0^{2\pi} f(t, x)\, dt = 0. \tag{68}$$

It follows from [17, Lemma 3.2] that, for sufficiently small $\varepsilon > 0$, if $(\alpha, I(x)) \le 0$, then every solution in the domain U_a intersects the plane $t = (\alpha, x) + b$ no more than once. Indeed, condition (68) guarantees that the following inequality holds:

$$t(x) \ge t(x + I(x)). \tag{69}$$

It is clear that $t(x)$ satisfies the Lipschitz condition with constant $|\alpha|$. Since $\max\limits_{|x|\le a} \le \varepsilon(a+M)$, then, for small $\varepsilon > 0$, the condition

$$\varepsilon(a+M)|\alpha| < 1 \tag{70}$$

is met.

Conditions (69) and (70), by Lemma 2, ensure that every solution of system (67) intersects the plane of impulse action $t = (\alpha, x) + b$ no more than once.

It is clear that, due to (68) and the nature of impulses, Conditions 1.2 and 1.4 of Theorem 1 are satisfied. We have $X_0(x) = x$ and $I_0(x) = 0$. It is also obvious that, for small ε, the plane $t = T/\varepsilon$ does not intersect the plane $t = (\alpha, x) + b$ in the domain U_a. Hence Condition A is also met.

We need to check Condition 1.3. The averaged boundary value problem

$$\dot{y} = \varepsilon y, \quad y(0) = y(T),$$

admits the trivial solution $y = 0$. In addition, we have $F_0(x_0) = F(x_0, y(T, x_0)) = x_0 + e^T x_0$ and $\det \frac{\partial F_0(x_0)}{\partial x_0} = 1 + e^T \neq 0$. Thus the condition (7) of Theorem 1 is also satisfied.

Finally, we conclude that there exists $\varepsilon_0 > 0$ such that, for $\varepsilon < \varepsilon_0$, the boundary value problem (67) has a solution $x(t, \varepsilon)$ and

$$\sup_{t\in[0,T/\varepsilon]} |x(t,\varepsilon)| \to 0, \quad \varepsilon \to 0.$$

The following examples demonstrate the importance of condition (7).

Example 2. We consider the boundary value problem with a fixed moment of impulse:

$$\begin{aligned} \dot{x} = \varepsilon x \cos t, \quad t \neq \pi, \\ \Delta x\big|_{t=\pi} = \varepsilon, \\ x(0) = x(T/\varepsilon). \end{aligned} \tag{71}$$

The averaged boundary value problem is of the form

$$\dot{y} = 0, \quad y(0) = y(T) = 0.$$

Clearly, problem (71) satisfies all conditions of Theorem 1, except for (7), since, in this case, $F_0(x_0) = x_0 - x_0 \equiv 0$.

The boundary value problem (71) has a solution if and only if the initial value $x_0 = x(0)$ satisfies the condition

$$x_0 = (x_0 + \varepsilon) e^{\varepsilon \sin \frac{T}{\varepsilon}}.$$

This condition is not met if $\frac{T}{\varepsilon} = k\pi$, $k \in \mathbb{N}$. Therefore, problem (71) has a solution not for all sufficiently small ε.

Example 3. If, instead of (71), we consider the problem

$$\begin{aligned} \dot{x} = \varepsilon x + \varepsilon x \cos t, \quad t \neq \pi, \\ \Delta x\big|_{t=\pi} = \varepsilon, \\ x(0) = x(T/\varepsilon), \end{aligned} \tag{72}$$

we get a different situation. The averaged boundary problem

$$\dot{y} = \varepsilon y, \quad y(0) = y(T).$$

has the trivial solution $y = 0$. In this case, $F_0(x_0) = x_0 - x_0 e^T$, and $\det F_0(x_0) = 1 - e^T \neq 0$ for $T > 0$. Hence, condition (7) is met. The remaining conditions of Theorem 1 are satisfied as well.

The boundary value problem (72) has a solution if and only if x_0 satisfies the condition

$$x_0\left(e^{\varepsilon \sin \frac{T}{\varepsilon} + T} - 1\right) = -\varepsilon e^{\varepsilon \sin \frac{T}{\varepsilon} + T - \varepsilon\pi}, \tag{73}$$

which, for small $\varepsilon > 0$, is always met since $\varepsilon \sin \frac{T}{\varepsilon} + T > 0$. The solution of problem (72) is represented as

$$x(t) = \begin{cases} x_0 e^{\varepsilon t}, & t \in [0, \pi], \\ \left(x_0 e^{\varepsilon\pi} + \varepsilon\right) e^{\varepsilon \sin t + \varepsilon(t-\pi)}, & t > \pi, \end{cases}$$

where x_0 is determined from (73).

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Метод усереднення застосовано до дослідження існування розв'язків крайових задач для систем диференціальних рівнянь із нефіксованими моментами імпульсної дії. Показано, що якщо усереднена крайова задача має розв'язок, то і початкова задача також має розв'язок. При цьому усереднена задача для імпульсної системи є простішою задачею для системи звичайних диференціальних рівнянь.

Ключові слова і фрази: малий параметр, метод усереднення, нерухома точка, імпульсна дія, крайова задача.