Conformal Ricci Soliton on 3-dimensional trans-Sasakian manifolds with respect to SVK connection

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In this study, we adapt the Schouten-van Kampen (SVK) connection on trans-Sasakian 3-manifolds. Then we consider semi-symmetric conditions on trans-Sasakian 3-manifolds with respect to the SVK connection admitting conformal Ricci soliton.

Key words and phrases: Ricci soliton, conformal Ricci soliton, Schouten-van Kampen connection, trans-Sasakian 3-manifold.

1 Introduction

In 1982, the concept of Ricci flow was introduced by R.S. Hamilton [12]. This concept was developed to answer Thurston’s geometric conjecture, which says that each closed three manifold admits a geometric decomposition. Also, R.S. Hamilton classified all compact manifolds with positive curvature operator in dimension four [12]. The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S,$$

on a compact Riemannian manifold $M$ with Riemannian metric $g$.

A self-similar soliton to the Ricci flow [12, 28] is known a Ricci soliton [13], if it moves only by a one parameter family of diffeomorphism and scaling. The Ricci soliton equation is given by

$$\mathcal{L}_V g + 2S = 2\lambda g,$$

where $\mathcal{L}_V$ is the Lie derivative, $S$ is Ricci tensor, $g$ is a Riemannian metric, $V$ is a vector field and $\lambda$ is a scalar. The Ricci soliton is said to be shrinking if $\lambda$ is positive, steady if $\lambda$ is zero and expanding if $\lambda$ is negative. Many studies on Ricci solitons have been reported by many geometers in different structure (see [3, 4, 7, 8, 14, 29, 30]).

The concept of conformal Ricci flow was studied by A.E. Fischer [10]. The conformal Ricci flow on $M$ is defined by the equation

$$\frac{\partial g}{\partial t} + 2 \left( S + \frac{g}{n} \right) = -pg$$

and $r(g) = -1$, where $p$ is a scalar non-dynamical field (time dependent scalar field), $r(g)$ is the scalar curvature of the manifold and $n$ is the dimension of manifold [10].

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Also, the notion of conformal Ricci soliton equation is given by

$$\mathcal{L}_V g + 2S = \left[2\lambda - \left(p + \frac{2}{n}\right)\right]g,$$  

where $\lambda$ is constant [1]. The equation is the generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation. In [21], on a 3-dimensional trans-Sasakian manifold conformal Ricci soliton was studied.

Moreover, the Schouten-van Kampen (SVK) connection defined as adapted to a linear connection for studying non holonomic manifolds and it is one of the most natural connections on a differentiable manifold [2, 15, 22]. A.F. Solov’ev studied hyperdistributions in Riemannian manifolds using the SVK connection [23, 25–27]. Then Z. Olszak studied SVK connection to almost (para) contact metric structures [17]. In recent times, S.Y. Perkta¸s and A. Yıldız studied some symmetry conditions and some soliton types of quasi-Sasakian manifolds and $f$-Kenmotsu manifolds with respect to SVK connection [19, 20].

In this study, we consider some curvature conditions on a 3-dimensional trans-Sasakian manifolds with respect to the SVK connection admitting conformal Ricci soliton and give an example of a 3-dimensional trans-Sasakian manifold with respect to SVK connection.

2 Preliminaries

Let $M$ be an almost contact metric manifold with an almost contact metric structure $(\phi, \xi, \eta, g)$, where $\phi$ is a (1,1) tensor field, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is the compatible Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi \xi = 0, \quad \phi \xi = 0,$$  

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$  

$$g(X, \phi Y) = -g(\phi X, Y),$$  

$$g(X, \xi) = \eta(X)$$

for all vector fields $X, Y \in \chi(M)$.

An almost contact metric structure $(\phi, \xi, \eta, g)$ on $M$ is called a trans-Sasakian structure [18] if $(M \times \mathbb{R}, J, G)$ belongs to the class $W_4$ [11], where $J$ is the almost complex structure on $M \times \mathbb{R}$ defined by $J \left( X, f \frac{d}{dt} \right) = \left( \phi X - f\xi, \eta(X) \frac{d}{dt} \right)$ for smooth functions $f$ on $M \times \mathbb{R}$.

It can be expressed [5] by the condition

$$(\nabla_X \phi) Y = a(\eta(Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X)$$

for some smooth functions $a, \beta$ on $M$ and we say that the trans-Sasakian structure is of type $(a, \beta)$.

From the above expression, we have

$$\nabla_X \xi = -a\phi X + \beta(X - \eta(X)\xi),$$  

$$\nabla_X \eta = -a g(\phi X, Y) + \beta g(\phi X, \phi Y).$$
For a 3-dimensional trans-Sasakian manifold the following relations hold [9]:

\[ 2\alpha \beta + \zeta \alpha = 0, \]

\[ S(X, Y) = \left( \frac{r}{2} - (\alpha^2 - \beta^2) \right) g(X, Y) - \left( \frac{r}{2} - 3(\alpha^2 - \beta^2) \right) \eta(X) \eta(Y), \]

\[ S(X, \zeta) = 2 \left( \frac{r}{2} - (\alpha^2 - \beta^2) \right) \eta(X), \]

\[ R(X, Y) \zeta = (\alpha^2 - \beta^2) \left[ \eta(Y) X - \eta(X) Y \right], \]

\[ R(\zeta, X) Y = (\alpha^2 - \beta^2) \left[ g(X, Y) \zeta - \eta(Y) X \right], \]

\[ \eta(R(X, Y) Z) = (\alpha^2 - \beta^2) \left[ g(Y, Z) \eta(X) - g(X, Z) \eta(Y) \right]. \]  

From (5) it follows that if \( \alpha \) and \( \beta \) are constants, then the manifold is either \( \alpha \)-Sasakian or \( \beta \)-Kenmotsu or cosymplectic, respectively.

Also, we have two naturally defined distributions in the tangent bundle \( TM \) of \( M \) as follows

\[ \hat{H} = \ker \eta, \quad \hat{\nu} = \text{span} \, \zeta, \]

which implies \( TM = \hat{H} \oplus \hat{\nu}, \hat{H} \cap \hat{\nu} = 0 \) and \( \hat{H} \perp \hat{\nu} \). This decomposition allows one to define the Schouten-van Kampen connection \( \hat{\nabla} \) over an almost contact metric structure. The Schouten-van Kampen connection \( \hat{\nabla} \) on an almost contact metric manifold with respect to Levi-Civita connection \( \nabla \) is defined [24] by

\[ \hat{\nabla}_X Y = \nabla_X Y - \eta(Y) \nabla_X \zeta + (\nabla_X \eta)(Y) \zeta. \]  

Moreover

\[ (\xi g) (X, Y) = (\nabla_\xi g)(X, Y) - \alpha g(\phi X, Y) + 2\beta g(X, Y) - 2\beta \eta(X) \eta(Y) - \alpha g(\phi X, Y) \]

\[ = 2\beta g(X, Y) - 2\beta \eta(X) \eta(Y). \]

Let \( M \) be a 3-dimensional trans-Sasakian manifold with constant \( \alpha \) and \( \beta \) with respect to SVK connection. Then using (6) and (7) in (9), we get

\[ \hat{\nabla}_X Y = \nabla_X Y + \alpha \left\{ \eta(Y) \phi X - g(\phi X, Y) \zeta \right\} + \beta \left\{ g(X, Y) \zeta - \eta(Y) X \right\}. \]  

Let \( R \) and \( \hat{R} \) be the curvature tensors of the Levi-Civita connection \( \nabla \) and SVK connection \( \hat{\nabla} \) are given by

\[ R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad \hat{R}(X, Y) = [\hat{\nabla}_X, \hat{\nabla}_Y] - \hat{\nabla}_{[X,Y]}. \]
Using (10), we obtain
\[
\tilde{R}(X,Y)Z = R(X,Y)Z + \alpha^2 \left\{ \begin{align*}
g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y + \eta(X)\eta(Z)Y \\
- \eta(Y)\eta(Z)X - g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi
\end{align*} \right\} + \beta^2 \{g(Y,Z)X - g(X,Z)Y\}. \tag{11}
\]

We will also consider the Riemannian curvature tensors \( \tilde{R}, R \), the Ricci tensors \( \tilde{S}, S \), the Ricci operators \( \tilde{Q}, Q \) and the scalar curvatures \( \tau, \tau \) of the connections \( \nabla \) and \( \tilde{\nabla} \) are given by
\[
\tilde{R}(X,Y,W,Z) = R(X,Y,W,Z) + \alpha^2 \left\{ \begin{align*}
g(\phi Y,W)g(\phi X,Z) - g(\phi X,W)g(\phi Y,Z) \\
+ g(Y,Z)\eta(W) - g(X,Z)\eta(Y)\eta(W) \\
- g(Y,W)\eta(X)\eta(Z) + g(X,W)\eta(Y)\eta(Z)
\end{align*} \right\} + \beta^2 \{g(Y,W)g(X,Z) - g(X,W)g(Y,Z)\},
\]
\[
\tilde{S}(X,Y) = S(X,Y) + 2\beta^2 g(X,Y) - 2\alpha^2 \eta(X)\eta(Y), \tag{12}
\]
\[
\tilde{Q}X = QX + 2\beta^2 X - 2\alpha^2 \eta(X)\xi,
\]
\[
\tau = \tau - 2\alpha^2 + 6\beta^2.
\]

In a 3-dimensional trans-Sasakian manifold \( M \) endowed with respect to the SVK connection bearing a conformal Ricci soliton, we can write
\[
\tilde{\mathcal{L}}_{V}g + 2\tilde{S} - \left\{ 2\lambda - \left( p + \frac{2}{3} \right) \right\} g = 0. \tag{13}
\]

From (9) and (13), since \( \nabla \tilde{g} = 0 \) and \( \tilde{T} \neq 0 \), we have
\[
\tilde{\mathcal{L}}_{V}g(X,Y) = g(\nabla_{X}V,Y) + g(X,\nabla_{Y}V) = (\mathcal{L}_{V}g)(X,Y),
\]
that is
\[
g(\nabla_{X}V,Y) + g(X,\nabla_{Y}V) + 2\tilde{S}(X,Y) - \left\{ 2\lambda - \left( p + \frac{2}{3} \right) \right\} g(X,Y) = 0. \tag{14}
\]

Putting \( V = \xi \) in (14), we obtain
\[
g(\nabla_{X}\xi,Y) + g(X,\nabla_{Y}\xi) + 2\tilde{S}(X,Y) - \left\{ 2\lambda - \left( p + \frac{2}{3} \right) \right\} g(X,Y) = 0. \tag{15}
\]

Now using (6) with (4) in (15), we get
\[
2\beta g(X,Y) - 2\beta \eta(X)\eta(Y) + 2\tilde{S}(X,Y) - \left\{ 2\lambda - \left( p + \frac{2}{3} \right) \right\} g(X,Y) = 0,
\]
which yields
\[
\tilde{S}(X,Y) = \left\{ \lambda - \frac{1}{2} \left( p + \frac{2}{3} \right) - \beta \right\} g(X,Y) + \beta \eta(X)\eta(Y). \tag{16}
\]

Thus \( M \) is an \( \eta \)-Einstein manifold with respect to the SVK connection.

Also using (12) in (16), we have
\[
S(X,Y) = \left\{ \lambda - \frac{1}{2} \left( p + \frac{2}{3} \right) - \beta - 2\beta^2 \right\} g(X,Y) + \left( \beta + 2\beta^2 \right) \eta(X)\eta(Y). \tag{17}
\]

Hence \( M \) is an \( \eta \)-Einstein manifold with respect to the Levi-Civita connection. Thus we have the following assertion.
Theorem 1. Let $M$ be a 3-dimensional trans-Sasakian manifold admitting a conformal Ricci soliton $(g, \xi, \lambda)$ with respect to SVK connection. Then $M$ is an $\eta$-Einstein manifold both with respect to the SVK connection and Levi-Civita connection.

Putting $Y = \xi$ and using (12) in (16), we get

$$\tilde{S}(X, \xi) = \left\{ \lambda - \frac{1}{2} \left( p + \frac{2}{3} \right) \right\} \eta(X).$$

(18)

So we discuss the following conditions:

i) assume that $p > \frac{2}{3}$ and therefore the equation (18) shows that $\lambda > 0$, thus the Ricci soliton $(g, \xi, \lambda)$ is expanding;

ii) assume that $p < \frac{2}{3}$ and therefore the equation (18) reveals that $\lambda < 0$, thus the Ricci soliton $(g, \xi, \lambda)$ is shrinking;

iii) assume that $p = \frac{2}{3}$ and therefore the equation (18) allows that $\lambda = 0$, thus the Ricci soliton $(g, \xi, \lambda)$ is steady.

Thus we state our results in the form of theorem as follows.

Theorem 2. A conformal Ricci soliton $(g, \xi, \lambda)$ on a 3-dimensional trans-Sasakian manifold with respect to SVK connection is said to be expanding, shrinking and steady if $p > \frac{2}{3}$, $p < \frac{2}{3}$ and $p = \frac{2}{3}$, respectively.

3 Conformal Ricci soliton on a 3-dimensional trans-Sasakian manifold satisfying $R(\xi, X).\tilde{B} = 0$

The C-Bochner curvature tensor $\tilde{B}$ in $M$ is defined [16] (see also [6]) by

$$\tilde{B}(X, Y)Z = \tilde{R}(X, Y)Z + \frac{1}{6} \left\{ \begin{array}{l}
g(X, Z)\hat{Q}Y - g(Y, Z)\hat{Q}X \\
- \tilde{S}(Y, Z)X + \tilde{S}(X, Z)Y \\
+ g(\phi X, Z)\hat{Q}\phi Y - g(\phi Y, Z)\hat{Q}\phi X \\
- \tilde{S}(\phi Y, Z)\phi X + \tilde{S}(\phi X, Z)\phi Y \\
+ 2\tilde{S}(\phi X, Y)\phi Z + 2g(\phi X, Y)\hat{Q}\phi Z \\
+ \eta(Y)\eta(Z)\hat{Q}X - \eta(X)\eta(Z)\hat{Q}Y \\
- \eta(Y)\tilde{S}(X, Z)\xi + \eta(X)\tilde{S}(Y, Z)\xi \\
\end{array} \right\}
$$

$$- \frac{D + 2}{6} \left\{ \begin{array}{l}
g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X \\
+ 2g(\phi X, Y)\phi Z \\
\end{array} \right\}
$$

$$+ \frac{D}{6} \left\{ \begin{array}{l}
\eta(Y)g(X, Z)\xi - \eta(Y)\eta(Z)X \\
+ \eta(X)\eta(Z)Y - \eta(X)g(Y, Z)\xi \\
\end{array} \right\}
$$

$$- \frac{D - 4}{6} \left\{ \begin{array}{l}
g(X, Z)Y - g(Y, Z)X \\
\end{array} \right\},$$

where $D = \frac{p + 2}{6}$. 

Using (11) with (12) in the above equation, we get

\[
\tilde{B}(X,Y)Z = R(X,Y)Z + \alpha^2 \left\{ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \right. \\
+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\
- g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi \\
\left. + \beta^2 \left\{ g(Y,Z)X - g(X,Z)Y \right. \right\} \\
+ \frac{1}{6} \left\{ S(X,Z)Y - S(Y,Z)X \\
+ g(X,Z)QY - g(Y,Z)QX \\
+ g(\phi X, Z)Q\phi Y - g(\phi Y, Z)Q\phi X \\
+ S(\phi X, Z)\phi Y - S(\phi Y, Z)\phi X \\
- g(Y,Z)\eta(X)\eta(Z)Y + 4\beta^2 g(X,Z)Y - 4\beta^2 g(Y,Z)X \\
+ 4\beta^2 (\phi X, Z)\phi Y - 4\beta^2 (\phi Y, Z)\phi X \\
- 2\beta^2 g(X,Z)\eta(Y)\xi + 2\beta^2 g(Y,Z)\eta(X)\xi \\
+ 8\beta^2 g(\phi X, Y)\phi Z \\
+ 2\alpha^2 g(Y,Z)\eta(X)\xi - 2\alpha^2 g(X,Z)\eta(Y)\xi \\
+ 2\alpha^2 g(\phi Y, Z)\eta(X)\xi - 2\alpha^2 g(X,Z)\eta(Y)\xi \\
- \frac{D + 2}{6} \left\{ g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X \\
+ 2g(\phi X, Y)\phi Z \right\} \\
+ \frac{D}{6} \left\{ g(X,Z)\eta(Y)\xi - \eta(Y)\eta(Z)X \\
+ \eta(X)\eta(Y)Z - g(Y,Z)\eta(X)\xi \right\} \\
- \frac{D - 4}{6} \{ g(X,Z)Y - g(Y,Z)X \}. \\
\right\} \\
(19)
\]

Taking \( Z = \xi \) in (19), we get

\[
\tilde{B}(X,Y)\xi = \left\{ \frac{1}{6} \left( \lambda - \frac{1}{2} \left( p + \frac{2}{3} \right) \right) + \frac{4}{6} \right\} [\eta(X)Y - \eta(Y)X], \\
(20)
\]

which gives

\[
\eta(\tilde{B}(X,Y)Z) = \left\{ \frac{1}{6} \left( \lambda - \frac{1}{2} \left( p + \frac{2}{3} \right) \right) + \frac{4}{6} \right\} \left[ g(X,Z)\eta(Y) \right. \\
\left. - g(Y,Z)\eta(X) \right]. \\
\]

Now, we assume that the condition \( R(\xi, X)\tilde{B} = 0 \) is satisfied, then we get

\[
R(\xi, X)\tilde{B}(Y,Z)W - \tilde{B}(R(\xi, X)Y, Z)W - \tilde{B}(Y, R(\xi, X)Z)W - \tilde{B}(Y, Z)R(\xi, X)W = 0. \\
(21)
\]

Using (8) in (21), we obtain

\[
\eta(\tilde{B}(Y,Z)W, X) - g(\tilde{B}(Y,Z)W, X)\xi + g(X,Y)\tilde{B}(\xi,Z)W - \eta(Y)\tilde{B}(X,Z)W \\
+ g(X,Z)\tilde{B}(Y,\xi)W - \eta(Z)\tilde{B}(Y,X)W + g(X,W)\tilde{B}(Y,Z)\xi - \eta(W)\tilde{B}(Y,Z)X = 0.
\]
By taking inner product with $\xi$ and using (2), we have

\[
\eta(B(Y, Z)W)\eta(X) - g(B(Y, Z)W, X)\xi + g(X, Y)\eta(B(\xi, Z)W) - \eta(Y)\eta(B(X, Z)W)
+ g(X, Z)\eta(B(Y, \xi)W) - \eta(Z)\eta(B(Y, X)W) + g(X, W)\eta(B(Y, Z)\xi) - \eta(W)\eta(B(Y, X)Z) = 0.
\]  

(22)

By using (20) in (22), we arrive at

\[
\left\{ \frac{1}{6}(\lambda - \frac{1}{2}(p + \frac{2}{3})) + \frac{4}{6} \right\} \left[ \begin{array}{c}
g(Y, W)g(X, Z) \\
-g(Z, W)g(X, Y)
\end{array} \right] - g(\tilde{B}(Y, Z)W, X) = 0.
\]  

(23)

Now using (19) in (23), we get

\[
\left\{ \frac{1}{6}(\lambda - \frac{1}{2}(p + \frac{2}{3})) + \frac{4}{6} \right\} \left[ \begin{array}{c}
g(Y, W)g(X, Z) \\
-g(Z, W)g(X, Y)
\end{array} \right] - g(B(Y, Z)W, X)
- \alpha^2 \left\{ \begin{array}{c}
S(Z, W)g(Y, W) - S(Z, W)g(Y, X) \\
+ S(Y, W)g(X, Z) - S(Y, X)g(Z, W)
\end{array} \right\} \\
- \beta^2 \left\{ \begin{array}{c}
g(Y, W)g(\phi Y, X) - g(\phi Y, W)g(\phi Y, X) \\
+ g(X, Z)\eta(Y)\eta(W) - g(X, Y)\eta(Z)\eta(W)
\end{array} \right\}
\]  

\[
- \frac{1}{6} \left\{ \begin{array}{c}
S(Z, W)\eta(Y)\eta(X) - S(Z, X)\eta(Y)\eta(W)
+ 4\beta^2 g(Y, W)g(X, Z) - 4\beta^2 g(Z, W)g(Y, X)
+ 2\alpha g(Y, W)\eta(X)\eta(Z) + 2\alpha g(Y, X)\eta(Z)\eta(W)
+ 2\alpha g(Z, W)\eta(Y)\eta(X) - 2\alpha g(X, Z)\eta(Y)\eta(W)
+ 4\beta^2 g(\phi Y, W)g(\phi Y, X) - 4\beta^2 g(\phi Z, W)g(\phi Y, X)
+ 2\beta^2 g(Y, X)\eta(Z)\eta(W) - 2\beta^2 g(Y, W)\eta(Z)\eta(X)
+ 2\beta^2 g(Z, W)\eta(Y)\eta(X) - 2\beta^2 g(Z, X)\eta(Y)\eta(W)
\end{array} \right\}
\]  

\[
+ \frac{D + 2}{6} \left\{ \begin{array}{c}
g(\phi Y, W)g(\phi Y, Z) - g(\phi Y, W)g(\phi Y, X)
+ 2g(\phi Y, Z)g(\phi W, X)
\end{array} \right\}
\]  

\[
- \frac{D}{6} \left\{ \begin{array}{c}
g(Y, W)\eta(Y)\eta(X) - \eta(Z)\eta(W)g(Y, X)
- g(Z, W)\eta(Y)\eta(X) + \eta(Y)\eta(Z)g(X, Z)
\end{array} \right\}
\]  

\[
+ \frac{D - 4}{6} \left\{ g(Y, W)g(Z, X) - g(Z, W)g(Y, X) \right\} = 0.
\]

Taking $Y = X = e_i$ and summing over $i = 1, 2, 3$, where $e_i$ is an orthonormal basis of $T_pM$, we obtain

\[
S(Z, W) = \left\{ \begin{array}{c}
\frac{1}{6} - \beta + \beta^2 - \alpha^2 \\
+ \frac{1}{3}(\lambda - \frac{1}{2}(p + \frac{2}{3})) - \frac{2\alpha}{3} - 1
\end{array} \right\} g(Z, W)
+ \left\{ \begin{array}{c}
\frac{2\alpha}{3} + 1 - \frac{1}{6}
\end{array} \right\} \eta(Z)\eta(W).
\]
Taking $Z = W = \xi$ in the above equation with using (17), we arrive at

$$\lambda - \frac{1}{2} \left( p + \frac{2}{3} \right) = 2 \left( \alpha^2 - \beta^2 \right).$$

Since $\alpha^2 \neq \beta^2$, in this case we have the following.

i) Assume that $\alpha^2 \geq \beta^2$, then $(\alpha - \beta)(\alpha + \beta) \geq 0$, which implies $\alpha$ greater than $\beta$. So, $\lambda > 0$ and Ricci soliton is shrinking.

ii) Assume that $\alpha^2 < \beta^2$ and $p + \frac{2}{3} > 4(\alpha^2 - \beta^2)$, then $(\alpha - \beta)(\alpha + \beta) < 0$, which gives $\alpha$ less than $-\beta$. So, $\lambda > 0$ and Ricci soliton is shrinking.

iii) Assume that $\alpha^2 < \beta^2$ and $p + \frac{2}{3} < 4(\alpha^2 - \beta^2)$, then $(\alpha - \beta)(\alpha + \beta) < 0$, which gives $\alpha$ less than $-\beta$. So, $\lambda < 0$ and Ricci soliton is expanding.

**Theorem 3.** A 3-dimensional trans-Sasakian manifold with respect to SVK connection satisfies $R(\xi, X).\tilde{B} = 0$ and admits conformal Ricci soliton, then

i) for $\alpha > \beta$, the Ricci soliton is shrinking;

ii) for $\alpha < -\beta$ and $p + \frac{2}{3} > 4(\alpha^2 - \beta^2)$, the Ricci soliton becomes shrinking;

iii) for $\alpha < -\beta$ and $p + \frac{2}{3} < 4(\alpha^2 - \beta^2)$, the Ricci soliton becomes expanding.

## 4 Conformal Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to SVK connection satisfying $\tilde{B}(\xi, X).S = 0$

The condition $\tilde{B}(\xi, X).S = 0$ implies that

$$S(\tilde{B}(\xi, X)Y, Z) + S(Y, \tilde{B}(\xi, X)Z) = 0. \tag{24}$$

Using (17) in (24), we get

$$\left\{ \begin{array}{c} \lambda - \frac{1}{2} \left( p + \frac{2}{3} \right) \\ -\beta - 2\beta^2 \end{array} \right\} \left[ g(\tilde{B}(\xi, X)Y, Z) + g(Y, \tilde{B}(\xi, X)Z) \right] + \left\{ 2\alpha^2 + \beta \right\} \left[ \eta(\tilde{B}(\xi, X)Y)\eta(Z) + \eta(\tilde{B}(\xi, X)Z)\eta(Y) \right] = 0.$$  

By using (20) in the above equation, we obtain

$$\left\{ 2\alpha^2 + \beta \right\} \left\{ \begin{array}{c} \lambda - \frac{1}{2} \left( p + \frac{2}{3} \right) \\ -\beta - 2\beta^2 \end{array} \right\} \left[ -g(X,Y)\eta(Z) - g(X,Z)\eta(Y) \right] = 0.$$  

Taking $Y = X = e_i$ and summing over $i = 1, 2, 3$, where $e_i$ is an orthonormal basis of $T_pM$ and taking condition $\eta(Z) \neq 0$, we obtain

$$\left\{ 2\alpha^2 + \beta \right\} \left[ \frac{1}{6} \left( \lambda - \frac{1}{2} \left( p + \frac{2}{3} \right) \right) + \frac{4}{6} \right] = 0.$$  

Thus we get the following result.

**Theorem 4.** A 3-dimensional trans-Sasakian manifold with respect to SVK connection satisfies $\tilde{B}(\xi, X).S = 0$ and admits conformal Ricci soliton with $2\alpha^2 \neq \beta$, then

i) if $p = \frac{27}{5}$, then $\lambda = 0$ and Ricci soliton is steady;

ii) if $p < \frac{27}{5}$, then $\lambda < 0$ and Ricci soliton is expanding;

iii) if $p > \frac{27}{5}$, then $\lambda > 0$ and Ricci soliton is shrinking.
5 Conformal Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to SVK connection satisfying $\ddot{S}(\xi, X).\tilde{R} = 0$

If we consider the condition $\ddot{S}(\xi, X).\tilde{R} = 0$, then we have

$$
(\ddot{S}(\xi, X).\tilde{R})(U, V)W = \ddot{S}(\xi, \tilde{R}(U, V)W)X - \ddot{S}(X, \tilde{R}(U, V)W)\xi + \ddot{S}(\xi, U)\tilde{R}(X, V)W \\
- \ddot{S}(\xi, U)\tilde{R}(\xi, V)W + \ddot{S}(\xi, V)\tilde{R}(U, X)W - \ddot{S}(\xi, V)\tilde{R}(U, \xi)W \\
+ \ddot{S}(\xi, W)\tilde{R}(U, V)X - \ddot{S}(X, W)\tilde{R}(U, V)\xi.
$$

(25)

By using (16) in (25), we have

$$
\left\{ \lambda - \frac{1}{2} \left( p + \frac{2}{3} \right) \right\} \left[ \eta(\tilde{R}(U, V)W)X + \eta(U)\tilde{R}(X, V)W \\
+ \eta(V)\tilde{R}(U, X)W + \eta(W)\tilde{R}(U, V)X \\
- \left\{ \lambda - \frac{1}{2} \left( p + \frac{2}{3} \right) - \beta \right\} \left[ g(X, \tilde{R}(U, V)W)\xi + g(X, U)\tilde{R}(\xi, V)W \\
+ g(X, V)\tilde{R}(U, \xi)W + g(W, \tilde{R}(U, V)\xi) \right] \\
- \beta \left[ \eta(X)\eta(\tilde{R}(U, V)W)\xi + \eta(X)\eta(U)\tilde{R}(\xi, V)W \\
+ \eta(X)\eta(V)\tilde{R}(U, \xi)W + \eta(X)\eta(W)\tilde{R}(U, V)\xi \right] = 0.
$$

By taking inner product with $\xi$ and using (11), we get

$$
\left\{ \lambda - \frac{1}{2} \left( p + \frac{2}{3} \right) \right\} \left[ \begin{array}{c}
\eta(X)\phi(U, V)W \\
g(X, \tilde{R}(U, V)W)\xi + g(X, U)\tilde{R}(\xi, V)W \\
+ g(X, V)\tilde{R}(U, \xi)W + g(W, \tilde{R}(U, V)\xi) \\
\eta(X)\eta(\tilde{R}(U, V)W)\xi + \eta(X)\eta(U)\tilde{R}(\xi, V)W \\
+ \eta(X)\eta(V)\tilde{R}(U, \xi)W + \eta(X)\eta(W)\tilde{R}(U, V)\xi
\end{array} \right] = 0.
$$

Taking $X = U = e_i$ and summing over $i = 1, 2, 3$, where $e_i$ is an orthonormal basis of $T_pM$, we find

$$
\left\{ \lambda - \frac{1}{2} \left( p + \frac{2}{3} \right) \right\} \left[ \begin{array}{c}
g(X, R(U, V)W) \\
g(\phi(U, V)W)\xi + g(\phi(U, V)W)\xi \\
+ g(X, \tilde{R}(U, V)W)\xi + g(X, \tilde{R}(U, V)W)\xi \\
\eta(X)\eta(U)\eta(V) - g(X, U)\eta(X)\eta(W) \\
\eta(X)\eta(U)\eta(V) + g(X, U)\eta(X)\eta(W) \\
+ \beta^2 g(X, V)g(X, U) - g(U, W)g(X, V)
\end{array} \right] = 0.
$$

Taking $V = W = \xi$ in the above equation, we arrive at

$$
\left\{ \lambda - \frac{1}{2} \left( p + \frac{2}{3} \right) \right\} \left\{ \lambda - \frac{1}{2} \left( p + \frac{2}{3} \right) - \alpha^2 \right\} = 0,
$$

which gives either $\lambda = \frac{1}{2} \left( p + \frac{2}{3} \right)$ or $\lambda = \alpha^2 + \frac{1}{2} \left( p + \frac{2}{3} \right)$.

So we can give the following result.

**Theorem 5.** A 3-dimensional trans-Sasakian manifold with respect to SVK connection satisfies $\ddot{S}(\xi, X).\tilde{R} = 0$ and admits conformal Ricci soliton, then

**Case I:** $\lambda$ depends only on $p$

i) if $p = -\frac{2}{3}$, then $\lambda = 0$ and Ricci soliton is steady;

ii) if $p > -\frac{2}{3}$, then $\lambda > 0$ and Ricci soliton is shrinking;

iii) if $p < -\frac{2}{3}$, then $\lambda < 0$ and Ricci soliton is expanding;

**Case II:** $\lambda$ depends on both $p$ and $\alpha$

i) if $p = -\frac{2}{3} - 2\alpha^2$, then $\lambda = 0$ and Ricci soliton is steady;

ii) if $p > -\frac{2}{3} - 2\alpha^2$, then $\lambda > 0$ and Ricci soliton is shrinking;

iii) if $p < -\frac{2}{3} - 2\alpha^2$, then $\lambda < 0$ and Ricci soliton is expanding.
6 Conformal Ricci soliton on a 3-dimensional trans-Sasakian manifold with respect to SVK connection satisfying $\tilde{B}.\phi = 0$

Now, we examine the curvature condition $\tilde{B}.\phi = 0$ on a 3-dimensional trans-Sasakian manifold with respect to SVK connection admitting a conformal Ricci soliton.

We know that if $\tilde{B}.\phi = 0$, then we have

$$(\tilde{B}.\phi)(X, Y)Z = 0,$$

which gives

$$\tilde{B}(X, Y)\phi Z - \phi \tilde{B}(X, Y)Z = 0. \quad (26)$$

If we take $Z = \xi$ in (26), we find

$$\phi \tilde{B}(X, Y)\xi = 0. \quad (27)$$

Using (20) in (27), we obtain

$$\left\{ \frac{1}{6} \left( \lambda - \frac{1}{2} \left( p + \frac{2}{3} \right) \right) + \frac{4}{6} \right\} \left[ \eta(X)\phi Y - \eta(Y)\phi X \right] = 0. \quad (28)$$

Replacing $X$ by $\phi X$ in (28) and using (2), we get

$$\left\{ \frac{1}{6} \left( \lambda - \frac{1}{2} \left( p + \frac{2}{3} \right) \right) + \frac{4}{6} \right\} \left[ (-X + \eta(X))\eta(Y) \right] = 0. \quad (29)$$

Taking $Y = \xi$ and replacing $X$ by $\phi X$ in (29), we have

$$\left\{ \frac{1}{6} \left( \lambda - \frac{1}{2} \left( p + \frac{2}{3} \right) \right) + \frac{4}{6} \right\} \phi X = 0.$$

Finally, taking inner product with $W$, we obtain

$$\left\{ \frac{1}{6} \left( \lambda - \frac{1}{2} \left( p + \frac{2}{3} \right) \right) + \frac{4}{6} \right\} g(\phi X, W) = 0.$$

It follows that

$$\frac{1}{6} \left( \lambda - \frac{1}{2} \left( p + \frac{2}{3} \right) \right) + \frac{4}{6} = 0.$$

So we can state the following result.

**Theorem 6.** A 3-dimensional trans-Sasakian manifold with respect to SVK connection satisfies $\tilde{B}.\phi = 0$ and admits conformal Ricci soliton, then

i) if $p = \frac{22}{3}$, then $\lambda = 0$ and Ricci soliton is steady;

ii) if $p < \frac{22}{3}$, then $\lambda < 0$ and Ricci soliton is expanding;

iii) if $p > \frac{22}{3}$, then $\lambda > 0$ and Ricci soliton is shrinking.
Example 1. We consider a 3-dimensional manifold \( \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \neq 0 \} \), where \((x_1, x_2, x_3)\) are standard coordinates in \(\mathbb{R}^3\). Let \(\{\omega_1, \omega_2, \omega_3\}\) be a linearly independent global frame on \(M\) defined by
\[
\omega_1 = e^{-\omega_3} \left( \frac{\partial}{\partial \omega_1} + \frac{\partial}{\partial \omega_2} \right), \quad \omega_2 = e^{-\omega_3} \left( -\frac{\partial}{\partial \omega_1} + \frac{\partial}{\partial \omega_2} \right), \quad \omega_3 = \frac{\partial}{\partial \omega_3}.
\]
Let \(g\) be the Riemannian metric defined by
\[
g(\omega_1, \omega_1) = g(\omega_2, \omega_2) = g(\omega_3, \omega_3) = 1, \quad g(\omega_1, \omega_2) = g(\omega_1, \omega_3) = g(\omega_2, \omega_3) = 0.
\]
If \(\eta\) is the 1-form defined by \(\eta(W) = g(W, \omega_3)\) and if \(\phi\) is the \((1, 1)\)-tensor field defined by
\[
\phi(\omega_1) = \omega_2, \quad \phi(\omega_2) = -\omega_1, \quad \phi(\omega_3) = 0.
\]
Also, we have
\[
\eta(\omega_3) = 1, \quad \phi Y = -Y + \eta(Y)\xi, \quad g(\phi Y, \phi V) = g(Y, V) + \eta(Y)\eta(V).
\]
Now, we get
\[
[\omega_1, \omega_2] = 0, \quad [\omega_1, \omega_3] = 0, \quad [\omega_2, \omega_3] = 0.
\]
From Koszul’s formula, we obtain
\[
\nabla_{\omega_1} \omega_1 = -\omega_3, \quad \nabla_{\omega_2} \omega_1 = 0, \quad \nabla_{\omega_3} \omega_1 = 0, \\
\nabla_{\omega_1} \omega_2 = 0, \quad \nabla_{\omega_2} \omega_2 = -\omega_3, \quad \nabla_{\omega_3} \omega_2 = 0, \quad \nabla_{\omega_1} \omega_3 = \omega_1, \quad \nabla_{\omega_2} \omega_3 = \omega_2, \quad \nabla_{\omega_3} \omega_3 = \omega_3.
\]
In view of the above equations, \((\phi, \xi, \eta, g)\) satisfy (2) and (3) with \(\alpha = 0\) and \(\beta = 1\). So, \(M\) is a trans-Sasakian manifold [31]. In view of (30), we find
\[
R(\omega_1, \omega_2) \omega_3 = 0, \quad R(\omega_2, \omega_3) \omega_3 = -\omega_2, \quad R(\omega_1, \omega_3) \omega_3 = -\omega_1, \\
R(\omega_1, \omega_2) \omega_2 = -\omega_1, \quad R(\omega_2, \omega_3) \omega_2 = \omega_3, \quad R(\omega_1, \omega_3) \omega_2 = 0, \\
R(\omega_1, \omega_2) \omega_1 = \omega_2, \quad R(\omega_2, \omega_3) \omega_1 = 0, \quad R(\omega_1, \omega_3) \omega_1 = \omega_3.
\]
If we consider SVK connection to this equation from (10) with (30), we obtain
\[
\tilde{\nabla}_{\omega_1} \omega_1 = (\beta - 1)\omega_3, \quad \tilde{\nabla}_{\omega_2} \omega_1 = \alpha \omega_3, \quad \tilde{\nabla}_{\omega_3} \omega_1 = 0, \\
\tilde{\nabla}_{\omega_1} \omega_2 = -\alpha \omega_3, \quad \tilde{\nabla}_{\omega_2} \omega_2 = (\beta - 1)\omega_3, \quad \tilde{\nabla}_{\omega_3} \omega_2 = 0, \quad \tilde{\nabla}_{\omega_1} \omega_3 = (1 - \beta)\omega_1 + \alpha \omega_2, \quad \tilde{\nabla}_{\omega_2} \omega_3 = (1 - \beta)\omega_2 + \alpha \omega_1, \quad \tilde{\nabla}_{\omega_3} \omega_3 = 0.
\]
We know that \(M\) is a trans-Sasakian manifold with respect to SVK connection. In view of (31), we get
\[
\tilde{R}(\omega_1, \omega_2) \omega_3 = 0, \quad \tilde{R}(\omega_2, \omega_3) \omega_3 = (\beta^2 - \alpha^2 - 1) \omega_2, \\
\tilde{R}(\omega_1, \omega_2) \omega_2 = (\beta^2 - \alpha^2 - 1) \omega_1, \quad \tilde{R}(\omega_2, \omega_3) \omega_2 = (1 + \alpha^2 - \beta^2) \omega_3, \\
\tilde{R}(\omega_1, \omega_2) \omega_1 = (1 + \alpha^2 - \beta^2) \omega_2, \quad \tilde{R}(\omega_2, \omega_3) \omega_1 = 0, \\
\tilde{R}(\omega_1, \omega_3) \omega_1 = \omega_3, \quad \tilde{R}(\omega_1, \omega_3) \omega_2 = 0, \\
\tilde{R}(\omega_1, \omega_3) \omega_3 = (\beta^2 - \alpha^2 - 1) \omega_1.
\]
From the above equation, we arrive at
\[
\tilde{S}(\omega_1, \omega_1) = 2(\beta^2 - 1), \quad \tilde{S}(\omega_2, \omega_2) = 2(\beta^2 - 1), \quad \tilde{S}(\omega_3, \omega_3) = 2(\beta^2 - \alpha^2 - 1).
\]
Finally, in view of (1), if \(\lambda = 2\beta^2 - \beta + 2 + \frac{p}{2} + \frac{1}{3}\) and \(\beta = -2\alpha^2\), then \(M\) admits a conformal Ricci soliton with respect to SVK connection.
References


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