ISSN 2075-9827 e-ISSN 2313-0210 https://journals.pnu.edu.ua/index.php/cmp
Carpathian Math. Publ. 2024, **16** (1), 309–319 Карпатські матем. публ. 2024, Т.16, №1, С.309–319
doi:10.15330/cmp.16.1.309-319

Algebras of polynomials generated by linear operators

Zaj F., Abtahi M.✉

Let E be a Banach space and A be a commutative Banach algebra with identity. Let $\mathbb{P}(E,A)$ be the space of A-valued polynomials on E generated by bounded linear operators (an n-homogenous polynomial in $\mathbb{P}(E,A)$ is of the form $P=\sum_{i=1}^{\infty} T_i^n$, where $T_i : E \to A$, $1 \le i < \infty$, are bounded linear operators and $\sum_{i=1}^{\infty} \|T_i\|^n < \infty$). For a compact set K in E, we let $\mathbb{P}(K,A)$ be the closure in $\mathscr{C}(K,A)$ of the restrictions $P|_K$ of polynomials P in $\mathbb{P}(E,A)$. It is proved that $\mathbb{P}(K,A)$ is an A-valued uniform algebra and that, under certain conditions, it is isometrically isomorphic to the injective tensor product $\mathcal{P}_N(K)\,\widehat{\otimes}_\epsilon\, A$, where $\mathcal{P}_N(K)$ is the uniform algebra on K generated by nuclear scalar-valued polynomials. The character space of $\mathbb{P}(K,A)$ is then identified with $\hat{K}_N \times \mathfrak{M}(A)$, where $\hat{K}_N$ is the nuclear polynomially convex hull of K in E, and $\mathfrak{M}(A)$ is the character space of A.

Key words and phrases: vector-valued uniform algebra, polynomial on a Banach space, nuclear polynomial, polynomial convexity, tensor product.

School of Mathematics and Computer Sciences, Damghan University, PO Box 36715-364, Damghan, Iran
✉ Corresponding author
E-mail: zaj.farzane@yahoo.com (Zaj F.), abtahi@du.ac.ir (Abtahi M.)

1 Introduction

Let E be a Banach space and A be a commutative unital Banach algebra, over the complex field $\mathbb{C}$. For a compact set K in E, let $\mathscr{C}(K,A)$ be the algebra of all continuous functions $f : K \to A$ equipped with uniform norm

$$\|f\|_K = \sup\{\|f(x)\| : x \in K\}. \tag{1}$$

When $A=\mathbb{C}$, we write $\mathscr{C}(K)$ instead of $\mathscr{C}(K,\mathbb{C})$. Given an element $a \in A$, the same notation a is used for the constant function given by $a(x)=a$ for all $x \in K$, and A is regarded as a closed subalgebra of $\mathscr{C}(K,A)$. We denote by $\mathbf{1}$ the unit element of A, and identify $\mathbb{C}$ with the closed subalgebra $\mathbb{C}\mathbf{1} = \{\alpha\mathbf{1} : \alpha \in \mathbb{C}\}$ of A. Therefore, every function $f \in \mathscr{C}(K)$ can be seen as the A-valued function $x \mapsto f(x)\mathbf{1}$. We use the same notation f for this A-valued function, and regard $\mathscr{C}(K)$ as a closed subalgebra of $\mathscr{C}(K,A)$. By an *A-valued uniform algebra on K* we mean a closed subalgebra $\boldsymbol{A}$ of $\mathscr{C}(K,A)$ that contains the constant functions and separates points of K (see [1,12]). A comprehensive discussion on complex function algebras appears in [5, Chapter 4].

We are mostly interested in those A-valued uniform algebras that are invariant under composition with characters of A. An A-valued uniform algebra $\boldsymbol{A}$ is called *admissible* if $\phi \circ f \in \boldsymbol{A}$ whenever $f \in \boldsymbol{A}$ and $\phi : A \to \mathbb{C}$ is a character of A. Recall that a character is just a nonzero multiplicative linear functional. Denoted by $\mathfrak{M}(A)$, the set of all characters of A, equipped with the Gelfand topology, is a compact Hausdorff space.

УДК 517.98
2020 *Mathematics Subject Classification:* 46G25, 46J10, 46H05, 46J20.

When $\boldsymbol{A}$ is admissible, we let $\mathfrak{A}$ be the subalgebra of $\boldsymbol{A}$ consisting of scalar-valued functions, that is, $\mathfrak{A} = \boldsymbol{A} \cap \mathscr{C}(K)$. In this case, $\mathfrak{A} = \{\phi \circ f : f \in \boldsymbol{A}\}$ for every $\phi \in \mathfrak{M}(A)$. The algebra $\mathscr{C}(K, A)$ is admissible with $\mathfrak{A} = \mathscr{C}(K)$. It is well-known that $\mathscr{C}(K, A)$ is isometrically isomorphic to the injective tensor product $\mathscr{C}(K) \widehat{\otimes}_\epsilon A$, and that $\mathfrak{M}(\mathscr{C}(K, A)) = K \times \mathfrak{M}(A)$ (see [8]). In general, given an admissible A-valued uniform algebra $\boldsymbol{A}$, it is natural to ask whether the analogous equalities $\boldsymbol{A} = \mathfrak{A} \widehat{\otimes}_\epsilon A$ and $\mathfrak{M}(\boldsymbol{A}) = \mathfrak{M}(\mathfrak{A}) \times \mathfrak{M}(A)$ hold. In this paper, we investigate these questions for a certain A-valued uniform algebra that is generated by linear operators.

Remark 1. *For an admissible A-valued uniform algebra $\boldsymbol{A}$ with $\mathfrak{A} = \mathscr{C}(K) \cap \boldsymbol{A}$, let $\mathfrak{A}A$ denote the subalgebra of $\boldsymbol{A}$ generated by $\mathfrak{A} \cup A$. Indeed, $\mathfrak{A}A$ consists of elements of the form $f = f_1 a_1 + \cdots + f_n a_n$, where $f_i \in \mathfrak{A}$ and $a_i \in A$ for $1 \le i \le n$, $n \in \mathbb{N}$. The following statements are equivalent.*

1. *$\boldsymbol{A} = \mathfrak{A} \widehat{\otimes}_\epsilon A$ (isometrically isomorphic).*
2. *The subalgebra $\mathfrak{A}A$ is dense in $\boldsymbol{A}$.*

In fact, the mapping $\Lambda_0 : f \otimes a \mapsto fa$ defines a homomorphism of $\mathfrak{A} \otimes A$ onto $\mathfrak{A}A$, and, using Hahn-Banach theorem, we have

$$\begin{aligned} \left\| \Lambda_0\left(\sum_{i=1}^n f_i \otimes a_i \right) \right\|_K &= \sup_{x \in K} \left\| \sum_{i=1}^n f_i(x) a_i \right\| = \sup_{x \in K} \sup_{\phi \in A_1^*} \left| \sum_{i=1}^n f_i(x) \phi(a_i) \right| \\ &= \sup_{\phi \in A_1^*} \left\| \sum_{i=1}^n f_i(\cdot) \phi(a_i) \right\|_K = \sup_{\phi \in A_1^*} \sup_{\psi \in \mathfrak{A}_1^*} \left| \psi\left(\sum_{i=1}^n f_i(\cdot)\phi(a_i) \right) \right| \\ &= \sup_{\psi \in \mathfrak{A}_1^*} \sup_{\phi \in A_1^*} \left| \sum_{i=1}^n \psi(f_i)\phi(a_i) \right| = \left\| \sum_{i=1}^n f_i \otimes a_i \right\|_\epsilon, \end{aligned}$$

where $\mathfrak{A}_1^*$ and A_1^* denote the unit balls of $\mathfrak{A}^*$ and A^*, respectively. Therefore, Λ_0 extends to an isometry of $\mathfrak{A} \widehat{\otimes}_\epsilon A$ onto $\overline{\mathfrak{A}A}$. Given $f \in \mathfrak{A}$ and $a \in A$, we may identify the A-valued function fa with the elementary tensor $f \otimes a$.

Remark 2 ([10])**.** *It is always possible to define a product $(a, b) \mapsto ab$ on the Banach space E in order to make it an algebra with identity. Indeed, take a nonzero functional $\psi \in E^*$ and a vector $e \in E$ such that $\psi(e) \neq 0$ and $\|e\| = 1$. Then $E = \ker\psi \oplus \mathbb{C}e$. For $a, b \in E$, write $a = x + \psi(a)e$ and $b = y + \psi(b)e$, with $x, y \in \ker\psi$, and define $ab = \psi(b)x + \psi(a)y + \psi(a)\psi(b)e$. This makes E an algebra with identity e. Define a norm on E by $\|a\|_1 = |\psi(a)| + \|x\|$, where $a = x + \psi(a)e$. Then, $\|\cdot\|_1$ is an equivalent norm on E making it a commutative unital Banach algebra. This observation allows us to consider $\mathscr{C}(K, E)$ as a Banach algebra, even if the Banach space E is not assumed to be an algebra in the first place.*

In this paper, we consider the space $\mathbb{P}(E, A)$ of all A-valued polynomials on E generated by bounded linear operators $T : E \to A$. By definition, an n-homogenous polynomial P in $\mathbb{P}(E, A)$ is of the form $P = \sum_{i=1}^\infty T_i^n$, where (T_i) is a sequence of bounded linear operators of E into A such that $\sum_{i=1}^\infty \|T_i\|^n < \infty$. This class of polynomials was introduced in [10] with further study carried out in [7]. For a compact set K in E, we let $\mathbb{P}(K, A)$ be the closure in $\mathscr{C}(K, A)$ of the restrictions $P|_K$ of polynomials $P \in \mathbb{P}(E, A)$. It is proved that $\mathbb{P}(K, A)$ is an admissible A-valued uniform algebra on K with $\mathfrak{A} = \mathcal{P}_N(K)$, where $\mathcal{P}_N(K)$ represents the (complex) uniform algebra on K generated by nuclear polynomials. In fact, $f \in \mathcal{P}_N(K)$ if and only if there is a sequence (P_k) of nuclear scalar-valued polynomials on E such that $P_k \to f$ uniformly on K. We prove that the following are equivalent:

(i) $\mathbb{P}(K, A) = \mathcal{P}_N(K) \widehat{\otimes}_\epsilon A$ for a Banach algebra A,

(ii) $\mathbb{P}(K, E) = \mathcal{P}_N(K) \widehat{\otimes}_\epsilon E$ (see Remark 2),

(iii) $I \in \mathcal{P}_N(K) \widehat{\otimes}_\epsilon E$, where $I : E \to E$ is the identity operator.

In this situation, the character space $\mathfrak{M}\big(\mathbb{P}(K, A)\big)$ is identified with $\hat{K}_N \times \mathfrak{M}(A)$, where $\hat{K}_N$ denotes the nuclear polynomially convex hull of K in E.

2 Algebras of polynomials generated by linear operators

First, let us recall basic definitions, notations and some results of the theory of polynomials on Banach spaces. For comprehensive texts, see [6, 11, 13].

Let E and F be Banach spaces over $\mathbb{C}$. The space of all continuous symmetric n-linear operators $T : E^n \to F$ is denoted by $\mathcal{L}_s(^nE, F)$. For $T \in \mathcal{L}_s(^nE, F)$, define $\hat{T}(x) = T(x, \ldots, x)$, $x \in E$. A mapping $P : E \to F$ is said to be an n-homogeneous polynomial if $P = \hat{T}$ for some $T \in \mathcal{L}_s(^nE, F)$. For convenience, 0-homogeneous polynomials are defined as constant mappings from E into F. The vector space of all n-homogeneous polynomials from E into F is denoted by $\mathcal{P}(^nE, F)$. The shortened notation $\mathcal{P}(^nE)$ is used when $F = \mathbb{C}$. A norm on $\mathcal{P}(^nE, F)$ is defined as

$$\|P\| = \sup\{\|P(x)\| : \|x\| \leq 1\}, \quad P \in \mathcal{P}(^nE, F). \tag{2}$$

Continuity of P (and of the corresponding symmetric n-linear operator T) is then equivalent to finiteness of $\|P\|$. Given an n-homogeneous polynomial $P \in \mathcal{P}(^nE, F)$, the symmetric n-linear mapping T that gives rise to P can be recovered by any of several polarization formulae, e.g.,

$$T(x_1, \ldots, x_n) = \frac{1}{2^n n!} \sum_{\epsilon_j = \pm 1} \epsilon_1 \cdots \epsilon_n P\Big(\sum_{j=1}^{n} \epsilon_j x_j\Big). \tag{3}$$

As a special case, for $a, b \in A$ and $m, n \in \mathbb{N}$, we get

$$a^m b^n = \frac{1}{2^{m+n}(m+n)!} \sum_{\epsilon_\ell = \pm 1} \epsilon_1 \cdots \epsilon_{m+n} \left(\sum_{\ell=1}^{m} \epsilon_\ell a + \sum_{\ell=m+1}^{m+n} \epsilon_\ell b\right)^{m+n}. \tag{4}$$

Therefore, n-homogeneous polynomials and symmetric n-linear operators are in one-to-one correspondence, and the polarization inequality $\|P\| \leq \|T\| \leq \frac{n^n}{n!}\|P\|$ shows that these spaces are isomorphic (see [6, Corollary 1.7 and Proposition 1.8]).

Notation. Let $T \in \mathcal{L}_s(^nE, F)$ and $x, y \in E$. For $0 \leq k \leq n$, let

$$T(x^k, y^{n-k}) = T(\underbrace{x, x, \ldots, x}_{k \text{ times}}, \underbrace{y, y, \ldots, y}_{n-k \text{ times}}). \tag{5}$$

Then, by [11, Theorem 1.8], we have the *Leibniz formula*

$$T\big((x+y)^n\big) = \sum_{k=1}^{n} \binom{n}{k} T(x^k, y^{n-k}). \tag{6}$$

2.1 The A-valued uniform algebra generated by linear operators

To continue, we restrict ourselves to Banach algebra valued polynomials. We let A be a commutative Banach algebra with identity $\mathbf{1}$. Adopting notations from [10], we denote by $\mathbb{P}\left({}^{n}E, A\right)$ the space of n-homogeneous A-valued polynomials on E of the form

$$P(x)=\sum_{i=1}^{\infty} T_i(x)^n, \quad x \in E, \tag{7}$$

where (T_i) is a sequence in $\mathcal{L}(E, A)$ such that $\sum_{i=1}^{\infty}\|T_i\|^n<\infty$. A norm on $\mathbb{P}\left({}^{n}E, A\right)$ is defined by

$$|||P|||=\inf \left\{\sum_{i=1}^{\infty}\|T_i\|^n : P=\sum_{i=1}^{\infty} T_i^n\right\}, \tag{8}$$

where the infimum is taken over all possible representations of P in (7). It is easy to verify that $|||\cdot|||$ is a norm and that $\|P\| \leq |||P|||$ for all $P \in \mathbb{P}\left({}^{n}E, A\right)$. The following shows that convergence with respect to this norm implies uniform convergence on compact sets.

Proposition 1. *Let K be a compact set in E. Then there is $M>0$ such that*

$$\|P\|_K \leq M^n |||P|||, \quad P \in \mathbb{P}\left({}^{n}E, A\right). \tag{9}$$

Consequently, the series in (7) *converges uniformly on K.*

Proof. Since K is compact, there is a constant M such that $\|x\| \leq M$ for all $x \in K$. Therefore, $\|Tx\| \leq M\|T\|$ for every $x \in K$ and $T \in \mathcal{L}(E, A)$. If a polynomial P is defined by (7), then

$$\|P\|_K=\sup_{x \in K}\left\|\sum_{i=1}^{\infty} T_i(x)^n\right\| \leq \sup_{x \in K} \sum_{i=1}^{\infty}\left\|T_i(x)\right\|^n \leq M^n \sum_{i=1}^{\infty}\|T_i\|^n.$$

The above inequality holds for any representation of P in (7). Taking infimum over all those representations, as in (8), we get

$$\|P\|_K \leq M^n |||P|||.$$

Also, we have

$$\left\|P-\sum_{i=1}^{s} T_i^n\right\|_K=\left\|\sum_{i=s+1}^{\infty} T_i^n\right\|_K \leq M^n \left|\left|\left|\sum_{i=s+1}^{\infty} T_i^n\right|\right|\right| \leq M^n \sum_{i=s+1}^{\infty}\|T_i\|^n \rightarrow 0 \quad \text{as } s \rightarrow \infty.$$

This shows that the series in (7) converges uniformly on K. □

If A is replaced by $\mathbb{C}$, we reach the nuclear (scalar-valued) polynomials. By definition, an n-homogenous polynomial $P: E \rightarrow \mathbb{C}$ is called *nuclear* if it can be written in a form

$$P(x)=\sum_{i=1}^{\infty} \psi_i(x)^n, \quad x \in E, \tag{10}$$

where (ψ_i) is a sequence in E^* with $\sum_{i=1}^{\infty}\|\psi_i\|^n<\infty$. We denote by $\mathcal{P}_N({}^{n}E)$ the space of all n-homogenous nuclear (scalar-valued) polynomials on E. In fact, $\mathcal{P}_N\left({}^{n}E\right)=\mathbb{P}\left({}^{n}E, \mathbb{C}\right)$. Nuclear polynomials between Banach spaces have been studied by many authors (see [2–4,15]).

We are now in a position to introduce the A-valued uniform algebras that these polynomials generate. Let $\mathbb{P}(E, A)$ be the space of all polynomials on E of the form $P=\sum_{k=0}^{n} P_k$, where $P_k \in \mathbb{P}\left({}^{k}E, A\right)$, $0 \leq k \leq n$, $n \in \mathbb{N}$. In the same fashion, the space $\mathcal{P}_N(E)$ of all linear combinations of homogenous nuclear polynomials on E is defined. In fact, $\mathcal{P}_N(E)=\mathbb{P}(E, \mathbb{C})$.

Definition 1. *Let K be a compact set in the Banach space E. Define*

$$\mathbb{P}_0(K,A)=\{P|_K : P\in\mathbb{P}(E,A)\},\quad \mathcal{P}_{N_0}(K)=\{P|_K : P\in\mathcal{P}_N(E)\}.$$

We define $\mathbb{P}(K,A)$ as the closure of $\mathbb{P}_0(K,A)$ in $\mathscr{C}(K,A)$. Similarly, $\mathcal{P}_N(K)$ is defined to be the closure of $\mathcal{P}_{N_0}(K)$ in $\mathscr{C}(K)$.

Therefore, $f\in\mathbb{P}(K,A)$ $\big($respectively, $f\in\mathcal{P}_N(K)\big)$ if and only if there is a sequence (P_k) of polynomials in $\mathbb{P}(E,A)$ $\big($respectively, $\mathcal{P}_N(E)\big)$, such that $P_k\to f$ uniformly on K.

Clearly, $\mathbb{P}(K,A)$ is a closed subspace of $\mathscr{C}(K,A)$. We are aiming to show that $\mathbb{P}(K,A)$ is a subalgebra of $\mathscr{C}(K,A)$. To achieve this, one may think of proving that the space $\mathbb{P}(E,A)$ itself is an algebra (that is, if $P\in\mathbb{P}(^mE,A)$ and $Q\in\mathbb{P}(^nE,A)$, then $PQ\in\mathbb{P}(^{m+n}E,A)$). This manner is naturally expected. However, the authors do not currently have strong evidence supporting or opposing the possibility that $\mathbb{P}(E,A)$ is an algebra. Our approach, therefore, is to give a direct proof of the fact that $\mathbb{P}(K,A)$ is an algebra, as follows.

Theorem 1. *For a compact set K in E, let $\boldsymbol{A}=\mathbb{P}(K,A)$. Then $\boldsymbol{A}$ is an admissible A-valued uniform algebra on K with $\mathfrak{A}=\mathcal{P}_N(K)$.*

Proof. Since $\mathbb{P}(E,A)$ contains 0-homogenous polynomials, we see that $\boldsymbol{A}$ contains the constant functions, and since $\mathbb{P}(E,A)$ contains the 1-homogenous polynomials of the form $P=\psi\mathbf{1}$ with $\psi\in E^*$, we see that $\boldsymbol{A}$ separates points of K. To prove that $\boldsymbol{A}$ is an algebra, we show that it is closed under multiplication, that is,

$$f,g\in\boldsymbol{A}\Rightarrow fg\in\boldsymbol{A}. \tag{11}$$

Given $f,g\in\boldsymbol{A}$, there exist sequences (P_k) and (Q_k) of polynomials in $\mathbb{P}(E,A)$, such that $P_k\to f$ and $Q_k\to g$ uniformly on K, from which we get $P_kQ_k\to fg$ uniformly on K. Therefore, (11) reduces to the following implication

$$P,Q\in\mathbb{P}(E,A)\Rightarrow PQ\in\boldsymbol{A}. \tag{12}$$

Since every polynomial in $\mathbb{P}(E,A)$ is a linear combination of homogenous polynomials, we may assume that $P\in\mathbb{P}(^mE,A)$ and $Q\in\mathbb{P}(^nE,A)$.

Consider two cases:

(1) $m,n\geq 1$,

(2) $m\geq 1$ and $n=0$.

In case (1), write $P=\sum_{i=1}^{\infty}S_i^m$ and $Q=\sum_{j=1}^{\infty}T_j^n$, where (S_i) and (T_j) are sequences of operators in $\mathcal{L}(E,A)$. By Proposition 1, these series converge uniformly on K, i.e.

$$\lim_{s\to\infty}\left\|P-\sum_{i=1}^{s}S_i^m\right\|_K=\lim_{s\to\infty}\left\|Q-\sum_{j=1}^{s}T_j^n\right\|_K=0.$$

Therefore,

$$\lim_{s\to\infty}\left\|PQ-\sum_{i=1}^{s}S_i^m\sum_{j=1}^{s}T_j^n\right\|_K=\lim_{s\to\infty}\left\|PQ-\sum_{i=1}^{s}\sum_{j=1}^{s}S_i^mT_j^n\right\|_K=0. \tag{13}$$

Applying polarization formula (4), we have

$$S_i^m T_j^n = \frac{1}{2^{m+n}(m+n)!} \sum_{\epsilon_\ell = \pm 1} \epsilon_1 \cdots \epsilon_{m+n} \left(\sum_{\ell=1}^{m} \epsilon_\ell S_i + \sum_{\ell=m+1}^{m+n} \epsilon_\ell T_j \right)^{m+n}.$$

With suitable choice of scalars α_k and operators U_k, $1 \le k \le 2^{m+n}$, we have

$$S_i^m T_j^n = \sum_{k=1}^{2^{m+n}} \alpha_k U_k^{m+n},$$

meaning that $S_i^m T_j^n \in \mathbb{P}(^{m+n}E, A)$. Now, (13) shows that PQ is approximated (uniformly on K) by elements of $\mathbb{P}(^{m+n}E, A)$ and thus $PQ \in \boldsymbol{A}$.

In case (2), write $P = \sum_{i=1}^{\infty} S_i^m$ and $Q = b$, where $b \in A$ is a constant. Since A is assumed to have an identity $\mathbf{1}$, the proof of [10, Proposition 2.1] shows that if $\lambda_k = \frac{1}{m^2} e^{\frac{2\pi k i}{m}}$ and $a_k = b + e^{\frac{2\pi k i}{m}} \mathbf{1}$, $1 \le k \le m$, then $b = \lambda_1 a_1^m + \cdots + \lambda_m a_m^m$. Therefore, if

$$b_k = \frac{e^{\frac{2\pi k i}{m^2}}}{\sqrt[m]{m^2}} \left(b + e^{\frac{2\pi k i}{m}} \mathbf{1} \right), \quad k = 1, 2, \ldots, m,$$

then $b = b_1^m + \cdots + b_m^m$. Now, for every $x \in E$, we have

$$(PQ)(x) = P(x)b = \sum_{j=1}^{\infty} \psi_j(x)^m \sum_{k=1}^{m} b_k^m = \sum_{j=1}^{\infty} \sum_{k=1}^{m} \left(\psi_j(x) b_k \right)^m = \sum_{k=1}^{m} \sum_{j=1}^{\infty} \left(\psi_j(x) b_k \right)^m.$$

Set $T_{jk} = \psi_j b_k$ and $P_k = \sum_{j=1}^{\infty} T_{jk}^m$. Then $P_k \in \mathbb{P}(^m E, A)$ and $PQ = P_1 + \cdots + P_m$. Therefore, PQ is an m-homogenous polynomial in $\mathbb{P}(E, A)$. In particular, $PQ \in \boldsymbol{A}$.

Finally, we show that $\mathbb{P}(K, A)$ is admissible, that is,

$$f \in \boldsymbol{A},\ \phi \in \mathfrak{M}(A) \Rightarrow \phi \circ f \in \boldsymbol{A}. \tag{14}$$

Given $f \in \boldsymbol{A}$, let (P_k) be a sequence of polynomials in $\mathbb{P}(E, A)$ such that $P_k \to f$ uniformly on K. For every $\phi \in \mathfrak{M}(A)$, since $\|\phi\| \le 1$, we have

$$\|\phi \circ P_k - \phi \circ f\|_K \le \|P_k - f\|_K,$$

and thus $\phi \circ P_k \to \phi \circ f$ uniformly on K. Therefore, (14) reduces to the following implication

$$P \in \mathbb{P}(E, A),\ \phi \in \mathfrak{M}(A) \Rightarrow \phi \circ P \in \boldsymbol{A}. \tag{15}$$

Given $P \in \mathbb{P}(E, A)$, since it is a linear combination of homogenous polynomials, we may assume that P itself is an n-homogenous polynomial and write $P = \sum_{i=1}^{\infty} T_i^n$ for $T_i \in \mathcal{L}(E, A)$. Then, for every $x \in E$,

$$(\phi \circ P)(x) = \phi\left(\sum_{i=1}^{\infty} T_i^n(x) \right) = \sum_{i=1}^{\infty} \phi\left(T_i(x)^n \right) = \sum_{i=1}^{\infty} \phi\left(T_i(x) \right)^n = \sum_{i=1}^{\infty} \left(\phi \circ T_i(x) \right)^n.$$

We see that $\phi \circ P$ is a nuclear polynomial, i.e. $\phi \circ P \in \mathcal{P}_N(E)$. Set $\psi_i = \phi \circ T_i$ and $S_i = \psi_i \mathbf{1}$. Then $S_i \in \mathcal{L}(E, A)$, $\|S_i\| \le \|T_i\|$, and

$$(\phi \circ P)\mathbf{1} = \sum_{i=1}^{\infty} (\psi_i \mathbf{1})^n = \sum_{i=1}^{\infty} S_i^n \in \mathbb{P}(K, A).$$

We conclude that $\mathbb{P}(K, A)$ is admissible with $\mathfrak{A} = \mathcal{P}_N(K)$. □

2.2 Representing as tensor product

We now investigate conditions that imply the equality $\mathbb{P}(K,A) = \mathcal{P}_N(K)\,\widehat{\otimes}_\epsilon\, A$. In the following, $I : E \to E$ is the identity operator.

Theorem 2. *Let K be a compact set in the Banach space E. The following statements are equivalent.*

(i) $\mathbb{P}(K,A) = \mathcal{P}_N(K)\,\widehat{\otimes}_\epsilon\, A$ *for every unital Banach algebra A.*

(ii) $\mathbb{P}(K,E) = \mathcal{P}_N(K)\,\widehat{\otimes}_\epsilon\, E$.

(iii) $I \in \mathcal{P}_N(K)\,\widehat{\otimes}_\epsilon\, E$.

Proof. The implication (i) ⇒ (ii) is trivial (see Remark 2). The implication (ii) ⇒ (iii) is also trivial since always $I \in \mathbb{P}(K,E)$.

We prove the implication (iii) ⇒ (i). Let A be a unital Banach algebra. In view of Remark 1 and Theorem 1, we always have $\mathcal{P}_N(K)\,\widehat{\otimes}_\epsilon\, A \subset \mathbb{P}(K,A)$. To prove the reverse inclusion, since $\overline{\mathbb{P}_0(K,A)} = \mathbb{P}(K,A)$, we just need to show that $\mathbb{P}_0(K,A) \subset \mathcal{P}_N(K)\,\widehat{\otimes}_\epsilon\, A$, this is, $P \in \mathcal{P}_N(K)\,\widehat{\otimes}_\epsilon\, A$ for every polynomial $P \in \mathbb{P}(E,A)$. We argue by induction on $n = \deg(P)$.

The base case, $n = 0$, trivially hold. In fact, if $\deg(P) = 0$ then P is a constant polynomial and belongs to $\mathcal{P}_N(K)\,\widehat{\otimes}_\epsilon\, A$. For the inductive step, take $n \in \mathbb{N}$ and assume that

$$\text{every polynomial } Q \text{ with } \deg(Q) < n \;\text{ belongs to } \mathcal{P}_N(K)\,\widehat{\otimes}_\epsilon\, A. \tag{16}$$

Let P be a polynomial with $\deg(P) = n$. Subtracting from P a polynomial Q with $\deg(Q) \le n-1$, if necessary, we may assume that P is an n-homogenous polynomial and that $P = \hat{T}$ for some $T \in \mathcal{L}_s({}^nE, A)$. Let $\epsilon > 0$. By the assumption, $I \in \mathcal{P}_N(K)\,\widehat{\otimes}_\epsilon\, E$. Therefore, by Remark 1, there exist functions $f_1, \ldots, f_m$ in $\mathcal{P}_N(K)$ and vectors $a_1, \ldots, a_m$ in E such that

$$\left\| I(x) - \sum_{i=1}^{m} f_i(x) a_i \right\| \le \epsilon, \quad x \in K. \tag{17}$$

Using notation (5) and Leibniz formula (6), we get

$$T\Bigg(\Big(x - \sum_{i=1}^{m} f_i(x)a_i\Big)^{n}\Bigg) = \sum_{k=0}^{n} \binom{n}{k} T\Bigg(x^k, \Big(-\sum_{i=1}^{m} f_i(x)a_i\Big)^{n-k}\Bigg). \tag{18}$$

Define

$$g_k(x) = \binom{n}{k} T\Bigg(x^k, \Big(-\sum_{i=1}^{m} f_i(x)a_i\Big)^{n-k}\Bigg) \quad \text{for } 0 \le k \le n-1 \text{ and } x \in K.$$

We have

$$\begin{aligned}
g_{n-1}(x) &= -n \sum_{i=1}^{m} f_i(x) T\big(x^{n-1}, a_i\big), \\
g_{n-2}(x) &= n(n-1) \sum_{i,j=1}^{m} f_i(x) f_j(x) T\big(x^{n-2}, a_i, a_j\big), \\
&\;\vdots \\
g_0(x) &= (-1)^n \sum_{\alpha} \frac{n!}{\alpha!} \prod_{i=1}^{m} f_i(x)^{\alpha_i} T\big(a_1^{\alpha_1}, a_2^{\alpha_2}, \ldots, a_m^{\alpha_m}\big),
\end{aligned}$$

where the last summation is taken over all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_m)$ in $\mathbb{N}_0^m$ such that $|\alpha| = \alpha_1 + \cdots + \alpha_m = n$.

We see that each g_j, $0 \leq j \leq n-1$, is an algebraic combination (multiplication and addition) of functions $f_1, f_2, \ldots, f_m$ in $\mathcal{P}_N(K)$, and some polynomials of degree j which, by assumption (16), belong to $\mathcal{P}_N(K) \widehat{\otimes}_\epsilon A$. Therefore, $g_0, g_1, \ldots, g_{n-1} \in \mathcal{P}_N(K) \hat{\otimes}_\epsilon A$.

On the other hand, for every $x \in K$,

$$\Big\| P(x) + \sum_{j=0}^{n-1} g_j(x) \Big\| = \Big\| T(x^n) + \sum_{k=0}^{n-1} \binom{n}{k} T\Big(x^k, \Big(-\sum_{i=1}^m f_i(x) a_i\Big)^{n-k}\Big) \Big\|$$
$$= \Big\| T\Big(\Big(x - \sum_{i=1}^m f_i(x) a_i\Big)^n\Big) \Big\| \leq \|T\| \Big\| x - \sum_{i=1}^m f_i(x) a_i \Big\|^n \leq \|T\| \epsilon^n,$$

where $\|T\|$ is the operator norm of T in $\mathcal{L}_s(E, A)$. Since ϵ is arbitrary, this means that P is approximated uniformly on K by functions in $\mathcal{P}_N(K) \widehat{\otimes}_\epsilon A$. Therefore, $P \in \mathcal{P}_N(K) \widehat{\otimes}_\epsilon A$ and the inductive argument is complete.

We conclude that $\mathbb{P}(K, A) = \mathcal{P}_N(K) \widehat{\otimes}_\epsilon A$. □

Recall (see, e.g., [11, Definition 27.3]) that a Banach space E has the *approximation property* if for every $\epsilon > 0$ and every compact set K in E there exists a finite rank operator $T : E \to E$ such that $\|T(x) - x\| < \epsilon$ for every $x \in K$.

Proposition 2. *Suppose that the Banach space E has the approximation property. Then $I \in \mathcal{P}_N(K) \widehat{\otimes}_\epsilon E$ for any compact set K in E.*

Proof. Let $\epsilon > 0$. By the approximation property, there is a finite-rank operator $T : E \to E$ such that $\|I - T\|_K \leq \epsilon$. The finite-rank operator T can be represented in a form

$$T = \psi_1 a_1 + \cdots + \psi_m a_m,$$

where $\psi_i \in E^*$, $a_i \in E$, $1 \leq i \leq m$. This means that $T \in \mathcal{P}_N(K) \widehat{\otimes}_\epsilon E$. Since ϵ is arbitrary, we conclude that $I \in \mathcal{P}_N(K) \widehat{\otimes}_\epsilon E$. □

For the Banach algebra A, let us denote by A^*_{cc} the space A^* equipped with the topology of compact convergence, i.e. the topology of uniform convergence on compact subsets of A. Given a set S in A^*, we say that S *generates* A^*_{cc} if $\langle S \rangle$ is dense in A^*_{cc}, where $\langle S \rangle$ denotes the linear span of S in A^*.

Proposition 3. *Suppose that the Banach algebra A has the approximation property. If $\mathfrak{M}(A)$ generates A^*_{cc}, then $\mathbb{P}(K, A) = \mathcal{P}_N(K) \hat{\otimes}_\epsilon A$ for any compact set K in any Banach space E.*

Proof. Let K be a compact set in a Banach space E and take a function $f \in \mathbb{P}(K, A)$. First, we show that $\phi \circ f \in \mathcal{P}_N(K)$ for every $\phi \in A^*$. By Theorem 1, we have $\phi \circ f \in \mathcal{P}_N(K)$, for every $\phi \in \mathfrak{M}(A)$, whence $\phi \circ f \in \mathcal{P}_N(K)$ for every $\phi \in \langle \mathfrak{M}(A) \rangle$, the linear span of $\mathfrak{M}(A)$ in A^*. Since $\mathfrak{M}(A)$ generates A^*_{cc}, given $\phi \in A^*$, there exists a net (ϕ_α) in $\langle \mathfrak{M}(A) \rangle$ such that $\phi_\alpha \to \phi$ in the compact convergence topology of A^*. The set $f(K)$ is compact in A, and thus $\phi_\alpha \to \phi$ uniformly on $f(K)$. This implies that $\phi_\alpha \circ f \to \phi \circ f$ uniformly on K. Since $\phi_\alpha \circ f \in \mathcal{P}_N(K)$ for all α, we get $\phi \circ f \in \mathcal{P}_N(K)$.

Now, let $\epsilon > 0$. Since A has the approximation property and $f(K)$ is compact in A, there exists a finite rank operator $T = \sum_{i=1}^m \phi_i b_i$ with $\phi_i \in A^*$, $b_i \in A$, $1 \le i \le m$, $m \in \mathbb{N}$, such that

$$\|y - T(y)\| = \left\| y - \sum_{i=1}^m \phi_i(y) b_i \right\| \le \epsilon, \quad y \in f(K).$$

Replacing y with $f(x)$, $x \in K$, we get

$$\left\| f(x) - \sum_{i=1}^m \phi_i \circ f(x) b_i \right\| \le \epsilon, \quad x \in K.$$

From the first part of the proof, we get $\sum_{i=1}^m (\phi_i \circ f) b_i \in \mathcal{P}_N(K) \widehat{\otimes}_\epsilon A$. Since $\epsilon > 0$ is arbitrary, we have $f \in \mathcal{P}_N(K) \widehat{\otimes}_\epsilon A$. □

To support the above result, we present some examples.

Example 1. *Suppose that X is a compact Hausdorff space and that $A = \mathscr{C}(X)$. It is well-known that A has the approximation property and that $\mathfrak{M}(A) = \{\delta_x : x \in X\}$, where $\delta_x : f \mapsto f(x)$ is the point mass measure at x. Indeed, $\mathfrak{M}(A)$ coincides with the set of extreme points of the unit ball A_1^*. By the Krein-Millman theorem, A_1^* equals the weak* closed convex hull of $\mathfrak{M}(A)$. Therefore, given $\phi \in A_1^*$, there exists a net (ϕ_α) in the convex hull of $\mathfrak{M}(A)$ such that $\phi_\alpha \to \phi$ in the weak* topology. Since (ϕ_α) is bounded, we get $\phi_\alpha \to \phi$ uniformly on compact sets. This means that $\phi_\alpha \to \phi$ in A_{cc}^*, and thus $\mathfrak{M}(A)$ generates A_{cc}^*. Now, by Proposition 3, we have $\mathbb{P}\big(K, \mathscr{C}(X)\big) = \mathcal{P}_N(K) \otimes_\epsilon \mathscr{C}(X)$ for any compact set K in a Banach space E. It is worth noting that*

$$\mathcal{P}_N(K) \widehat{\otimes}_\epsilon \mathscr{C}(X) = \mathscr{C}(X) \widehat{\otimes}_\epsilon \mathcal{P}_N(K) = \mathscr{C}\big(X, \mathcal{P}_N(K)\big).$$

Example 2. *Let S be any nonempty set, and $A = \ell^p(S)$ for some $p \in (1, \infty)$. Then $A^* = \ell^q(S)$ with $1/p + 1/q = 1$. Define multiplication on A point-wise, that is, $(fg)(s) = f(s)g(s)$ for all $s \in S$. It is a matter of calculation to verify that*

$$\sum_{s \in S} |f(s)|^p |g(s)|^p \le \sum_{s \in S} |f(s)|^p \sum_{s \in S} |g(s)|^p, \quad f, g \in A.$$

Therefore $\big(A, \|\cdot\|_p\big)$ is a commutative Banach algebra. For every $s \in S$, the evaluation homomorphism $\phi_s : f \mapsto f(s)$ is a character of A. Conversely, let $\phi : A \to \mathbb{C}$ be a character. Suppose that χ_s is the characteristic function at $s \in S$. Then, given $f \in A$, we have $f = \sum_{s \in S} f(s)\chi_s$, and thus $\phi(f) = \sum_{s \in S} f(s)\phi(\chi_s)$. Since $\phi \neq 0$, there must be a point $s_0 \in S$ such that $\phi(\chi_{s_0}) \neq 0$. If $s \neq s_0$, then $\phi(\chi_s)\phi(\chi_{s_0}) = \phi(\chi_s \chi_{s_0}) = 0$, so that $\phi(\chi_s) = 0$. Also $\phi(\chi_{s_0})^2 = \phi(\chi_{s_0})$ and thus $\phi(\chi_{s_0}) = 1$. Therefore, we have

$$\phi(f) = \sum_{s \in S} f(s)\phi(\chi_s) = f(s_0) = \phi_{s_0}(f).$$

We conclude that $\mathfrak{M}(A) = \{\phi_s : s \in S\}$. The fact that $\langle \mathfrak{M}(A) \rangle$ is dense in $A^* = \ell^q(S)$ in the norm topology yields that $\mathfrak{M}(A)$ generates A_{cc}^*. Moreover, if S is countable then $\ell^p(S)$ has a Schauder basis and thus it has the approximation property.

2.3 The character space

Let K be a compact set in the Banach space E. The character space of a function algebra on K generated by a certain class $\mathcal{P}$ of polynomials is closely related to the polynomially convex hull of K with respect to the given class $\mathcal{P}$.

Definition 2. *The nuclear polynomially convex hull of K is defined as*

$$\hat{K}_N = \{a \in E : |P(a)| \leq \|P\|_K,\ P \in \mathcal{P}_N(E)\}. \tag{19}$$

It is said that K is nuclear polynomially convex if $K = \hat{K}_N$.

Theorem 3. *The character space of $\mathcal{P}_N(K)$ is homeomorphic to $\hat{K}_N$, and $\mathcal{P}_N(K)$ is isometrically isomorphic to $\mathcal{P}_N(\hat{K}_N)$.*

Proof. Let $a \in \hat{K}_N$. Given $f \in \mathcal{P}_N(K)$, there is a sequence (P_k) of polynomials in $\mathcal{P}_{N_0}(K)$ such that $P_k \to f$ uniformly on K, and thus $|P_k(a) - P_j(a)| \leq \|P_k - P_j\|_K \to 0$ as $k, j \to \infty$.

This means that $(P_k(a))$ is a Cauchy sequence in $\mathbb{C}$. Define $\phi_a(f) = \lim_{k\to\infty} P_k(a)$. If (Q_k) is another sequence of polynomials in $\mathcal{P}_{N_0}(K)$ such that $Q_k \to f$ uniformly on K, then $|Q_k(a) - P_k(a)| \leq \|Q_k - P_k\|_K \to 0$. This shows that $\phi_a(f)$ is well-defined. An standard argument shows that $\phi_a(f+g) = \phi_a(f) + \phi_a(g)$ and $\phi_a(fg) = \phi_a(f)\phi_a(g)$ for all $f, g \in \mathcal{P}_N(K)$. Therefore, $\phi_a \in \mathfrak{M}(\mathcal{P}_N(K))$.

Conversely, assume that $\phi : \mathcal{P}_N(K) \to \mathbb{C}$ is a character. Consider the dual space E^* as a subspace of $\mathcal{P}_N(E)$ consisting of 1-homogenous polynomials. Then the restriction of ϕ to E^* is a linear functional on E^*. We show that ϕ is weak* continuous on norm bounded subsets of E^*. Let (ψ_α) be a bounded net in E^* that converges in the weak* topology to some $\psi_0 \in E^*$. Then $\psi_\alpha \to \psi_0$ uniformly on K. Since ϕ is continuous with respect to $\|\cdot\|_K$, we get $\phi(\psi_\alpha) \to \phi(\psi_0)$. This shows that ϕ is weak* continuous on bounded subsets of E^*, as desired. By [9, Corollary 4], ϕ is weak* continuous on E^* and thus there is $a \in E$ such that $\phi(\psi) = \psi(a)$ for all $\psi \in E^*$. Now, take an n-homogenous nuclear polynomial $P = \sum_{i=1}^\infty \psi_i^n$ with $\psi_i \in E^*$ and $\sum_{i=1}^\infty \|\psi_i\|^n < \infty$. By Proposition 1, the series converges uniformly on K, and thus

$$\phi(P) = \phi\left(\lim_{s\to\infty} \sum_{i=1}^{s} \psi_i^n\right) = \lim_{s\to\infty} \sum_{i=1}^{s} \phi(\psi_i)^n = \lim_{s\to\infty} \sum_{i=1}^{s} \psi_i(a)^n = P(a).$$

Note that $|P(a)| = |\phi(P)| \leq \|P\|_K$ for every $P \in \mathcal{P}_N(E)$, which shows that $a \in \hat{K}_N$. Thus $\phi = \phi_a$ on $\mathcal{P}_{N_0}(K)$, a dense subspace of $\mathcal{P}_N(K)$, whence $\phi = \phi_a$ on $\mathcal{P}_N(K)$. Finally, the mapping $\hat{K}_N \ni a \mapsto \phi_a \in \mathfrak{M}(\mathcal{P}_N(K))$ is an embedding of K onto $\mathfrak{M}(\mathcal{P}_N(K))$ (see [5, Chapter 4]). □

We conclude this paper with the following result on the character space of $\mathbb{P}(K, A)$.

Theorem 4. *The character space $\mathfrak{M}(\mathbb{P}(K, A))$ contains $\hat{K}_N \times \mathfrak{M}(A)$ as a closed subset. If either of the conditions in Theorem 2, Proposition 2 or Proposition 3 hold, then*

$$\mathfrak{M}(\mathbb{P}(K, A)) = \hat{K}_N \times \mathfrak{M}(A).$$

Proof. It follows from previous results and from Tomiyama theorem, proved in [14]. □

Acknowledgment

The authors are very thankful to the reviewer(s) for their careful reading and suggestions that improve the presentation of the paper.

References

[1] Abtahi M. *Vector-valued characters on vector-valued function algebras*. Banach J. Math. Anal. 2016, **10** (3), 608–620. doi:10.1215/17358787-3607486

[2] Carando D. *Extendible polynomials on Banach spaces*. J. Math. Anal. Appl. 1999, **233** (1), 359–372. doi: 10.1006/jmaa.1999.6319

[3] Carando D., Dimant V. *Duality in spaces of nuclear and integral polynomials*. J. Math. Anal. Appl. 2000, **241** (1), 107–121. doi:10.1006/jmaa.1999.6626

[4] Cilia R., Gutiérrez J.M. *Nuclear and integral polynomials*. J. Aust. Math. Soc. 2004, **76** (2), 269–280.

[5] Dales H.G. Banach Algebras and Automatic Continuity. In: London Mathematical Society Monographs, 24. Oxford Science Publications, Clarendon Press, Oxford University Press, New York, 2000.

[6] Dineen S. Complex Analysis on Infinite Dimensional Spaces. In: Kim M., Wendland K. (Eds.) Springer Monographs in Mathematics. Springer-Verlag, London, 1999.

[7] Galindo P., Lourenco M., Moraes L. *Polynomials generated by linear operators*. Proc. Amer. Math. Soc. 2004, **132** (10), 2917–2927. doi:10.1090/S0002-9939-04-07442-8

[8] Hausner A. *Ideals in a certain Banach algebra*. Proc. Amer. Math. Soc. 1957, **8** (2), 246–249. doi:10.2307/2033720

[9] Horváth J. Topological vector spaces and distributions. Vol. I. Addison-Wesley Publ., London, 1966.

[10] Lourenco M.L., Moraes L.A. *A class of polynomials from Banach spaces into Banach algebras Klein-Gordon equations is studied in the Sobolev space $Hs = Hs$*. Publ. Res. Inst. Math. Sci. 2001, **37** (4), 521–529. doi: 10.2977/prims/1145477328

[11] Mujica J. Complex Analysis in Banach Spaces. North-Holland Mathematical Studies 120. In: Dover Books on Mathematics. North-Holland, Amesterdam, 1986.

[12] Nikou A., O'Farrell A.G. *Banach algebras of vector-valued functions*. Glasg. Math. J. 2014, **56** (2), 419–246. doi:10.1017/S0017089513000359

[13] Prolla J.B. Approximation of Vector-Valued Functions. North-Holland, Amsterdam, NewYork, Oxford, 1977.

[14] Tomiyama J. *Tensor products of commutative Banach algebras*. Tohoku Math. J. (2) 1960, **12**, 147–154.

[15] Zalduendo I. *Extending polynomials on Banach spaces – a survey*. Rev. Un. Mat. Argentina 2005, **46** (2), 45–72.

Received 11.03.2022

Зай Ф., Абтахі М. *Алгебри поліномів, породжені лінійними операторами* // Карпатські матем. публ. — 2024. — Т.16, №1. — С. 309–319.

Нехай E — банаховий простір, а A — комутативна банахова алгебра з одиницею. Нехай $\mathbb{P}(E,A)$ — простір A-значних поліномів на E, породжених обмеженими лінійними операторами (n-однорідний поліном в $\mathbb{P}(E,A)$ має вигляд $P=\sum_{i=1}^{\infty} T_i^n$, де $T_i : E \to A$, $1 \le i < \infty$, є обмеженими лінійними операторами і $\sum_{i=1}^{\infty} \|T_i\|^n < \infty$). Для довільної компактної множини K в E позначимо через $\mathbb{P}(K,A)$ замикання в $\mathscr{C}(K,A)$ звужень $P|_K$ поліномів P в $\mathbb{P}(E,A)$. Доведено, що $\mathbb{P}(K,A)$ є A-значною рівномірною алгеброю, яка за певних умов є ізометрично ізоморфною ін'єктивному тензорному добутку $\mathcal{P}_N(K)\,\widehat{\otimes}_\epsilon\, A$, де $\mathcal{P}_N(K)$ — рівномірна алгебра на K, породжена ядерними скалярними поліномами. Тоді простір характерів простору $\mathbb{P}(K,A)$ ототожнюється з $\hat{K}_N \times \mathfrak{M}(A)$, де $\hat{K}_N$ — ядерна поліноміальна опукла оболонка K в E, а $\mathfrak{M}(A)$ — простір характерів алгебри A.

Ключові слова і фрази: векторно-значна рівномірна алгебра, поліном на банаховому просторі, ядерний поліном, поліноміальна опуклість, тензорний добуток.