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Notes on the tangent bundles over F-Kählerian manifolds

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Let TM be the tangent bundle over an F-Kählerian manifold endowed with Berger type deformed Sasaki metric g_{BS}. In this paper, we obtain the Levi-Civita connection of this metric and study geodesics on the tangent bundle TM and F-unit tangent bundle $T_{1,F}M$. Secondly, we characterize the geodesic curvatures on $T_{1,F}M$. Finally, we present some conditions for a vector field $\xi : M \to TM$ to be harmonic and study the harmonicity of the canonical projection $\pi : TM \to M$. In addition, we search the harmonicity of the Berger type deformed Sasaki metric g_{BS} and the Sasaki metric g_S with respect to each other.

Key words and phrases: horizontal lift, vertical lift, geodesic, harmonic map.

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1 Introduction

Let M be an n-dimensional manifold. We point out here and once that all geometric objects considered in this paper are supposed to be of class C^∞.

An F-structure is a $(1,1)$-tensor field F which satisfies

$$F^3 + F = 0. \tag{1}$$

An almost complex structure $\left(F^2 + I = 0\right)$ is an example of F-structures. Also, note that an F-structure is a polynomial structure with the structural polynomial $Q(F) = F^3 + F$. Recall that a polynomial structure is integrable if the Nijenhuis tensor vanishes [22]. Then, the integrability of an F-structure is equivalent to the vanishing of the Nijenhuis tensor

$$N_F(X,Y) = [FX, FY] - F\,[FX, Y] - F\,[X, FY] + F^2\,[X,Y]\,.$$

A Riemannian metric (M, g) is hyperbolic with respect to F if it satisfies

$$g(FX, Y) = -g(X, FY) \tag{2}$$

for any vector fields X, Y on M (see [4–7, 19]). We refer to the condition (2) as the hyperbolic compatibility of g and F and call g a hyperbolic metric. The manifold (M, F, g) equipped with a hyperbolic metric g is called an almost F-Hermitian manifold [20]. According to J. Bureš

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and J. Vanžura in [8], the manifold (M, F, g) is a metric polynomial manifold. We will use the terminology in [20].

The 2-covariant skew-symmetric tensor field ω defined by $\omega(X, Y) = g(FX, Y)$ is the fundamental 2-form of the almost F-Hermitian manifold (M, F, g). If the fundamental 2-form ω is closed, i.e. $d\omega = 0$, then the triple (M, F, g) will be called an almost F-Kählerian manifold. Moreover, if $d\omega = 0$ and $N_F = 0$, the triple (M, F, g) will be called an F-Kählerian manifold. In [20], B. Opozda proved that $d\omega = 0$ and $N_F = 0$ is equivalent to $\nabla F = 0$, where ∇ is the Levi-Civita connection of g.

The first thing that comes to mind when the tangent bundle of any Riemannian manifold is mentioned is the Sasaki metric. When the geometric properties of the tangent bundle with the Sasaki metric are investigated, we usually encounter the flatness of the base manifold. This is the reason why researchers have attempted to search for different metrics on the tangent bundle which are different deformations of the Sasaki metric (for example, see [1–3, 10, 24, 25]).

We mention that in [18] the authors provide a complete classification of the natural lifted metrics on the tangent bundle of a Riemannian manifold, and in [21] the author introduce a metric which generalizes all the types of metrics obtained as natural lifts of the metric from the base manifold to the tangent bundle, called general natural metric. S.L. Druţă-Romaniuc in [12] characterizes in an unified way the general natural (almost) Hermitian, para-Hermitian, anti-Hermitian and product Riemannian structures on the tangent bundle.

In this paper, we consider the tangent bundle TM over an F-Kählerian manifold endowed with Berger type deformed Sasaki metric g_{BS}, thus meaning that the horizontal distribution defined by the Levi-Civita connection of g projects isometrically over M, the vertical and the horizontal distributions are orthogonal, and the metric acting on vertical vectors depends on F and on a constant δ. Firstly, we obtain the formulas describing the Levi-Civita connection of this metric (Theorem 1) and we study the geodesics on the tangent bundle TM and on the F-unit tangent bundle $T_{1,F}M$ (Theorem 3 and Theorem 4). Secondly, we characterize the geodesic curvatures on the F-unit tangent bundle $T_{1,F}M$ (Theorem 6). In the last section, we present some conditions for a vector field $\xi : M \to TM$ to be harmonic (Theorem 8 and Theorem 9) and search the harmonicity of the canonical projection $\pi : TM \to M$ (Theorem 10). Also, we investigate the harmonicity of the Berger type deformed Sasaki metric g_{BS} and the Sasaki metric g_S with respect to each other.

2 Lifts to tangent bundles

Let M be an n-dimensional Riemannian manifold with a Riemannian metric g and TM be its tangent bundle denoted by $\pi : TM \to M$. A system of local coordinates $\left(U, x^i\right)$ in M induces on TM a system of local coordinates $\left(\pi^{-1}(U), x^i, x^{\bar{i}} = u^i\right)$, $\bar{i} = n + i = n + 1, \ldots, 2n$, where $\left(u^i\right)$ is the cartesian coordinates in each tangent space $T_P M$ at $P \in M$ with respect to the natural base $\left\{\frac{\partial}{\partial x^i}|_P\right\}$, P being an arbitrary point in U whose coordinates are $\left(x^i\right)$.

Given a vector field $X = X^i \frac{\partial}{\partial x^i}$ on M, the vertical lift ${}^V X$ and the horizontal lift ${}^H X$ of X are given, with respect to the induced coordinates, by

$$X^V = X^i \partial_{\bar{i}}, \qquad X^H = X^i \partial_i - u^s \Gamma^i_{sk} X^k \partial_{\bar{i}},$$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial u^i}$ and Γ^i_{sk} are the coefficients of the Levi-Civita connection ∇ of g [23].

In particular, we have the vertical spray ${}^{V}u$ and the horizontal spray ${}^{H}u$ on TM defined by

$$u^V = u^i\,(\partial_i)^V = u^i\partial_{\bar{i}}, \qquad u^H = u^i\,(\partial_i)^H = u^i\overline{\partial}_i.$$

Note, that u^V is also called the canonical or Liouville vector field on TM.

Let f be smooth function of M to $\mathbb{R}$ and X,Y,Z be any vector fields on M. We have (see [23])

$$X^H(f^V) = X(f), \qquad X^V(f^V) = 0,$$
$$X^H\Big((g(Y,Z))^V\Big) = X(g(Y,Z)), \qquad X^V\Big((g(Y,Z))^V\Big) = 0,$$

where $f^V = f \circ \pi$.

The bracket operation of vertical and horizontal vector fields is given by the following formulas (see [11,23])

$$\begin{cases} [X^H, Y^H] = [X,Y]^H - (R(X,Y)u)^V, \\ [X^H, Y^V] = (\nabla_X Y)^V, \\ [X^V, Y^V] = 0 \end{cases}$$

for all vector fields X and Y on M, where R is the Riemannian curvature tensor of g defined by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Proposition 1 ([1,24]). *Let (M,g) be a Riemannian manifold and J be a $(1,1)$-tensor field on M such that $\nabla J = 0$. If $u = u^i\frac{\partial}{\partial x^i} \in T_xM$ for all $x \in M$, then we have the following equalities:*

1) $X^H\big(g(Ju,Ju)\big)_{(x,u)} = 0,$

2) $X^H\big(g(Y,Ju)\big)_{(x,u)} = g\,(\nabla_X Y, Ju)_x,$

3) $X^V\big(g(Ju,Ju)\big)_{(x,u)} = X^V\,(|Ju|^2)_{(x,u)} = 2g(JX,Ju)_x,$

4) $X^V\big(g(Y,Ju)\big)_{(x,u)} = g(JX,Y)_x,$

5) $X^H\big(f\,(r^2)\,\big) = 0,$

6) $X^V\big(f\,(r^2)\,\big) = 2f'\,(r^2)\,g(JX,Ju)_x,$ *where* $r^2 = g(Ju,Ju) = |Ju|^2.$

3 The Levi-Civita connection of the Berger type deformed Sasaki metric on the tangent bundle

Let M be an n-dimensional manifold. An F-structure is a $(1,1)$-tensor field F, which satisfies

$$F^3 + F = 0.$$

A Riemannian metric (M,g) is hyperbolic with respect to F if it satisfies

$$g(FX,Y) = -g(X,FY)$$

for any vector fields X,Y on M. The manifold (M,F,g) equipped with a hyperbolic metric g is called an almost F-Hermitian manifold. Moreover, if $\nabla F = 0$, where ∇ is the Levi-Civita connection of g, the triple (M,F,g) will be called an F-Kählerian manifold.

Example 1. *Let* $M=\mathbb{R}^3$, $F=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$ *and* $g=dx^2+dy^2+dz^2$. *Then* (M,F,g) *is an* F*-Kählerian manifold, on which we have*

$$\begin{aligned}&g(X,Y)=X_1Y_1+X_2Y_2+X_3Y_3,\\&g(FX,FY)=X_1Y_1+X_2Y_2,\\&g(FX,FY)\neq g(X,Y),\quad (F \text{ is non-isometric}),\\&g(FX,Y)=-X_2Y_1+X_1Y_2=-g(X,FY)\end{aligned}$$

for all vector fields $X=X_1\partial_1+X_2\partial_2+X_3\partial_3$ *and* $Y=Y_1\partial_1+Y_2\partial_2+Y_3\partial_3$.

Definition 1. *Let* (M,F,g) *be an almost* F*-Kählerian manifold and* TM *be its tangent bundle. A Berger type deformed Sasaki metric on* TM *is defined as follows*

$$\begin{aligned}&g_{BS}\left(X^H,Y^H\right)=g(X,Y),\\&g_{BS}\left(X^V,Y^H\right)=g_{BS}\left(X^H,Y^V\right)=0,\\&g_{BS}\left(X^V,Y^V\right)=g(X,Y)+\delta^2 g(X,Fu)g(Y,Fu)\end{aligned}$$

for all vector fields X,Y *on* M*, where* δ *is some constant. The metric is said to be a Berger type deformed Sasaki metric.*

Lemma 1. *Let* (M,F,g) *be an* F*-Kählerian manifold. Then we have*

$$\begin{aligned}g(FX,X)&=0,\\ g\left(F^2X,F^2Y\right)&=g(FX,FY),\\ R(FX,Y)&=-R(X,FY)\end{aligned}\tag{3}$$

for all vector fields X *and* Y *on* M*, where* R *is the Riemannian curvature of the Levi-Civita connection* ∇ *of* g *defined by*

$$R(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}.$$

Proof. The first and second relation comes directly from (2) and (1). We will only show the third relation. For all vector fields X,Y,Z,W, we have

$$\begin{aligned}g\big(R(FX,Y)Z,W\big)&=g\big(R(Z,W)FX,Y\big)=g\big([\nabla_Z,\nabla_W]FX-\nabla_{[Z,W]}FX,Y\big)\\&=g\big(F([\nabla_Z,\nabla_W]X)-F(\nabla_{[Z,W]}X),Y\big)=g\big(F(R(Z,W)X),Y\big)\\&=-g\big(R(Z,W)X,FY\big)=-g\big(R(X,FY)Z,W\big).\end{aligned}$$

□

Lemma 2. *Let* (M,F,g) *be an* F*-Kählerian manifold and* g_{BS} *be the Berger type deformed Sasaki metric on* TM*. Then*

$$\begin{aligned}g(Z,Fu)&=\frac{1}{\lambda}g_{BS}\left(Z^V,(Fu)^V\right),\\ g(Z,FX)&=g_{BS}\left(Z^V,(FX)^V\right)-\frac{\delta^2}{\lambda}g(FX,Fu)g_{BS}\left(Z^V,(Fu)^V\right)\\&=g_{BS}\left((FX)^V-\frac{\delta^2}{\lambda}g(FX,Fu)(Fu)^V,Z^V\right),\end{aligned}$$

where $\lambda=1+\delta^2|Fu|^2$, X,Z *are vector fields and* $u\in TM$.

Proof. Using Definition 1, it follows that

$$\begin{aligned} g_{BS}\left(Z^V,(Fu)^V\right) &= g(Z,Fu)+\delta^2 g(Z,Fu)g(Fu,Fu) = g(Z,Fu)+\delta^2|Fu|^2 g(Z,Fu) \\ &= \left(1+\delta^2|Fu|^2\right)g(Z,Fu) = \lambda\, g(Z,Fu), \\ g_{BS}\left(Z^V,(FX)^V\right) &= g(Z,FX)+\delta^2 g(Z,Fu)g(FX,Fu) \\ &= g(Z,FX)+\frac{\delta^2}{\lambda}g(FX,Fu)g_{BS}\left(Z^V,(Fu)^V\right). \end{aligned}$$

Hence

$$g(Z,FX) = g_{BS}\left(Z^V,(FX)^V\right)-\frac{\delta^2}{\lambda}g(FX,Fu)g_{BS}\left(Z^V,(Fu)^V\right).$$

□

Using Definition 1, Proposition 1, Lemma 2 and Koszul formula, we obtain the following theorem.

Theorem 1. *Let (M,F,g) be an F-Kählerian manifold and g_{BS} be the Berger type deformed Sasaki metric on TM. If $\overline{\nabla}$ and ∇ denote the Levi-Civita connection of (TM,g_{BS}) and (M,g), respectively, then*

(1) $\overline{\nabla}_{X^H}Y^H = \left(\nabla_X Y\right)^H - \frac{1}{2}\left(R(X,Y)u\right)^V$,

(2) $\overline{\nabla}_{X^H}Y^V = \left(\nabla_X Y\right)^V + \frac{1}{2}\left[R(u,Y)X+\delta^2 g(Y,Fu)R(u,Fu)X\right]^H$,

(3) $\overline{\nabla}_{X^V}Y^H = \frac{1}{2}\left[R(u,X)Y+\delta^2 g(X,Fu)R(u,Fu)Y\right]^H$,

(4) $\overline{\nabla}_{X^V}Y^V = \delta^2\left[g(Y,Fu)(FX)^V+g(X,Fu)(FY)^V\right]$
$-\frac{\delta^4}{\lambda}\Big[g\big(FX,Fu\big)g\big(Y,Fu\big)+g\big(FY,Fu\big)g\big(X,Fu\big)\Big](Fu)^V$, *where* $\lambda = 1+\delta^2|Fu|^2$.

4 Geodesics of the Berger type deformed Sasaki metric

The section is devoted to the characterizations of geodesics on the tangent bundle and F-unit tangent bundle with respect to the Berger type deformed Sasaki metric.

Lemma 3 ([17]). *Let (M,g) be a Riemannian manifold. If X,Y are vector fields and $(x,u)\in TM$ such that $X_x = u$, then we have*

$$d_x X\left(Y_x\right) = Y^H_{(x,u)} + \left(\nabla_Y X\right)^V_{(x,u)}.$$

4.1 Geodesics of the Berger type deformed Sasaki metric on tangent bundle TM

Lemma 4 ([24,25]). *Let (M,g) be a Riemannian manifold and $x: I\to M$ be a curve on M. If $C: t\in I\to C(t) = \big(x(t),y(t)\big)\in TM$ is a curve on TM such that $y(t)\in T_{x(t)}M$, i.e. $y(t)$ is a vector field along $x(t)$, then*

$$\dot{C} = \dot{x}^H + \left(\nabla_{\dot{x}}y\right)^V,$$

where $\dot{x} = \frac{dx}{dt}$ *and* $\dot{C} = \frac{dC}{dt}$.

Proof. If Y is a vector field such that $Y\big(x(t)\big) = y(t)$, then we have

$$\dot{C}(t) = dC(t) = dY\big(x(t)\big).$$

Using Lemma 3, we find

$$\dot{C}(t) = dY\big(x(t)\big) = \dot{x}^H + (\nabla_{\dot{x}} y)^V. \tag{4}$$

□

Theorem 2. *Let (M, F, g) be an F-Kählerian manifold and g_{BS} be the Berger type deformed Sasaki metric on TM. If $C(t) = \big(x(t), y(t)\big)$ is a curve on TM, where $y(t)$ is a vector field along $x(t)$, then*

$$\begin{aligned}\overline{\nabla}_{\dot{C}}\dot{C} = &\Big[\nabla_{\dot{x}}\dot{x} + R\left(y, \nabla_{\dot{x}}y\right)\dot{x} + \delta^2 g\left(\nabla_{\dot{x}}y, Fy\right) R(y, Fy)\dot{x}\Big]^H \\ &+ \Big[\nabla_{\dot{x}}\nabla_{\dot{x}}y + 2\delta^2 g\left(\nabla_{\dot{x}}y, Fy\right) F\left(\nabla_{\dot{x}}y\right) - \frac{2\delta^4}{\lambda} g\left(\nabla_{\dot{x}}y, Fy\right) g\left(F\left(\nabla_{\dot{x}}y\right), Fy\right) Fy\Big]^V.\end{aligned}$$

Proof. Form (4) and Theorem 1, we find

$$\begin{aligned}\overline{\nabla}_{\dot{C}}\dot{C} &= \overline{\nabla}_{\left[\dot{x}^H + (\nabla_{\dot{x}}y)^V\right]}\left[\dot{x}^H + (\nabla_{\dot{x}}y)^V\right] \\ &= \overline{\nabla}_{\dot{x}^H}\dot{x}^H + \overline{\nabla}_{\dot{x}^H}\left(\nabla_{\dot{x}}y\right)^V + \overline{\nabla}_{(\nabla_{\dot{x}}y)^V}\dot{x}^H + \overline{\nabla}_{(\nabla_{\dot{x}}y)^V}\left(\nabla_{\dot{x}}y\right)^V \\ &= \left(\nabla_{\dot{x}}\dot{x}\right)^H + \left(\nabla_{\dot{x}}\nabla_{\dot{x}}y\right)^V + \frac{1}{2}\Big[R\left(y, \nabla_{\dot{x}}y\right)\dot{x} + \delta^2 g\left(\nabla_{\dot{x}}y, Fy\right) R(y, Fy)\dot{x}\Big]^H \\ &\quad + \frac{1}{2}\Big[R\left(y, \nabla_{\dot{x}}y\right)\dot{x} + \delta^2 g\left(\nabla_{\dot{x}}y, Fy\right) R(y, Fy)\dot{x}\Big]^H \\ &\quad + 2\delta^2 g\left(\nabla_{\dot{x}}y, Fy\right)\left(F\left(\nabla_{\dot{x}}y\right)\right)^V - \frac{2\delta^4}{\lambda} g\left(\nabla_{\dot{x}}y, Fy\right) g\left(F\left(\nabla_{\dot{x}}y\right), Fy\right)(Fy)^V \\ &= \Big[\nabla_{\dot{x}}\dot{x} + R\left(y, \nabla_{\dot{x}}y\right)\dot{x} + \delta^2 g\left(\nabla_{\dot{x}}y, Fy\right) R(y, Fy)\dot{x}\Big]^H \\ &\quad + \Big[\nabla_{\dot{x}}\nabla_{\dot{x}}y + 2\delta^2 g\left(\nabla_{\dot{x}}y, Fy\right) F\left(\nabla_{\dot{x}}y\right) - \frac{2\delta^4}{\lambda} g\left(\nabla_{\dot{x}}y, Fy\right) g\left(F\left(\nabla_{\dot{x}}y\right), Fy\right) Fy\Big]^V.\end{aligned}$$

□

As a direct consequence of Theorem 2, we get the following assertion.

Theorem 3. *Let (M, F, g) be an F-Kählerian manifold, g_{BS} be the Berger type deformed Sasaki metric on TM and $C(t) = \big(x(t), y(t)\big)$ be a curve on TM, where $y(t)$ is a vector field along $x(t)$. Then C is a geodesic on TM if and only if*

$$\begin{cases}\nabla_{\dot{x}}\dot{x} = -R\left(y, \nabla_{\dot{x}}y\right)\dot{x} - \delta^2 g\left(\nabla_{\dot{x}}y, Fy\right) R(y, Fy)\dot{x}, \\ \nabla_{\dot{x}}\nabla_{\dot{x}}y = -2\delta^2 g\left(\nabla_{\dot{x}}y, Fy\right) F\left(\nabla_{\dot{x}}y\right) + \frac{2\delta^4}{\lambda} g\left(\nabla_{\dot{x}}y, Fy\right) g\left(F\left(\nabla_{\dot{x}}y\right), Fy\right) Fy.\end{cases}$$

If we put $x'' = \nabla_{\dot{x}}\dot{x}$, $y' = \nabla_{\dot{x}}y$ and $y'' = \nabla_{\dot{x}}\nabla_{\dot{x}}y$, then the above conditions can be rewritten as follows

$$\begin{cases}x'' = -R\left(y, y'\right)\dot{x} - \delta^2 g\left(y', Fy\right) R(y, Fy)\dot{x}, \\ y'' = 2\delta^2 g\left(y', Fy\right)\Big[-Fy' + \dfrac{\delta^2}{\lambda} g\left(Fy', Fy\right) Fy\Big].\end{cases} \tag{5}$$

A curve $C(t) = \big(x(t), y(t)\big)$ on TM is said to be a horizontal lift of the curve $x(t)$ on M if $\nabla_{\dot{x}} y = 0$. Thus, we have the following result.

Corollary 1. *Let (M, F, g) be an F-Kählerian manifold, g_{BS} be the Berger type deformed Sasaki metric on TM and $C(t) = \big(x(t), y(t)\big)$ be the horizontal lift of the curve $x(t)$. Then $C(t)$ is a geodesic on (TM, g_{BS}) if and only if $x(t)$ is a geodesic on (M, F, g).*

If $x(t)$ is a curve on M, then the curve $C(t) = \big(x(t), \dot{x}(t)\big)$ is called a natural lift of the curve $x(t)$. From Corollary 1, we deduce the following result.

Corollary 2. *Let (M, F, g) be an F-Kählerian manifold and g_{BS} be the Berger type deformed Sasaki metric on TM. The natural lift $C(t) = \big(x(t), \dot{x}(t)\big)$ of any geodesic $x(t)$ is a geodesic on (TM, g_{BS}).*

4.2 Geodesics of the Berger type deformed Sasaki metric on F-unit tangent bundle $T_{1,F}M$

Let $T_{1,F}M$ be the F-unit tangent bundle of TM. Then $T_{1,F}M$ is a hypersurface in TM defined by

$$T_{1,F}M = \Big\{(x,u) \in TM,\ g(Fu, Fu) = |Fu|^2 = 1\Big\} = \Big\{(x,u) \in TM,\ g\big(u, -F^2u\big) = 1\Big\}.$$

In the general case, $T_{1,F}M \neq T_1M$. Indeed, using Example 1, we get

$$T_{1,F}M = \mathbb{R}^3 \times S^1 \times \mathbb{R} \qquad \text{and} \qquad T_1M = \mathbb{R}^3 \times S^2.$$

The unit normal vector field to $T_{1,F}M$ is given by

$$\mathcal{U} : TM \ni (x,u) \mapsto \mathcal{U}_{(x,u)} = \big(-F^2u\big)^V \in T(TM).$$

Indeed, for $(x,u) \in T_{1,F}M$, we have

$$g_{BS}(\mathcal{U}, \mathcal{U})_{(x,u)} = g\big(F^2u, F^2u\big) + \delta^2 g\big(F^2u, Fu\big)^2 = -g\big(F^3u, Fu\big) = g\big(Fu, Fu\big) = 1.$$

On the other hand, if we set

$$\varphi : TM \ni (x,u) \mapsto g\big(Fu, Fu\big) = g\big(u, -F^2u\big) \in \mathbb{R},$$

then the hypersurface $T_{1,F}M$ is given by

$$T_{1,F}M = \big\{(x,u) \in TM : \varphi(x,u) = 1\big\},$$

where $grad_{g_{BS}}(\varphi)$ is a vector field normal to $T_{1,F}M$. From the Proposition 1, for any vector field X on M, we get

$$\begin{aligned} g_{BS}\big(X^H, grad_{g_{BS}}(\varphi)\big) &= X^H(\varphi) = X^H\big(g(Fu, Fu)\big) = 0 = g_{BS}\big(X^H, \mathcal{U}\big), \\ g_{BS}\big(X^V, grad_{g_{BS}}(\varphi)\big) &= X^V(\varphi) = X^V\big(g(Fu, Fu)\big) = 2g(FX, Fu) = 2g_{BS}\big(X^V, \mathcal{U}\big). \end{aligned}$$

So $\mathcal{U} = \frac{1}{2} grad_{g_{BS}}(\varphi)$ is a vector field orthonormal to $T_{1,F}M$.

Given a vector field X on M, the tangential lift X^T of X is given by

$$X^T(x,u) = X^V_{(x,u)} + g_x\big(X_x, F^2u\big)\mathcal{U}_{(x,u)} = \Big[X^V - g_{BS}\big(X^V, \mathcal{U}\big)\mathcal{U}\Big]_{(x,u)}.$$

If $\widetilde{\nabla}$ is the induced connection on T_1M, then we have

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = \overline{\nabla}_{\widetilde{X}}\widetilde{Y} - g_{BS}\left(\overline{\nabla}_{\widetilde{X}}\widetilde{Y}, \mathcal{U}\right)\mathcal{U} \tag{6}$$

for all vector fields $\widetilde{X}, \widetilde{Y}$ on T_1M.

Lemma 5. *Let (M, F, g) be an F-Kählerian manifold, g_{BS} be the Berger type deformed Sasaki metric on TM and $C(t) = (x(t), y(t))$ be a curve on $T_{1,F}M$ such that $y(t)$ is a vector field along $x(t)$. Then we have*

$$|Fy| = 1 \quad \text{and} \quad g\big(F(\nabla_{\dot{x}} y), Fy\big) = 0.$$

Proof. The proof follows from the definition of $T_{1,F}M$, namely

$$0 = \nabla_{\dot{x}} g(Fy, Fy) = 2g\left(\nabla_{\dot{x}}(Fy), Fy\right) = 2g\left(F\left(\nabla_{\dot{x}} y\right), Fy\right).$$

□

As $T_{1,F}M$ is a hypersurface in TM, the curve on $T_{1,F}M$ is a geodesic if and only if its second covariant derivative in TM is collinear to the unit normal $\Big(-\left(F^2 y\right)^V\Big)$. From Theorem 2, the formula (6) and Lemma 5, we obtain the following lemma.

Lemma 6. *Let (M, F, g) be an F-Kählerian manifold, g_{BS} be the Berger type deformed Sasaki metric on TM and $C(t) = (x(t), y(t))$ be a curve on $T_{1,F}M$ such that $y(t)$ is a vector field along $x(t)$. Then C is a geodesic on $T_{1,F}M$ if and only if*

$$\begin{aligned} \nabla_{\dot{x}} \dot{x} &= -R\left(y, \nabla_{\dot{x}} y\right) \dot{x} - \delta^2 g\left(\nabla_{\dot{x}} y, Fy\right) R(y, Fy) \dot{x}, \\ \nabla_{\dot{x}} \nabla_{\dot{x}} y &= -2\delta^2 g\left(\nabla_{\dot{x}} y, Fy\right) F\left(\nabla_{\dot{x}} y\right) - \rho F^2 y, \end{aligned} \tag{7}$$

where ρ is any function.

If we put $x'' = \nabla_{\dot{x}} \dot{x}$, $y' = \nabla_{\dot{x}} y$ and $y'' = \nabla_{\dot{x}} \nabla_{\dot{x}} y$, then equation (7) is rewritten as follows

$$\begin{aligned} x'' &= -R\left(y, y'\right) \dot{x} - \delta^2 g\left(y', Fy\right) R(y, Fy) \dot{x}, \\ y'' &= -\rho F^2 y - 2\delta^2 g\left(y', Fy\right) Fy'. \end{aligned} \tag{8}$$

Remark 1. *(i) As $|Fy|^2 = 1$, i.e. $y \in T_{1,F}M$, then $g\left(Fy', Fy\right) = 0$.*

(ii) We have $0 = \nabla_{\dot{x}} g\left(Fy', Fy\right) = g\left(Fy'', Fy\right) + g\left(Fy', Fy'\right) = g\left(Fy'', Fy\right) + |Fy'|^2$, so $g\left(Fy'', Fy\right) = -\left|Fy'\right|^2$.

Lemma 7. *Let (M, F, g) be an F-Kählerian manifold, g_{BS} be the Berger type deformed Sasaki metric on TM and $C(t) = (x(t), y(t))$ be a geodesic on $T_{1,F}M$ such that $y(t)$ is a vector field along $x(t)$. If we put $c = |Fy'|$, $\mu = g\left(y', Fy\right)$, then we get*

$$\rho = -c^2 - 2\delta^2 \mu^2, \quad c' = 0, \quad \mu' = 0.$$

Proof. From formula (8), we obtain

$$\begin{aligned} y'' &= -\rho F^2 y - 2\delta^2 \mu\, Fy', \\ g\left(Fy'', Fy\right) &= -2\delta^2 \mu\, g\left(F^2 y', Fy\right) - \rho\, g\left(F^3 y, Fy\right) \\ &= -2\delta^2 \mu\, g\left(F^2 y', Fy\right) + \rho\, g(Fy, Fy) - \left|Fy'\right|^2 \\ &= -2\delta^2 \mu\, g\left(F^2 y', Fy\right) + \rho = 2\delta^2 \mu\, g\left(F^3 y', y\right) + \rho \\ &= 2\delta^2 \mu\, g\left(y', Fy\right) + \rho = 2\delta^2 \mu^2 + \rho, \\ \frac{1}{2}\left(c^2\right)' &= g\left(Fy'', Fy'\right) \\ &= -\rho\, g\left(F^3 y, Fy'\right) - 2\delta^2 \mu\, g\left(F^2 y', Fy'\right) \\ &= \rho\, g\left(Fy, Fy'\right) = 0, \qquad \text{(from Remark 1 } (i)) \\ \mu' &= g\left(y'', Fy\right) + g\left(y', Fy'\right) = g\left(y'', Fy\right) \\ &= -\rho\, g\left(F^2 y, Fy\right) - 2\delta^2 \mu\, g\left(Fy', Fy\right) = 0. \end{aligned}$$

□

Using Lemma 6 and Lemma 7, we obtain the following theorem.

Theorem 4. *Let (M, F, g) be an F-Kählerian manifold, on which F satisfies the equation (1), g_{BS} be the Berger type deformed Sasaki metric on TM and $C(t) = \big(x(t), y(t)\big)$ be a curve on $T_{1,F}M$ such that $y(t)$ is a vector field along $x(t)$. If we put $c = |Fy'|$, $\mu = g\,(y', Fy)$, then the curve $C(t) = \big(x(t), y(t)\big)$ is a geodesic on $T_{1,F}M$ if and only if*

$$\begin{aligned} c = const, \quad \mu = const, \quad \rho = -c^2 - 2\delta^2\mu^2 = const,\\ x'' = -R\,(y, y')\,\dot{x} - \delta^2\mu\, R(y, Fy)\dot{x},\\ y'' = c^2F^2y + 2\delta^2\mu \left[\mu\, F^2y - Fy'\right]. \end{aligned} \tag{9}$$

Theorem 5. *Let (M, F, g) be a locally symmetric F-Kählerian manifold ($\nabla\, R = 0$), g_{BS} be the Berger type deformed Sasaki metric on TM and $C(t) = \big(x(t), y(t)\big)$ be a geodesic on $T_{1,F}M$ (respectively, on TM) and $\gamma = \pi \circ C$. Then the tensor field $\mathcal{R}$ defined by the relation*

$$\mathcal{R}\,(y, y') = R\,(y, y') + \delta^2\mu\, R(y, Fy)$$

is parallel along γ for the case of $T_{1,F}M$ (respectively, TM).

Proof. 1) Let $C(t) = \big(x(t), y(t)\big)$ be a geodesic on $T_{1,F}M$. Using (3) and (9), we get

$$\begin{aligned} \mathcal{R}'\,(y, y') &= \nabla_{\dot{x}}\mathcal{R}\,(y, y') = (\nabla_{\dot{x}}R)\,(y, y') + R\,(y', y') + R\,(y, y'') + \delta^2\mu\, R\,(y', Fy)\\ &\quad + \delta^2\mu\, R\,(y, Fy') = R\,(y, y'') + \delta^2\mu\, R\,(y', Fy) + \delta^2\mu\, R\,(y, Fy') = R\,(y, y'')\\ &\quad - \delta^2\mu\, R\,(Fy', y) + \delta^2\mu\, R\,(y, Fy') = R\,(y, y'') + 2\delta^2\mu\, R\,(y, Fy')\\ &= R\left(y, c^2F^2y + 2\delta^2\mu^2\, F^2y\right) - 2\delta^2\mu\, R\,(y, Fy') + 2\delta^2\mu\, R\,(y, Fy')\\ &= \left(c^2 + 2\delta^2\mu^2\right)\, R\left(y, F^2y\right) = -\left(c^2 + 2\delta^2\mu^2\right)\, R(Fy, Fy) = 0. \end{aligned}$$

2) Let $C(t) = \big(x(t), y(t)\big)$ be a geodesic on TM. Using (3) and the second condition in (5), we get

$$\begin{aligned} \mathcal{R}'\,(y, y') &= \nabla_{\dot{x}}\left[R\,(y, y') + \delta^2 g\,(y', Fy)\;R(y, Fy)\right]\\ &= R\,(y', y') + R\,(y, y'') + \delta^2 g\,(y'', Fy)\,R(y, Fy)\\ &\quad + \delta^2 g\,(y', Fy')\;R(y, Fy) + \delta^2 g\,(y', Fy)\;R\,(y', Fy) + \delta^2 g\,(y', Fy)\;R\,(y, Fy')\\ &= R\,(y, y'') + \delta^2 g\,(y'', Fy)\,R(y, Fy) + \delta^2 g\,(y', Fy)\;R\,(y', Fy) + \delta^2 g\,(y', Fy)\;R\,(y, Fy')\\ &= R\,(y, y'') + \delta^2 g\,(y'', Fy)\,R(y, Fy) + 2\delta^2 g\,(y', Fy)\;R\,(y, Fy')\\ &= -2\delta^2 g\,(y', Fy)\,R\,(y, Fy') + \frac{2\delta^4}{\lambda} g\,(y', Fy)\,g\,(Fy', Fy)\,R(y, Fy)\\ &\quad + \delta^2 g\,(y'', Fy)\,R(y, Fy) + 2\delta^2 g\,(y', Fy)\;R\,(y, Fy')\\ &= \frac{2\delta^4}{\lambda} g\,(y', Fy)\,g\,(Fy', Fy)\,R(y, Fy) + \delta^2 g\,(y'', Fy)\;R(y, Fy)\\ &= \frac{2\delta^4}{\lambda} g\,(y', Fy)\,g\,(Fy', Fy)\,R(y, Fy) - 2\delta^4 g\,(y', Fy)\,g\,(Fy', Fy)\,R(y, Fy)\\ &\quad + \frac{2\delta^6}{\lambda} g\,(y', Fy)\,g\,(Fy', Fy)\,g(Fy, Fy)\,R(y, Fy)\\ &= \left[\frac{2\delta^4}{\lambda} - 2\delta^4 + \frac{2\delta^6}{\lambda} g(Fy, Fy)\right] g\,(y', Fy)\,g\,(Fy', Fy)\;R(y, Fy)\\ &= \frac{2\delta^4}{\lambda}\left[1 - \lambda + \delta^2|Fy|^2\right] g\,(y', Fy)\,g\,(Fy', Fy)\;R(y, Fy) = 0. \end{aligned}$$

□

Remark 2. *Let (M, F, g) be a locally symmetric F-Kählerian manifold ($\nabla R = 0$), g_{BS} be the Berger type deformed Sasaki metric on TM. If $C(t) = (x(t), y(t))$ is a curve on $T_{1,F}M$ such that $y(t)$ is a vector field along the curve $x(t)$, then $\overline{C}(t) = (x(t), Fy(t))$ is a curve on $T_{1,F}M$ such that $Fy(t)$ is a vector field along the curve $x(t)$. Indeed,*

$$\left|F^2y(t)\right|^2 = g\big(F^2y(t), F^2y(t)\big) = -g\big(F^3y(t), Fy(t)\big) = g\big(Fy(t), Fy(t)\big) = |Fy(t)|^2.$$

Theorem 6. *Let (M, F, g) be a locally symmetric F-Kählerian manifold ($\nabla R = 0$), g_{BS} be the Berger type deformed Sasaki metric on TM and $\overline{C}(t) = (x(t), Fy(t)) = (x(t), z(t))$ be a geodesic on $T_{1,F}M$. If $\gamma = \pi \circ \overline{C}$, then all geodesic curvaures of γ are constants.*

Proof. We have

$$\begin{aligned}
z' &= \nabla_{\dot{x}}(Fy) = F(\nabla_{\dot{x}}y) = Fy', \\
c^2 &= \left|Fz'\right|^2 = g(Fz', Fz') = g\left(F^2y', F^2y'\right) = g(Fy', Fy') = \left|Fy'\right|^2 = \left|z'\right|^2, \\
\mu &= g(z', Fz) = g\left(Fy', F^2y\right) = g(y', Fy), \\
\rho &= -c^2 - 2\delta^2\mu^2.
\end{aligned}$$

From Theorem 4 and Theorem 5, we find

$$x'' = -\mathcal{R}(z, z')\,x' = -R(z, z')\,\dot{x} - \delta^2\mu\, R(z, Fz)\,\dot{x}\mathcal{R}'(z, z') = \nabla_{\dot{\gamma}}\mathcal{R}(z, z') = 0.$$

So $x^{(p+1)} = -\mathcal{R}(z, z')\,x^{(p)}$, $p \geq 1$, and

$$\frac{d}{dt}|x^{(p)}|^2 = 2g\left(x^{(p+1)}, x^{(p)}\right) = -2g\left(\mathcal{R}(z, z')\,x^{(p)}, x^{(p)}\right) = 0.$$

Thus, we obtain

$$\left|x^{(p)}\right| = const \qquad \forall\, p \geq 1. \tag{10}$$

Denote by s an arc length parameter on γ. Then $\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt}$. Since $\overline{C}$ is a geodesic, we get $|\dot{\overline{C}}| = \left|\frac{d}{dt}\overline{C}\right| = \kappa = const$ and

$$\kappa^2 = \left|\dot{\overline{C}}\right|^2 = \left|\frac{ds}{dt}\right|^2 + \left|z'\right|^2 + \delta^2 g(z', Fz)^2 = \left|\frac{ds}{dt}\right|^2 + c^2 + \delta^2\mu^2.$$

Hence

$$\left|\frac{ds}{dt}\right| = \sqrt{\kappa^2 - c^2 - \delta^2\mu^2} = \beta = const, \tag{11}$$

where $\beta^2 = \kappa^2 - c^2 - \delta^2\mu^2 = const$.

Denote by $\nu_1, \ldots, \nu_{2n-1}$ the Frenet frame along γ and by $k_1, \ldots, k_{2n-1}$ the geodesic curvatures of γ. From (11), we obtain

$$\begin{aligned}
x' &= \beta\nu_1, \\
x'' &= \beta^2 k_1 \nu_2, \\
x^{(3)} &= \beta^3 k_1(-k_1\nu_1 + k_2\nu_3), \\
&\ldots
\end{aligned}$$

Using (10), we deduce $k_1 = const$, $k_2 = const$, …, $k_{2n-1} = const$, which completes the proof. □

5 Harmonicity

In the section, we will study some harmonicity problems on the tangent bundle equipped with the Berger type deformed Sasaki metric. Given a smooth map $\phi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds, the second fundamental form of ϕ is defined by

$$(\nabla d\phi)(X,Y) = \nabla^{\phi}_X d\phi(Y) - d\phi(\nabla_X Y).$$

Here ∇ is the Riemannian connection on M and ∇^{ϕ} is the pull-back connection on the pull-back bundle $\phi^{-1}TN$, and $\tau(\phi) = trace_g \nabla d\phi$ is the tension field of ϕ.

The energy functional of ϕ is defined by

$$E(\phi) = \int_K e(\phi) dv_g$$

such that K is any compact of M, where

$$e(\phi) = \frac{1}{2} trace_g h(d\phi, d\phi)$$

is the energy density of ϕ.

A map is called harmonic if it is a critical point of the energy functional E. For any smooth variation $\{\phi_t\}_{t\in I}$ of ϕ with $\phi_0 = \phi$ and $V = \frac{d}{dt}\phi_t\Big|_{t=0}$, we have

$$\frac{d}{dt} E(\phi_t)\Big|_{t=0} = -\int_K h\big(\tau(\phi), V\big) dv_g.$$

Then ϕ is harmonic if and only if $\tau(\phi) = 0$. We refer to [13–16] for background on harmonic maps.

Theorem 7. *Let (M, F, g) be an F-Kählerian manifold and g_{BS} be the Berger type deformed Sasaki metric on TM. If $\xi : (M, g, F) \to (TM, g_{BS})$ is a vector field on M, then the tension field of ξ is given by*

$$\begin{aligned}\tau(\xi)_x = {} & Tr_g \Big[R(\xi, \nabla_* \xi) * + \delta^2 g(\nabla_* \xi, F\xi) R(\xi, F\xi) *\Big]^H \\ & + Tr_g \Big[\nabla^2_* \xi + 2\delta^2 g(\nabla_* \xi, F\xi) F(\nabla_* \xi) - \frac{2\delta^4}{\lambda} g\Big(F(\nabla_* \xi), F\xi\Big) g(\nabla_* \xi, F\xi) F\xi\Big]^V.\end{aligned}$$

Proof. Let $\{E_i\}_{i=1}^{2k}$ be a local orthonormal frame on M. From Lemma 3 and Theorem 1, at $x \in M$, we have

$$\begin{aligned}\overline{\nabla}_{d\xi(E_i)} d\xi(E_i) &= \overline{\nabla}_{E_i^H + (\nabla_{E_i}\xi)^V} \Big(E_i^H + (\nabla_{E_i}\xi)^V\Big) \\ &= \overline{\nabla}_{E_i^H} E_i^H + \overline{\nabla}_{E_i^H} (\nabla_{E_i}\xi)^V + \overline{\nabla}_{(\nabla_{E_i}\xi)^V} E_i^H + \overline{\nabla}_{(\nabla_{E_i}\xi)^V} (\nabla_{E_i}\xi)^V \\ &= (\nabla_{E_i} E_i)^H + \Big(\nabla^2_{E_i}\xi\Big)^V + \frac{1}{2}\Big[R(u, \nabla_{E_i}\xi) E_i + \delta^2 g(\nabla_{E_i}\xi, Fu) R(u, Fu) E_i\Big]^H \\ &\quad + \frac{1}{2}\Big[R(u, \nabla_{E_i}\xi) E_i + \delta^2 g(\nabla_{E_i}\xi, Fu) R(u, Fu) E_i\Big]^H \\ &\quad + 2\delta^2 \left[g(\nabla_{E_i}\xi, Fu) F(\nabla_{E_i}\xi)\right]^V - \frac{2\delta^4}{\lambda}\left[g(F(\nabla_{E_i}\xi), Fu) g(\nabla_{E_i}\xi, Fu) Fu\right]^V \\ &= (\nabla_{E_i} E_i)^H + \Big(\nabla^2_{E_i}\xi\Big)^V + \Big[R(u, \nabla_{E_i}\xi) E_i + \delta^2 g(\nabla_{E_i}\xi, Fu) R(u, Fu) E_i\Big]^H \\ &\quad + 2\delta^2 \left[g(\nabla_{E_i}\xi, Fu) F(\nabla_{E_i}\xi)\right]^V - \frac{2\delta^4}{\lambda}\left[g(F(\nabla_{E_i}\xi), Fu) g(\nabla_{E_i}\xi, Fu) Fu\right]^V,\end{aligned}$$

where $u = \xi_x$. So

$$\begin{aligned}\overline{\nabla}_{d\xi(E_i)} d\xi\left(E_i\right) - d\xi\left(\nabla_{E_i} E_i\right) &= \overline{\nabla}_{d\xi(E_i)} d\xi\left(E_i\right) - \left(\nabla_{E_i} E_i\right)^H - \left(\nabla_{\nabla_{E_i} E_i} \xi\right)^V \\ &= -\left(\nabla_{\nabla_{E_i} E_i} \xi\right)^V + \left(\nabla^2_{E_i} \xi\right)^V + \left[R\left(u, \nabla_{E_i} \xi\right) E_i + \delta^2 g\left(\nabla_{E_i} \xi, Fu\right) R(u, Fu) E_i\right]^H \\ &\quad + 2\delta^2 \left[g\left(\nabla_{E_i} \xi, Fu\right) F\left(\nabla_{E_i}\right)\right]^V - \frac{2\delta^4}{\lambda} \left[g\left(F\left(\nabla_{E_i} \xi\right), Fu\right) g\left(\nabla_{E_i} \xi, Fu\right) Fu\right]^V.\end{aligned}$$

□

The Theorem 7 gives the following theorem.

Theorem 8. *Let (M, F, g) be an F-Kählerian manifold and g_{BS} be the Berger type deformed Sasaki metric on TM. The map $\xi : (M, F, g) \to (TM, g_{BS})$ is harmonic if and only if*

$$Tr_g \left[R\left(\xi, \nabla_* \xi\right) * + \delta^2 g\left(\nabla_* \xi, F\xi\right) R(\xi, F\xi) *\right] = 0,$$
$$Tr_g \left[\frac{2\delta^4}{\lambda} g\Big(F\left(\nabla_* \xi\right), F\xi\Big) g\left(\nabla_* \xi, F\xi\right) F\xi\right] = Tr_g \left[\nabla^2_* \xi + 2\delta^2 g\left(\nabla_* \xi, F\xi\right) F\left(\nabla_* \xi\right)\right].$$

Proposition 2. *Let (M, F, g) be an F-Kählerian manifold and g_{BS} be the Berger type deformed Sasaki metric on TM. The map $\xi : (M, F, g) \to (TM, g_{BS})$ is an isometric immersion if and only if $\nabla \xi = 0$.*

Proof. Let X, Y be vector fields. From Lemma 3 we have

$$\begin{aligned} g_{BS}\big(d\xi(X), d\xi(Y)\big) &= g_{BS}\big(X^H + (\nabla_X \xi)^V, Y^H + (\nabla_Y \xi)^V\big) \\ &= g_{BS}\big(X^H, Y^H\big) + g_{BS}\big((\nabla_X \xi)^V, (\nabla_Y \xi)^V\big) \\ &= g(X, Y) + g\big(\nabla_X \xi, \nabla_Y \xi\big) + \delta^2 g\big(\nabla_X \xi, Fu\big) g\big(\nabla_Y \xi, Fu\big), \end{aligned}$$

from which it follows that ξ is an isometric immersion, i.e $g_{BS}\left(d\xi(X), d\xi(Y)\right) = g(X, Y)$, if and only if

$$g\left(\nabla_X \xi, \nabla_Y \xi\right) + \delta^2 g\left(\nabla_X \xi, Fu\right) g\left(\nabla_Y \xi, Fu\right) = 0,$$

which is equivalent to $\nabla \xi = 0$. □

As a direct consequence of Theorem 8 and Proposition 2, we obtain the following theorem.

Theorem 9. *Let (M, F, g) be an F-Kählerian manifold and g_{BS} be the Berger type deformed Sasaki metric on TM. If $\xi : (M, F, g) \to (TM, g_{BS})$ is isometric immersion, then ξ is totally geodesic. Furthermore, ξ is harmonic.*

Let $\{E_i\}_{i=1}^{2k}$ be a local frame on M. Then $\left\{E_i^H, E_i^V\right\}_{i=1}^{2k}$ is a local frame on TM. The matrix of the Berger type deformed Sasaki metric with respect to the local frame is as follows

$$g_{BS} \begin{pmatrix} I & 0 \\ 0 & I + a \end{pmatrix}$$

and its inverse

$$g_{BS}^{-1} \begin{pmatrix} I & 0 \\ 0 & I - b \end{pmatrix},$$

where $a = \left((Fu)^i (Fu)^j\right)_{i,j \leq 2k}$ and $b = a(I + a)^{-1}$.

Proposition 3. *Let $\pi : (TM, g_{Bs}) \to (M, F, g)$ be the canonical projection, then the second fundamental form β of π is given by*

$$\beta(E_i^H, E_j^H) = \beta(E_i^V, E_j^V) = 0, \qquad \beta(E_i^H, E_j^V)_{(x,u)} = -\frac{1}{2}R\left(u, E_j\right)E_i - \frac{\delta^2}{2}g\left(E_j, Fu\right)R(u, Fu)E_i.$$

From Proposition 3, we obtain the following theorem.

Theorem 10. *Let (M, F, g) be an F-Kählerian manifold and g_{BS} be the Berger type deformed Sasaki metric on TM. The Riemannian immersion $\pi : (TM, g_{Bs}) \to (M, g, F)$ has the following properties:*

1) π is harmonic,

2) if (M, F, g) is locally flat then π is totally geodesic,

3) π is totally geodesic if and only if

$$R\left(u, E_j\right)E_i = -\delta^2 g\left(E_j, Fu\right)R(u, Fu)E_i \qquad \forall\, 1 \le i, j \le 2k.$$

Let h be another F-Kählerian metric on M with respect to an F-structure J_1 and let $\pi : (TM, g_{BS}) \to (M, J_1, h)$ be the canonical projection. Then we have

$$\beta(E_i^H, E_j^H) = {}^h\nabla_{E_i}E_j - \nabla_{E_i}E_j, \tag{12}$$

$$\beta(E_i^V, E_j^V) = 0, \qquad \beta(E_i^H, E_j^V)_{(x,u)} = -\frac{1}{2}R\left(u, E_j\right)E_i - \frac{\delta^2}{2}g\left(E_j, Fu\right)R(u, Fu)E_i,$$

where ${}^h\nabla$ is the Levi-Civita connection of the metric h. It is known [9] that h is harmonic with respect g if $Tr_g\left({}^h\nabla - \nabla\right) = g^{ij}\left({}^h\Gamma_{ij}^h - \Gamma_{ij}^h\right) = 0$. The equations in (12) give the following proposition.

Proposition 4. *Let (M, F, g) be an F-Kählerian manifold and g_{BS} be the Berger type deformed Sasaki metric on TM. The projection $\pi : (TM, g_{BS}) \to (M_m, J_1, h)$ is harmonic if and only if h is harmonic with respect to g.*

Now, we will investigate the harmonicity of the Berger type deformed Sasaki metric g_{BS} and the Sasaki metric g_S with respect to each other.

Proposition 5. *Let (M, F, g) be an F-Kählerian manifold and g_{BS} be the Berger type deformed Sasaki metric on TM. If $I : (TM, g_S) \to (TM, g_{BS})$ is the identity map, then the following assertions hold:*

1) the identity map I can not be totally geodesic;

2) the tension field is $\tau_{g_B}I = 2\delta^2\left(F^2u\right)^V$. So, g_{BS} can not be harmonic with respect to g_S.

Proof. Let $\{E_i\}_{i=1}^{2k}$ be a local orthonormal frame on M. Then $\left\{E_i^H, E_i^V\right\}_{i=1}^{2k}$ is a local orthonormal frame on (TM, g_S). If $\widehat{\nabla}$ denotes the Levi-Civita connection of (TM, g_S), then we have

$$\begin{aligned}\tau_{g_B}I &= \sum_i\left[\overline{\nabla}_{E_i^H}E_i^H + \overline{\nabla}_{E_i^V}E_i^V - \widehat{\nabla}_{E_i^H}E_i^H - \widehat{\nabla}_{E_i^V}E_i^V\right] = \sum_i\overline{\nabla}_{E_i^V}E_i^V\\ &= \sum_i 2\delta^2 g\left(E_i, Fu\right)\left(FE_i\right)^V - 2\frac{\delta^4}{\lambda}\sum_i g\left(FE_i, Fu\right)g\left(E_i, Fu\right)\left(Fu\right)^V\\ &= 2\delta^2\sum_i g\left(E_i, Fu\right)\left(FE_i\right)^V + 2\frac{\delta^4}{\lambda}g\left(F^2u, Fu\right)(Fu)^V = 2\delta^2\left(F^2u\right)^V.\end{aligned}$$

□

Proposition 6. *Let (M, F, g) be an F-Kählerian manifold and g_{BS} be the Berger type deformed Sasaki metric on TM. If $I : (TM, g_{BS}) \to (TM, g_S)$ is the identity map, then the tension field of I is given by*

$$\tau_{g_{BS}} I = -2\delta^2 \left(F^2 u\right)^V + 2\delta^2 \sum_{ij} b^{ij} (Fu)^i F E_j^V - 2\frac{\delta^4}{\lambda} \sum_{ij} b^{ij} (Fu)^i g\left(FE_j, Fu\right) (Fu)^V.$$

So, g_B can not be harmonic with respect to g_{BS}.

Proof. Let $\{E_i\}_{i=1}^{2k}$ be a local orthonormal frame on M. Then $\{E_i^H, E_i^V\}_{i=1}^{2k}$ is a local orthonormal frame on (TM, g_S). If $\widehat{\nabla}$ denote the Levi-Civita connection of (TM, g_S), then we have

$$\begin{aligned}
\tau_{g_{BS}} I &= \sum_i \left[\widehat{\nabla}_{E_i^H} E_i^H - \overline{\nabla}_{E_i^H} E_i^H + \widehat{\nabla}_{E_i^V} E_i^V - \overline{\nabla}_{E_i^V} E_i^V\right] - \sum_{ij} b^{ij} \left[\widehat{\nabla}_{E_i^V} E_j^V - \overline{\nabla}_{E_i^V} E_j^V\right] \\
&= -\sum_i \overline{\nabla}_{E_i^V} E_i^V + \sum_{ij} b^{ij} \overline{\nabla}_{E_i^V} E_j^V \\
&= -2\delta^2 \left(F^2 u\right)^V + 2\delta^2 \sum_{ij} b^{ij} \left[(Fu)^i F E_j\right]^V \\
&- 2\frac{\delta^4}{\lambda} \sum_{ij} b^{ij} (Fu)^i g\left(FE_j, Fu\right) (Fu)^V.
\end{aligned}$$

□

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Джаа Н.Е.Х., Ґезер А., Загане А. *Нотатки про дотичні розшарування над F-келеровими многовидами* // Карпатські матем. публ. — 2025. — Т.17, №2. — С. 550–564.

Нехай TM — дотичне розшарування над F-келеровим многовидом, наділене деформованою за типом Бержера метрикою Сасакі g_{BS}. У цій статті ми отримуємо зв'язність Леві-Чівіти для цієї метрики та досліджуємо геодезичні на дотичному розшаруванні TM і на F-одиничному дотичному розшаруванні $T_{1,F}M$. Далі ми описуємо геодезичні кривини на $T_{1,F}M$. Нарешті, подаємо деякі умови, за яких векторне поле $\xi : M \to TM$ є гармонійним, і досліджуємо гармонійність канонічної проекції $\pi : TM \to M$. Крім того, ми розглядаємо гармонійність деформованої за типом Бержера метрики Сасакі g_{BS} та метрики Сасакі g_S відносно одна одної.

Ключові слова і фрази: горизонтальний підйом, вертикальний підйом, геодезична, гармонійне відображення.