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Best orthogonal trigonometric approximations of the Nikol'skii-Besov-type classes of periodic functions of one and several variables

Fedunyk-Yaremchuk O.V., Hembars'ka S.B.

We obtained the exact order estimates of the best orthogonal trigonometric approximations of periodic functions of one and several variables from the Nikol'skii-Besov-type classes $B^{\omega}_{1,\theta}$ ($B^{\Omega}_{1,\theta}$ in the multivariate case $d \geq 2$) in the space $B_{\infty,1}$. We observe that in the multivariate case the orders of mentioned approximation characteristics of the functional classes $B^{\Omega}_{1,\theta}$ are realized by their approximations by step hyperbolic Fourier sums that contain the necessary number of harmonics. In the univariate case, an optimal in the sense of order estimates for the best orthogonal trigonometric approximations of the corresponding functional classes are the ordinary partial sums of their Fourier series. As a consequence of the obtained results, the exact order estimates of the orthowidths of the classes $B^{\omega}_{1,\theta}$ ($B^{\Omega}_{1,\theta}$ for $d \geq 2$) in the space $B_{\infty,1}$ are also established. Besides, we note that in the univariate case, in contrast to the multivariate one, the estimates of the considered approximation characteristics do not depend on the parameter θ.

Key words and phrases: Nikol'skii-Besov-type class, step hyperbolic Fourier sum, best orthogonal trigonometric approximation, orthowidth.

Lesya Ukrainka Volyn National University, Potapova Str. 9, 43025, Lutsk, Ukraine
E-mail: fedunyk.o.v@gmail.com (Fedunyk-Yaremchuk O.V.), gembarskaya72@gmail.com (Hembars'ka S.B.)

Introduction

In this paper, we continue to study the approximation characteristics of the classes $B^{\Omega}_{p,\theta}$ of periodic functions of several variables and classes $B^{\omega}_{p,\theta}$ of one variable in the space $B_{\infty,1}$, which norm is stronger than the norm in L_{∞}. We recall that some approximative characteristics of functional classes in the space $B_{\infty,1}$ were studied in [8, 13, 16, 17, 31–33, 41]. As noted in these papers, the motivation to study the approximation characteristics (best approximation, widths, best M-term approximation, etc.) of classes $B^{r}_{p,\theta}$ and $B^{\Omega}_{p,\theta}$ in the space $B_{\infty,1}$ was the fact that the questions of their orders, especially in the multidimensional case, in the space L_{∞} still remain open (see also [11]).

The exact order estimates of approximation characteristics of the classes $B^{\omega}_{1,\theta}$ and $B^{\Omega}_{1,\theta}$, which are established in the paper, complement the results of [33, 41].

1 Notations, definitions and auxiliary assertions

Let $\mathbb{R}^d$ denotes d-dimensional space with elements $x = (x_1, \dots, x_d)$, and let

$$(x, y) = x_1 y_1 + \ldots + x_d y_d$$

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be a scalar product of elements $x, y \in \mathbb{R}^d$. Denote $\mathbb{T}^d = \prod_{j=1}^{d}[0; 2\pi)$. Let $L_p(\mathbb{T}^d)$ be the space of functions f which are 2π-periodic in each variable and such that

$$\|f\|_p = \left((2\pi)^{-d} \int_{\mathbb{T}^d} |f(x)|^p dx \right)^{1/p} < \infty, \quad 1 \le p < \infty, \qquad \|f\|_\infty = \operatorname*{ess\,sup}_{x \in \mathbb{T}^d} |f(x)| < \infty.$$

In what follows, we assume that for $f \in L_p(\mathbb{T}^d)$ the condition

$$\int_0^{2\pi} f(x) dx_j = 0, \quad j = \overline{1, d},$$

is satisfied. We denote the set of such functions by $L_p^0(\mathbb{T}^d)$. Sometimes instead of $L_p(\mathbb{T}^d)$ and $L_p^0(\mathbb{T}^d)$ we use the simpler notations L_p and L_p^0, respectively.

We denote the l-th difference of a function $f \in L_p^0$, $1 \le p \le \infty$, with a step h_j in the variable x_j by the formula

$$\Delta_{h_j}^l f(x) = \sum_{n=0}^{l} (-1)^{l-n} C_l^n f(x_1, \ldots, x_{j-1}, x_j + n h_j, x_{j+1}, \ldots, x_d).$$

For $f \in L_p^0, 1 \le p \le \infty$, $h = (h_1, \ldots, h_d)$ and $t \in \mathbb{R}_+^d$ we introduce a mixed l-th difference $\Delta_h^l f(x) = \Delta_{h_1}^l \ldots \Delta_{h_d}^l f(x) = \Delta_{h_d}^l (\ldots (\Delta_{h_1}^l f(x)))$ and we denote the mixed modulus of continuity of order l by

$$\Omega_l(f, t)_p = \sup_{|h_j| \le t_j,\ j = \overline{1,d}} \|\Delta_h^l f(\cdot)\|_p.$$

Let $\Omega(t) = \Omega(t_1, \ldots, t_d)$ be a given function of the type of mixed modulus of continuity of the l-th order. This means that the function Ω satisfies the following conditions:

1) $\Omega(t) > 0$, $t_j > 0$, $j = \overline{1, d}$ and $\Omega(t) = 0$ if $\prod_{j=1}^{d} t_j = 0$;

2) $\Omega(t)$ increases in each variable;

3) $\Omega(m_1 t_1, \ldots, m_d t_d) \le \left(\prod_{j=1}^{d} m_j \right)^l \Omega(t)$, $m_j \in \mathbb{N}$, $j = \overline{1, d}$;

4) $\Omega(t)$ is continuous at $t_j \ge 0$, $j = \overline{1, d}$.

Following S.N. Bernstein [9] we call the function $\varphi(\tau)$ of one variable almost increasing on $[a, b]$, if there exists a constant $C_1 > 0$, which does not depend on τ_1, τ_2, such that

$$\varphi(\tau_1) \le C_1 \varphi(\tau_2), \quad a \le \tau_1 \le \tau_2 \le b,$$

and almost decreasing on $[a, b]$, if there exists a constant $C_2 > 0$, which does not depend on τ_1, τ_2, such that

$$\varphi(\tau_1) \ge C_2 \varphi(\tau_2), \quad a \le \tau_1 \le \tau_2 \le b.$$

We assume that the function $\Omega(t)$, $t \in \mathbb{R}_+^d$, satisfies also the conditions (S^α) and (S_l), which are called the Bari-Stechkin conditions [5,35]. This means the following.

A function $\varphi(\tau) \ge 0$ of one variable $\tau \in [0, 1]$ satisfies the condition (S^α) if $\varphi(\tau)/\tau^\alpha$ almost increases for some $\alpha > 0$.

A function $\varphi(\tau) \geq 0$, $\tau \in [0,1]$, satisfies the condition (S_l) if $\varphi(\tau)/\tau^{\gamma}$ almost decreases for some $0 < \gamma < l$, $l \in \mathbb{N}$.

In the case of $d > 1$ we say that $\Omega(t), t \in \mathbb{R}_+^d$, satisfies these conditions if $\Omega(t)$ satisfies these conditions in each variable t_j for fixed t_i, $i \neq j$.

We now define the functional classes $B^{\Omega}_{p,\theta}$, which were considered in the paper [40] by S. Yongsheng and W. Heping.

Let $1 \leq p, \theta \leq \infty$ and let $\Omega(t)$ be a given function of the type of mixed modulus of continuity of the order l, which satisfies conditions 1)–4), (S^{α}) and (S_l). Then the classes $B^{\Omega}_{p,\theta}$ are defined as follows

$$B^{\Omega}_{p,\theta} = \left\{ f \in L_p^0 : \ \|f\|_{B^{\Omega}_{p,\theta}} \leq 1 \right\},$$

where

$$\|f\|_{B^{\Omega}_{p,\theta}} = \left\{ \int_{\mathbb{T}^d} \left(\frac{\Omega_l(f,t)_p}{\Omega(t)} \right)^{\theta} \prod_{j=1}^{d} \frac{dt_j}{t_j} \right\}^{1/\theta}, \quad 1 \leq \theta < \infty, \qquad \|f\|_{B^{\Omega}_{p,\infty}} = \sup_{t>0} \frac{\Omega_l(f,t)_p}{\Omega(t)}.$$

Here and subsequently the expression $t > 0$ for $t = (t_1, \dots, t_d)$ is equivalent to $t_j > 0, j = \overline{1,d}$.

We note that, in the case $r = (r_1, \dots, r_d), 0 < r_j < l$, $j = \overline{1,d}$, and $\Omega(t) = \prod_{j=1}^{d} t_j^{r_j}$, the classes $B^{\Omega}_{p,\theta}$ are identical to analogs of the Besov classes $B^r_{p,\theta}$, which were considered in the papers [2,19]. In turn, for $\theta = \infty$ the classes $B^r_{p,\infty} = H^r_p$ are analogs of the Nikol'skii classes [20]. The classes $B^{\Omega}_{p,\infty} = H^{\Omega}_p$ were studied by N.N. Pustovoitov in [21].

In the following considerations we will use the definition of classes $B^{\Omega}_{p,\theta}$ in a slightly different form. To do this, we recall the definition of order relation.

For two non-negative sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ the relation (order inequality) $a_n \ll b_n$ means that there exists a constant $C_3 > 0$, which does not depend on n and such that $a_n \leq C_3 b_n$. The relation $a_n \asymp b_n$ is equivalent to $a_n \ll b_n$ and $b_n \ll a_n$.

To every vector $s \in \mathbb{N}^d$ we put the set

$$\rho(s) = \left\{ k \in \mathbb{Z}^d : 2^{s_j - 1} \leq |k_j| < 2^{s_j}, j = \overline{1,d} \right\}$$

in correspondence and for $f \in L_p^0, 1 < p < \infty$, we denote

$$\delta_s(f) := \delta_s(f,x) = \sum_{k \in \rho(s)} \widehat{f}(k) e^{i(k,x)},$$

where

$$\widehat{f}(k) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(t) e^{-i(k,t)} dt$$

are the Fourier coefficients of the function f.

Therefore, for $f \in B^{\Omega}_{p,\theta}$, $1 < p < \infty$, $1 \leq \theta \leq \infty$, where $\Omega(t)$ is a given function of the type of mixed modulus of continuity of order l, which satisfies conditions 1)–4), (S^{α}), (S_l) and the relations

$$\|f\|_{B^{\Omega}_{p,\theta}} \asymp \begin{cases} \left(\sum_s \Omega^{-\theta}(2^{-s}) \|\delta_s(f)\|_p^{\theta} \right)^{1/\theta}, & 1 \leq \theta < \infty, \\ \sup_s \dfrac{\|\delta_s(f)\|_p}{\Omega(2^{-s})}, & \theta = \infty, \end{cases} \tag{1}$$

hold. Here and below, $\Omega(2^{-s}) = \Omega(2^{-s_1}, \dots, 2^{-s_d})$, $s_j \in \mathbb{N}$, $j = \overline{1,d}$.

Note that the case $1 \le \theta < \infty$ in (1) was considered in [40] and the case $\theta = \infty$ in [21].

For the norms of functions from the classes $B^{\Omega}_{p,\theta}$ for $p = 1$ and $p = \infty$ we can write relations analogous to (1) by replacing the "blocks" $\delta_s(f)$ by others. Namely, by $V_m(t), m \in \mathbb{N}, t \in \mathbb{R}$, we denote the Vall'ee-Poussin kernel

$$V_m(t) = 1 + 2\sum_{k=1}^{m} \cos kt + 2\sum_{k=m+1}^{2m-1} \left(\frac{2m-k}{m}\right) \cos kt$$

(for the correctness of the definition of $V_m(t)$, we should assume that the last sum in this formula vanishes for $m = 1$).

To every vector $s \in \mathbb{N}^d$, we put the polynomial

$$A_s(x) = \prod_{j=1}^{d} \Big(V_{2^{s_j}}(x_j) - V_{2^{s_j-1}}(x_j) \Big), \quad x \in \mathbb{R}^d,$$

in correspondence and for $f \in L^0_p, 1 \le p \le \infty$, we set $A_s(f) := A_s(f,x) = (f * A_s)(x)$, where $*$ means the convolution operation. Then the following relations hold:

$$\|f\|_{B^{\Omega}_{p,\theta}} \asymp \begin{cases} \left(\sum_s \Omega^{-\theta}(2^{-s}) \|A_s(f)\|_p^\theta \right)^{1/\theta}, & 1 \le \theta < \infty, \\ \sup_s \dfrac{\|A_s(f)\|_p}{\Omega(2^{-s})}, & \theta = \infty. \end{cases} \tag{2}$$

Note that the case $1 \le \theta < \infty$ in (2) was considered in [34], and the case $\theta = \infty$ in [21].

In what follows, we consider the classes $B^{\Omega}_{p,\theta}$ defined by a function of the type of a mixed modulus of continuity of order l of the special form

$$\Omega(t) = \omega\left(\prod_{j=1}^{d} t_j\right), \tag{3}$$

where $\omega(\tau)$ is a given function (of one variable) of the type of a modulus of continuity of order l that satisfies conditions (S^α) and (S_l).

It is easy to verify that the function $\Omega(t)$ of the form (3) satisfies the properties 1)–4) of a function of the type of mixed modulus of continuity of order l, and satisfies conditions (S^α) and (S_l). Therefore, the above mentioned relations (1), (2) for the norms of functions of the class $B^{\Omega}_{p,\theta}$ remain true.

Now we define the norm in the subspace $B_{\infty,1}$. For any trigonometric polynomial t, it is defined by the formula

$$\|t\|_{B_{\infty,1}} = \sum_{s \in \mathbb{N}^d \bigcup \{0\}} \|A_s(t)\|_\infty.$$

In the same way, the norm $\|f\|_{B_{\infty,1}}$ for the functions $f \in L_1$ under the condition of convergence of the series $\sum_{s\in\mathbb{N}^d \bigcup\{0\}} \|A_s(f)\|_\infty$ is defined. We note that, in this case, the relation

$$\|\cdot\|_\infty \ll \|\cdot\|_{B_{\infty,1}} \tag{4}$$

holds.

Next, we define the approximation characteristics to be studied.

Let X be some normalized functional space with norm $\|\cdot\|_X$ and θ_M be any set of M d-dimensional vectors $k^j = (k_1^j, \dots, k_d^j), j = \overline{1, M}$, with integer coordinates. For $f \in X$ we denote

$$S_{\theta_M}(f) := S_{\theta_M}(f, x) = \sum_{j=1}^{M} \widehat{f}(k^j) e^{i(k^j, x)}, \quad x \in \mathbb{R}^d,$$

where

$$\widehat{f}(k^j) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(t) e^{-i(k^j, t)}\, dt$$

are the Fourier coefficients of a function f, which correspond to the set of vectors θ_M.

We consider the approximation characteristic

$$e_M^{\perp}(f)_X := \inf_{\theta_M} \|f - S_{\theta_M}(f)\|_X$$

and for the functional class $F \subset X$ we put

$$e_M^{\perp}(F)_X = \sup_{f \in F} e_M^{\perp}(f)_X.$$

Quantity $e_M^{\perp}(F)_X$ is called the best orthogonal trigonometric approximation of class F in the space X. The quantities $e_M^{\perp}(F)_X$ for classes of functions $W^r_{p,\alpha}$, $B^r_{p,\theta}$ and $B^{\Omega}_{p,\theta}$ in spaces L_q, $1 \le q < \infty$, and $B_{\infty,1}$ were studied in the works [7, 18, 23–27, 33, 41], which contain a more detailed bibliography.

Recall the well-known assertion that will be used below.

Theorem 1 ([18]). *Let $1 \le p < \infty$, $1 \le \theta \le \infty$, $\Omega(t) = \omega\Big(\prod_{j=1}^{d} t_j\Big)$, where $\omega(\tau)$ satisfies condition (S^{α}) with some $\alpha > 1/p$ and condition (S_l). Then for any sequence $M = (M_n)_{n=1}^{\infty}$ of natural numbers such that $M \asymp 2^n n^{d-1}$, the relation*

$$e_M^{\perp}(B^{\Omega}_{p,\theta})_{\infty} \asymp \omega(2^{-n}) 2^{n/p} n^{(d-1)(1-1/\theta)}$$

holds.

2 Main results

First, consider a one-dimensional case.

Theorem 2. *Let $d = 1$, $1 \le \theta \le \infty$, and $\omega(\tau)$ satisfies condition (S^{α}) with some $\alpha > 1$ and condition (S_l). Then the relation*

$$e_M^{\perp}(B^{\omega}_{1,\theta})_{B_{\infty,1}} \asymp \omega(M^{-1}) M \tag{5}$$

holds.

Proof. Firstly, we establish upper bound in (5). Note that for $1 \le \theta < \infty$, the embedding $B^{\omega}_{1,\theta} \subset H^{\omega}_1$ is valid. Then it is sufficient to obtain the upper bound for $\theta = \infty$, i.e. for classes H^{ω}_1.

Hence, let $M \in \mathbb{N}$, $f \in H^{\omega}_1$. Consider an approximation of the function f by polynomials

$$S_n(f) := S_n(f, x) = \sum_{s=1}^{n} \delta_s(f, x), \quad n \in \mathbb{N}, \quad x \in \mathbb{R}, \tag{6}$$

where the number n is coupled to M via the relation $2^n \leq M \leq 2^{n+1}$.

Then, according to the definition of the norm in the space $B_{\infty,1}$ and the convolution property, we can write

$$\begin{aligned} e_M^{\perp}(f)_{B_{\infty,1}} &\leq ||f - S_n(f)||_{B_{\infty,1}} \leq \left\| \sum_{s=n+1}^{\infty} \delta_s(f) \right\|_{B_{\infty,1}} \\ &= \sum_{s=n+1}^{\infty} \left\| A_s * \sum_{s'=s-1}^{s+1} \delta_{s'}(f) \right\|_{\infty} \leq \sum_{s=n+1}^{\infty} \|A_s\|_1 \left\| \sum_{s'=s-1}^{s+1} \delta_{s'}(f) \right\|_{\infty} = I_1. \end{aligned} \tag{7}$$

To continue the estimation of the quantity I_1, we should note that by the relation $\|V_{2^s}\|_1 \leq C_4$ (see, e.g., [36, Ch.1, §1]), we have

$$\|A_s\|_1 = \|V_{2^s} - V_{2^{s-1}}\|_1 \leq \|V_{2^s}\|_1 + \|V_{2^{s-1}}\|_1 \leq C_5. \tag{8}$$

Thus, for the quantity I_1 we obtain an estimate

$$\begin{aligned} I_1 &\ll \sum_{s=n+1}^{\infty} \left\| \sum_{s'=s-1}^{s+1} \delta_{s'}(f) \right\|_{\infty} \leq \sum_{s=n+1}^{\infty} \sum_{s'=s-1}^{s+1} \|\delta_{s'}(f)\|_{\infty} \\ &\ll \sum_{s=n}^{\infty} \|\delta_s(f)\|_{\infty} = \sum_{s=n}^{\infty} \left\| \delta_s \left(\sum_{s'=s-1}^{s+1} A_{s'}(f) \right) \right\|_{\infty} = I_2. \end{aligned} \tag{9}$$

Next, given that the norm of the operator δ_s, $\delta_s f := \delta_s(f)$, as an operator from $L_1(\mathbb{T})$ into the $L_\infty(\mathbb{T})$ does not exceed in order of 2^s, we continue to estimate the quantity I_2 as follows

$$I_2 \ll \sum_{s=n}^{\infty} 2^s \left\| \sum_{s'=s-1}^{s+1} A_{s'}(f) \right\|_1 \ll \sum_{s=n}^{\infty} 2^s \sum_{s'=s-1}^{s+1} \|A_{s'}(f)\|_1 .$$

Besides, taking into account that $\|A_{s'}(f)\|_1 \ll \omega(2^{-s'})$, $s' \in \mathbb{N}$, $f \in H_1^{\omega}$, we may write

$$\left\| \sum_{s'=s-1}^{s+1} A_{s'}(f) \right\|_1 \leq \sum_{s'=s-1}^{s+1} \|A_{s'}(f)\|_1 \ll \sum_{s'=s-1}^{s+1} \omega(2^{-s'}) \ll \omega(2^{-s}). \tag{10}$$

Hence, taking into account (9) and (10), from (7) it follows that

$$I_2 \ll \sum_{s=n}^{\infty} 2^s \omega(2^{-s}). \tag{11}$$

Now, noting that

$$\frac{\omega(2^{-s})}{2^{-\alpha s}} \leq C_6 \frac{\omega(2^{-n})}{2^{-\alpha n}}, \quad C_6 > 0,$$

let us estimate the quantity I_2 as follows

$$I_2 \ll \frac{\omega(2^{-n})}{2^{-\alpha n}} \sum_{s=n}^{\infty} 2^{-s(\alpha-1)} \ll \omega(2^{-n}) 2^n.$$

Finally, taking the relationship between the numbers M and n into account, we arrive at the estimate

$$e_M^{\perp}(B_{1,\theta}^{\omega})_{B_{\infty,1}} \ll \omega(M^{-1}) M.$$

Concerning the lower bound in (5), note that it is a consequence of Theorem 1 under the conditions $p = 1, d = 1$, since according to relation (4), the following is fulfilled

$$e_M^{\perp}(B_{1,\theta}^{\omega})_{B_{\infty,1}} \gg e_M^{\perp}(B_{1,\theta}^{\omega})_{\infty}.$$

□

To formulate the consequence of the Theorem 2, we introduce the notation.

If $F \subset X \subset L_\infty(\mathbb{T})$ is some functional class, we set

$$\mathcal{E}_n(F)_X = \sup_{f \in F} \|f - S_n(f)\|_X,$$

where the polynomial $S_n(f)$ is defined by formula (6).

Corollary 1. *Let $d = 1$, $1 \le \theta \le \infty$, and $\omega(\tau)$ satisfies condition (S^α) with some $\alpha > 1$ and condition (S_l). Then the relation*

$$\mathcal{E}_n(B^\omega_{1,\theta})_{B_{\infty,1}} \asymp \omega(2^{-n})2^n \tag{12}$$

holds.

Note that the upper bound in (12) was established when proving Theorem 2. The corresponding lower bound is also a consequence of this theorem, since for $2^{n+1} \le M \le 2^{n+2}$, the following relations hold

$$\mathcal{E}_n(B^\omega_{1,\theta})_{B_{\infty,1}} \gg e^\perp_M(B^\omega_{1,\theta})_{B_{\infty,1}} \asymp \omega(2^{-n})2^n.$$

Remark 1. *Analyzing the proof of Theorem 2 and Corollary 1, we conclude that for $1 \le \theta \le \infty$ and $\omega(\tau)$ that satisfies condition (S^α) with some $\alpha > 1$ and condition (S_l), the following relations hold*

$$e^\perp_M(B^\omega_{1,\theta})_{B_{\infty,1}} \asymp e^\perp_M(B^\omega_{1,\theta})_\infty; \qquad \mathcal{E}_n(B^\omega_{1,\theta})_{B_{\infty,1}} \asymp \mathcal{E}_n(B^\omega_{1,\theta})_\infty.$$

In the following assertion we consider a multidimensional case ($d \ge 2$). We introduce the additional notations.

For $s = (s_1, \dots, s_d)$, $s_j \in \mathbb{N}$, $j = \overline{1,d}$, $d \ge 2$, and $n \in \mathbb{N}$, we set

$$Q_n = \bigcup_{(s,1)<n} \rho(s), \quad (s,1) = s_1 + \ldots + s_d.$$

The set Q_n is called a step hyperbolic cross. By $S_{Q_n}(f)$ we denote the so-called step hyperbolic Fourier sum of a function $f \in L_1(\mathbb{T}^d)$ of the form

$$S_{Q_n}(f) := S_{Q_n}(f,x) = \sum_{(s,1)<n} \delta_s(f,x), \quad x \in \mathbb{R}^d.$$

Let $X \in L_1(\mathbb{T}^d)$ be some functional space with norm $\|\cdot\|_X$. Then for the class of the functions $F \subset X$, we set

$$\mathcal{E}_{Q_n}(F)_X := \sup_{f \in F} \|f - S_{Q_n}(f)\|_X.$$

In the case, where $X = L_q(\mathbb{T}^d)$, we write $\mathcal{E}_{Q_n}(F)_q$ instead of $\mathcal{E}_{Q_n}(F)_X$.

Theorem 3. *Let $d \ge 2$, $1 \le \theta \le \infty$, $\Omega(t) = \omega\Big(\prod_{j=1}^d t_j\Big)$, where $\omega(\tau)$ satisfies condition (S^α) with some $\alpha > 1$ and condition (S_l). Then for any sequence $M = (M_n)_{n=1}^\infty$ of natural numbers such that $M \asymp 2^n n^{d-1}$, the relation*

$$e^\perp_M(B^\Omega_{1,\theta})_{B_{\infty,1}} \asymp \omega(2^{-n})2^n n^{(d-1)(1-1/\theta)} \tag{13}$$

holds.

Proof. First, we establish the upper bound in (13). We show that this estimate is realized by approximation of functions $f \in B^{\Omega}_{1,\theta}$ by their step hyperbolic Fourier sums $S_{Q_n}(f)$ under the condition $M \asymp 2^n n^{d-1}$.

Hence, according to the definition of the norm in the space $B_{\infty,1}$ and taking into account the property of convolution, we can write

$$\begin{aligned}\Big\|f - \sum_{(s,1)<n} \delta_s(f)\Big\|_{B_{\infty,1}} &= \Big\| \sum_{(s,1)\geq n} \delta_s(f)\Big\|_{B_{\infty,1}} = \sum_{s\in\mathbb{N}^d} \Big\|A_s * \sum_{\substack{s'\in\mathbb{N}^d \\ (s',1)\geq n}} \delta_{s'}(f)\Big\|_\infty \\ &\leq \sum_{(s,1)\geq n-d} \Big\|A_s * \sum_{\|s-s'\|_\infty\leq 1} \delta_{s'}(f)\Big\|_\infty \leq \sum_{(s,1)\geq n-d} \|A_s\|_1 \Big\| \sum_{\|s-s'\|_\infty \leq 1} \delta_{s'}(f)\Big\|_\infty \\ &\ll \sum_{(s,1)\geq n-d} \sum_{\|s-s'\|_\infty\leq 1} \|\delta_{s'}(f)\|_\infty \ll \sum_{(s,1)\geq n-2d} \|\delta_s(f)\|_\infty \\ &= \sum_{(s,1)\geq n-2d} \Big\|\delta_s\Big(\sum_{\|s-s'\|_\infty\leq 1} A_{s'}(f)\Big)\Big\|_\infty = I_3.\end{aligned} \tag{14}$$

Further, given that the norm of the operator δ_s, $\delta_s f := \delta_s(f)$, as an operator from $L_1(\mathbb{T}^d)$ into the $L_\infty(\mathbb{T}^d)$ does not exceed in order of $2^{(s,1)}$, for the quantity I_3 we obtain

$$\begin{aligned}I_3 &\ll \sum_{(s,1)\geq n-2d} 2^{(s,1)} \|A_{s'}(f)\|_1 \ll \sum_{(s,1)\geq n-3d} 2^{(s,1)}\|A_s(f)\|_1 \\ &= \sum_{(s,1)\geq n-3d} \omega^{-1}(2^{-(s,1)}) 2^{(s,1)} \|A_s(f)\|_1 \omega(2^{-(s,1)}) = I_4.\end{aligned} \tag{15}$$

Consider several cases depending on the value of the parameter θ.

1. Let $\theta \in (1,\infty)$. Then, taking advantage of the Hölder inequality, we have

$$\begin{aligned}I_4 &\leq \Big(\sum_{(s,1)\geq n-3d} \omega^{-\theta}(2^{-(s,1)}) \|A_s(f)\|_1^\theta\Big)^{1/\theta} \Big(\sum_{(s,1)\geq n-3d} \omega^{\theta'}(2^{-(s,1)}) 2^{(s,1)\theta'}\Big)^{1/\theta'} \\ &\ll \|f\|_{B^{\Omega}_{1,\theta}} \Big(\sum_{(s,1)\geq n-3d} \omega^{\theta'}(2^{-(s,1)}) 2^{(s,1)\theta'}\Big)^{1/\theta'} \leq \Big(\sum_{(s,1)\geq n-3d} \omega^{\theta'}(2^{-(s,1)}) 2^{(s,1)\theta'}\Big)^{1/\theta'} = I_5,\end{aligned} \tag{16}$$

where $1/\theta + 1/\theta' = 1$.

Let m denote $n-3d$. Then, taking into account that

$$\frac{\omega(2^{-(s,1)})}{2^{-\alpha(s,1)}} \leq C_7 \frac{\omega(2^{-m})}{2^{-\alpha m}}, \quad C_7 > 0, \quad (s,1) \geq m, \tag{17}$$

we get

$$\begin{aligned}I_5 &\ll \frac{\omega(2^{-m})}{2^{-\alpha m}} \Big(\sum_{(s,1)\geq m} 2^{-(s,1)\alpha\theta'} 2^{(s,1)\theta'}\Big)^{1/\theta'} = \frac{\omega(2^{-m})}{2^{-\alpha m}} \Big(\sum_{(s,1)\geq m} 2^{-(s,1)(\alpha-1)\theta'}\Big)^{1/\theta'} \\ &= \frac{\omega(2^{-m})}{2^{-\alpha m}} \Big(\sum_{j\geq m} 2^{-j(\alpha-1)\theta'} \sum_{(s,1)=j} 1\Big)^{1/\theta'} \asymp \frac{\omega(2^{-m})}{2^{-\alpha m}} \Big(\sum_{j\geq m} 2^{-j(\alpha-1)\theta'} j^{d-1}\Big)^{1/\theta'} \\ &\ll \omega(2^{-m}) 2^m m^{(d-1)(1-1/\theta)}.\end{aligned} \tag{18}$$

2. In the case $\theta = \infty$, we may write

$$I_4 \leq \sup_{s:(s,1)\geq m} \frac{||A_s(f)||_1}{\omega(2^{-(s,1)})} \sum_{(s,1)\geq m} \omega(2^{-(s,1)})2^{(s,1)} \ll ||f||_{B^{\Omega}_{1,\infty}} \sum_{(s,1)\geq m} \omega(2^{-(s,1)})2^{(s,1)} \\ \leq \sum_{(s,1)\geq m} \omega(2^{-(s,1)})2^{(s,1)} = I_6. \tag{19}$$

Now, taking into account (17), the estimation (19) can be continued as follows

$$I_6 \ll \frac{\omega(2^{-m})}{2^{-\alpha m}} \sum_{(s,1)\geq m} 2^{-\alpha(s,1)}2^{(s,1)} = \frac{\omega(2^{-m})}{2^{-\alpha m}} \sum_{j\geq m} 2^{-j(\alpha-1)} \sum_{(s,1)=j} 1 \\ \asymp \frac{\omega(2^{-m})}{2^{-\alpha m}} \sum_{j\geq m} 2^{-j(\alpha-1)} j^{d-1} \ll \omega(2^{-m})2^m m^{d-1}. \tag{20}$$

3. If $\theta = 1$, the estimate for the quantity I_4 is as follows

$$I_4 \leq \sup_{s:(s,1)\geq m} 2^{(s,1)}\omega(2^{-(s,1)}) \sum_{(s,1)\geq m} \omega^{-1}(2^{-(s,1)})\|A_s(f)\|_1 \\ \leq \sup_{s:(s,1)\geq m} 2^{(s,1)}\omega(2^{-(s,1)})\|f\|_{B^{\Omega}_{1,1}} \leq \sup_{s:(s,1)\geq m} 2^{(s,1)}\omega(2^{-(s,1)}). \tag{21}$$

Taking into account (17) from (21) we have

$$I_4 \leq \frac{\omega(2^{-m})}{2^{-\alpha m}} \sup_{s:(s,1)\geq m} 2^{-(s,1)(\alpha-1)} \asymp \omega(2^{-m})2^m. \tag{22}$$

By comparing (14)–(22), from the above we obtain the desired estimate for the quantity $e^{\perp}_M(B^{\Omega}_{1,\theta})_{B_{\infty,1}}$ at $1 \leq \theta \leq \infty$.

Concerning the lower bound in (13), we note that, according to relation $\|\cdot\|_{B_{\infty,1}} \gg \|\cdot\|_\infty$, it is a consequence of Theorem 1. □

Next, as a consequence of Theorems 2, 3 and the previously known result, we obtain an exact order estimate of another important approximation characteristic. We give a definition and formulate an auxiliary assertion.

Let $\{u_i\}_{i=1}^M$ be an orthonormal system of functions $u_i \in L_\infty(\mathbb{T}^d)$, $i = \overline{1, M}$, in the space $L_2(\mathbb{T}^d)$.

In correspondence to each function $f \in L_q(\mathbb{T}^d)$, $1 \leq q \leq \infty$, we put an approximation aggregate in the form $\sum_{i=1}^M (f, u_i)u_i$, i.e. the orthogonal projection of the function f on the subspace generated by the system of functions $\{u_i\}_{i=1}^M$. Hereafter,

$$(f, u_i) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(x)\overline{u}_i(x)dx,$$

where $\overline{u}_i(x)$ are functions complex-conjugate to the functions $u_i(x)$. If $F \subset L_q(\mathbb{T}^d)$, then the quantity

$$d^{\perp}_M(F, L_q) = \inf_{\{u_i\}_{i=1}^M \subset L_\infty(\mathbb{T}^d)} \sup_{f\in F} \left\| f - \sum_{i=1}^M (f, u_i)u_i \right\|_q \tag{23}$$

is called the orthowidth (the Fourier-width) of class F in the space $L_q(\mathbb{T}^d)$. The width $d_M^{\perp}(F, L_q)$ was introduced by V.N. Temlyakov [38]. In addition, in [37] V.N. Temlyakov considered the quantity $d_M^B(F, L_q)$, which is close to the Fourier-width and defined by

$$d_M^B(F, L_q) = \inf_{G \in L_M(B)_q} \sup_{f \in F \cap \mathcal{D}(G)} \|f - Gf\|_q . \tag{24}$$

Here $L_M(B)_q$ stands for the set of linear operators satisfying the following conditions:

a) the domain $\mathcal{D}(G)$ of those operators contains all trigonometric polynomials, and their range of values is contained in an M-dimensional subspace of the space $L_q(\mathbb{T}^d)$;

b) there exist such a number $B \geq 1$ that the inequality $\left\| G e^{i(k,\cdot)} \right\|_2 \leq B$ holds for all vectors $k = (k_1, \ldots, k_d)$, $k_j \in \mathbb{Z}$, $j = \overline{1, d}$.

Note that $L_M(1)_2$ includes operators of orthogonal projection on M-dimensional spaces, as well as operators defined on an orthonormal system of functions using a multiplier that is defined by such a sequence $\{\lambda_l\}$ that $|\lambda_l| \leq 1$ for all l. It is easy to see that according to the definitions, the relation

$$d_M^B(F, L_q) \leq d_M^{\perp}(F, L_q) \tag{25}$$

is valid. Clearly, the same relation is obeyed in the case of the space $B_{\infty,1}$, i.e.

$$d_M^B(F, B_{\infty,1}) \leq d_M^{\perp}(F, B_{\infty,1}). \tag{26}$$

Quantities (23) and (24) for various functional classes F-both in the Lebesgue spaces $L_q(\mathbb{T}^d)$, $1 \leq q \leq \infty$, and in other functional spaces, were studied in the works [1, 3, 4, 6, 10, 12, 14, 15, 22, 28–30, 39]. A more detailed bibliography can be found in the monographs [11, 23, 36, 37].

Before proceeding directly to the presentation of the results obtained, we quote a well-known statement that will be used below.

Theorem 4 ([13]). *Let $d = 1$, $1 \leq \theta \leq \infty$, and $\omega(\tau)$ satisfies condition (S^α) with some $\alpha > 1$ and condition (S_l). Then the relation*

$$d_M^B(B_{1,\theta}^{\omega}, B_{\infty,1}) \asymp \omega(M^{-1}) M \tag{27}$$

holds.

Theorem 5 ([13]). *Let $d \geq 2$, $1 \leq \theta \leq \infty$, $\Omega(t) = \omega\Big(\prod_{j=1}^d t_j\Big)$, where $\omega(\tau)$ satisfies condition (S^α) with some $\alpha > 1$ and condition (S_l). Then for any sequence $M = (M_n)_{n=1}^{\infty}$ of natural numbers such that $M \asymp 2^n n^{d-1}$, the following estimate is valid*

$$d_M^B(B_{1,\theta}^{\Omega}, B_{\infty,1}) \asymp \omega(2^{-n}) 2^n n^{(d-1)(1-1/\theta)}. \tag{28}$$

Now we formulate some consequences of Theorems 1, 2, 3 and 4.

Corollary 2. *Let $d = 1$, $1 \leq \theta \leq \infty$, and $\omega(\tau)$ satisfies condition (S^α) with some $\alpha > 1$ and condition (S_l). Then the relation*

$$d_M^{\perp}(B_{1,\theta}^{\omega}, B_{\infty,1}) \asymp \omega(M^{-1}) M \tag{29}$$

holds.

The upper bound follows from (12) under the condition $2^{n+1} \le M \le 2^{n+2}$, since the following relations are fulfilled

$$d_M^{\perp}(B_{1,\theta}^{\omega}, B_{\infty,1}) \ll \mathcal{E}_n(B_{1,\theta}^{\omega})_{B_{\infty,1}} \asymp \omega(2^{-n})2^n \asymp \omega(M^{-1})M.$$

The lower bound in (29) is a consequence of the Theorem 4 according to relation (25), i.e.

$$d_M^{\perp}(B_{1,\theta}^{\omega}, B_{\infty,1}) \ge d_M^{B}(B_{1,\theta}^{\omega}, B_{\infty,1}) \asymp \omega(M^{-1})M.$$

Corollary 3. *Let $d \ge 2$, $1 \le \theta \le \infty$, $\Omega(t) = \omega\Big(\prod_{j=1}^{d} t_j\Big)$, where $\omega(\tau)$ satisfies condition (S^{α}) with some $\alpha > 1$ and condition (S_l). Then the following estimate is valid*

$$\mathcal{E}_{Q_n}(B_{1,\theta}^{\Omega})_{B_{\infty,1}} \asymp \omega(2^{-n})2^n n^{(d-1)(1-1/\theta)}. \tag{30}$$

Note that the upper bound in (30) was determined when proving Theorem 3, and the corresponding lower bound can be obtained as a corollary of Theorem 3, since under the condition $M \asymp 2^n n^{d-1}$ the following relations are fulfilled

$$\mathcal{E}_{Q_n}(B_{1,\theta}^{\Omega})_{B_{\infty,1}} \gg e_M^{\perp}(B_{1,\theta}^{\Omega})_{B_{\infty,1}} \asymp \omega(2^{-n})2^n n^{(d-1)(1-1/\theta)}.$$

Corollary 4. *Let $d \ge 2$, $1 \le \theta \le \infty$, $\Omega(t) = \omega\Big(\prod_{j=1}^{d} t_j\Big)$, where $\omega(\tau)$ satisfies condition (S^{α}) with some $\alpha > 1$ and condition (S_l). Then for any sequence $M = (M_n)_{n=1}^{\infty}$ of natural numbers such that $M \asymp 2^n n^{d-1}$, the relation*

$$d_M^{\perp}(B_{1,\theta}^{\Omega}, B_{\infty,1}) \asymp \omega(2^{-n})2^n n^{(d-1)(1-1/\theta)} \tag{31}$$

holds.

The upper bound follows from (30) under the condition $M \asymp 2^n n^{d-1}$. The lower bound in (31) is a consequence of the Theorem 5 according to relation (25).

Remark 2. *Comparing the results of Theorems 1 and 3 with Corollaries 3 and 4, we can conclude that, for $1 \le \theta \le \infty$ and $\Omega(t) = \omega\Big(\prod_{j=1}^{d} t_j\Big)$, where $\omega(\tau)$ satisfies condition (S^{α}) with some $\alpha > 1$ and condition (S_l), the following relations hold:*

$$e_M^{\perp}(B_{1,\theta}^{\Omega})_{B_{\infty,1}} \asymp e_M^{\perp}(B_{1,\theta}^{\Omega})_{\infty};$$

$$\mathcal{E}_{Q_n}(B_{1,\theta}^{\Omega})_{B_{\infty,1}} \asymp \mathcal{E}_{Q_n}(B_{1,\theta}^{\Omega})_{\infty};$$

$$d_M^{\perp}(B_{1,\theta}^{\Omega}, B_{\infty,1}) \asymp d_M^{\perp}(B_{1,\theta}^{\Omega}, L_{\infty}).$$

Remark 3. *At the end of the work, we would like to note that in the case $\Omega(t) = \prod_{j=1}^{d} t_j^{r_j}$, $r_j > 0$, $j = \overline{1,d}$, i.e. for the Nikol'skii-Besov classes $B_{p,\theta}^{r}$, the corresponding statements were obtained in the work [33].*

Remark 4. *Analyzing the results obtained in this part of the paper, we get the following conclusion. In the one-dimensional case, in contrast to the multidimensional one, the obtained estimates of the corresponding approximation characteristics are independent of the parameter θ.*

References

[1] Akishev G.A. *The ortho-diameters of Nikol'skii and Besov classes in the Lorentz spaces*. Russian Math. (Iz. VUZ) 2009, **53** (2), 21–29. doi:10.3103/S1066369X09020029 (translation of Izv. Vyssh. Uchebn. Zaved. Mat. 2009, **2**, 25–33. (in Russian))

[2] Amanov T.I. *Representation and embedding theorems for function spaces* $S^{(r)}_{p,\theta}B(\mathbb{R}_n)$ *and* $S^{(r)}_{p,\theta^*}B$, $(0 \leq x_j \leq 2\pi;$ $j = 1, \dots, n)$. Tr. Mat. Inst. Steklova 1965, **77**, 5–34. (in Russian)

[3] Andrianov A.V., Temlyakov V.N. *On two methods of generalization of properties of univariate function systems to their tensor product*. Proc. Steklov Inst. Math. 1997, **219**, 25–35. (translation of Tr. Mat. Inst. Steklova 1997, **219**, 32–43. (in Russian))

[4] Balgimbayeva S.A., Smirnov T.I. *Estimates of the Fourier widths of the classes of periodic functions with given majorant of the mixed modulus of smoothness*. Sib. Math. J. 2018, **59** (2), 217–230. doi:10.1134/S0037446618020040 (translation of Sibirsk. Mat. Zh. 2018, **59** (2), 277–292. doi:10.17377/smzh.2018.59.204 (in Russian))

[5] Bari N.K., Stechkin S.B. *The best approximations and differential properties of two conjugate functions*. Trans. Moscow Math. Soc. 1956, **5**, 483–522. (in Russian)

[6] Bazarkhanov D.B. *Estimates of the Fourier widths of classes of Nikol'skii-Besov and Lizorkin-Triebel types of periodic functions of several variables*. Math. Notes 2010, **87** (1-2), 281–284. doi:10.1134/S0001434610010359 (translation of Mat. Zametki 2010, **87** (2), 305–308. doi:10.4213/mzm8592 (in Russian))

[7] Belinskii E.S. *Approximation by a "floating" system of exponentials on classes of periodic functions with a bounded mixed derivative*. Research on the theory of functions of many real variables: Proc. of Yaroslavl' State University 1988, 16–33. (in Russian).

[8] Belinsky E.S. *Estimates of entropy numbers and Gaussian measures for classes of functions with bounded mixed derivative*. J. Approx. Theory 1998, **93**, 114–127. doi: 10.1006/jath.1997.3157

[9] Bernstein S.N. Collected work, Vol. II. Constructive theory of functions (1931-1953). Nauka, Moscow, 1954. (in Russian)

[10] Dũng D. *Approximation by trigonometric polynomials of functions of several variables on the torus*. Sb. Math. 1988, **59** (1), 247–267. doi:10.1070/SM1988v059n01ABEH003134 (translation of Mat. Sb. 1986, **131(173)** (2), 251–271. (in Russian))

[11] Dũng D., Temlyakov V.N., Ullrich T. Hyperbolic Cross Approximation. Birkhauser, Basel, 2018.

[12] Fedunyk-Yaremchuk O.V., Hembars'ka S.B. *Estimates of approximative characteristics of the classes* $B^{\Omega}_{p,\theta}$ *of periodic functions of several variables with given majorant of mixed moduli of continuity in the space* L_q. Carpathian Math. Publ. 2019, **11** (2), 281–295. doi:10.15330/cmp.11.2.281-295

[13] Fedunyk-Yaremchuk O.V., Hembars'kyi M.V., Hembars'ka S.B. *Approximative characteristics of the Nikol'skii-Besov-type classes of periodic functions in the space* $B_{\infty,1}$. Carpathian Math. Publ. 2020, **12** (2), 376–391. doi:10.15330/cmp.12.2.376-391

[14] Fedunyk-Yaremchuk O.V., Solich K.V. *Estimates of approximative characteristics of the classes* $B^{\Omega}_{p,\theta}$ *of periodic functions of many variables with given majorant of mixed continuity moduli in the space* L_∞. J. Math. Sci. (N.Y.) 2018, **231** (1), 28–40. doi:10.1007/s10958-018-3803-3 (translation of Ukr. Mat. Visn. 2017, **14** (3), 345–360. (in Ukrainian))

[15] Galeev E.M. *Orders of the orthoprojection widths of classes of periodic functions of one and of several variables*. Math. Notes 1988, **43** (2), 110–118. doi:10.1007/BF01152547 (translation of Mat. Zametki 1988, **43** (2), 197–211. (in Russian))

[16] Hembars'kyi M.V., Hembars'ka S.B. *Approximate characteristics of the classes* $B^{\Omega}_{p,\theta}$ *of periodic functions of one variable and many ones*. J. Math. Sci. (N.Y.) 2019, **242** (6), 820–832. doi:10.1007/s10958-019-04518-0 (translation of Ukr. Mat. Visn. 2019, **16** (1), 88–104. (in Ukrainian))

[17] Hembars'kyi M.V., Hembars'ka S.B., Solich K.V. *The best approximations and widths of the classes of periodic functions of one and several variables in the space* $B_{\infty,1}$. Mat. Stud. 2019, **51** (1), 74–85. doi:10.15330/ms.51.1.74-85 (in Ukrainian)

[18] Konograi A.F., Stasyuk S.A. *Best trigonometric approximations of the classes $B^{\Omega}_{p,\theta}$ of periodic functions of many variables*. Approx. Theory of Functions and Related Problems: Proc. Inst. Math. NAS Ukr. 2007, **4** (1), 151–171. (in Ukrainian)

[19] Lizorkin P.I., Nikol'skii S.M. *Function spaces of mixed smoothness from the decomposition point of view*. Proc. Steklov Inst. Math. 1990, **187**, 163–184. (translation of Tr. Mat. Inst. Steklova 1989, **187**, 143–161. (in Russian))

[20] Nikol'skii S.M. *Functions with dominant mixed derivative, satisfying a multiple Holder condition*. Sibirsk. Mat. Zh. 1963, **4** (6), 1342–1364. (in Russian)

[21] Pustovoitov N.N. *Representation and approximation of periodic functions of several variables with given mixed modulus of continuity*. Anal. Math. 1994, **20**, 35–48. doi:10.1007/BF01908917 (in Russian)

[22] Pustovoitov N.N. *On the widths of multivariate periodic classes of functions whose mixed moduli of continuity are bounded by a product of power-and logarithmic-type functions*. Anal. Math. 2008, **34**, 187–224. doi:10.1007/s10476-008-0303-6 (in Russian)

[23] Romanyuk A.S. Approximative characteristics of the classes of periodic functions of many variables. Proc. Inst. Math. NAS Ukr., Kiev, 2012, **93**. (in Russian)

[24] Romanyuk A.S. *Approximation of classes of functions of many variables by their orthogonal projections onto subspaces of trigonometric polynomials*. Ukrainian Math. J. 1996, **48** (1), 90–100. doi:10.1007/BF02390986 (translation of Ukrain. Mat. Zh. 1996, **48** (1), 80–89. (in Russian))

[25] Romanyuk A.S. *Approximation of classes of periodic functions in several variables*. Math. Notes 2002, **71** (1), 98–109. doi:10.1023/A:1013982425195 (translation of Mat. Zametki 2002, **71** (1), 109–121. doi:10.4213/mzm332 (in Russian))

[26] Romanyuk A.S. *Best trigonometric approximations for some classes of periodic functions of several variables in the uniform metric*. Math. Notes 2007, **82** (2), 216–228. doi:10.1134/S0001434607070279 (translation of Mat. Zametki 2007, **82** (2), 247–261. doi:10.4213/mzm3797 (in Russian))

[27] Romanyuk A.S. *Bilinear and trigonometric approximations of periodic functions of several variables of Besov classes $B^{r}_{p,\theta}$*. Izv. Math. 2006, **70** (2), 277–306. doi:10.1070/IM2006v070n02ABEH002313 (translation of Izv. Ross. Akad. Nauk Ser. Mat. 2006, **70** (2), 69–98. doi:10.4213/im558 (in Russian))

[28] Romanyuk A.S. *Diameters and best approximation of the classes $B^{r}_{p,\theta}$ of periodic functions of several variables*. Anal. Math. 2011, **37**, 181–213. doi:10.1007/s10476-011-0303-9 (in Russian)

[29] Romanyuk A.S. *Estimates for approximation characteristics of the Besov classes $B^{r}_{p,\theta}$ of periodic functions of many variables in the space L_q. I*. Ukrainian Math. J. 2001, **53** (9), 1473–1482. doi:10.1023/A:1014314708184 (translation of Ukrain. Mat. Zh. 2001, **53** (9), 1224–1231. (in Russian))

[30] Romanyuk A.S. *Estimates for approximation characteristics of the Besov classes $B^{r}_{p,\theta}$ of periodic functions of many variables in the space L_q. II*. Ukrainian Math. J. 2001, **53** (10), 1703–1711. doi:10.1023/A:1015200128349 (translation of Ukrain. Mat. Zh. 2001, **53** (10), 1402–1408. (in Russian))

[31] Romanyuk A.S., Romanyuk V.S. *Approximating characteristics of the classes of periodic multivariate functions in the space $B_{\infty,1}$*. Ukrainian Math. J. 2019, **71** (2), 308–321. doi: 10.1007/s11253-019-01646-3 (translation of Ukrain. Mat. Zh. 2019, **71** (2), 271–282. (in Ukrainian))

[32] Romanyuk A.S., Romanyuk V.S. *Estimation of some approximating characteristics of the classes of periodic functions of one and many variables*. Ukrainian Math. J. 2020, **71** (8), 1257–1272. doi:10.1007/s11253-019-01711-x (translation of Ukrain. Mat. Zh. 2019, **71** (8), 1102–1115. (in Ukrainian))

[33] Romanyuk A.S., Yanchenko S. Ya. *Approximation of classes of periodic functions of one and many variables from the Nikol'skii-Besov and Sobolev spaces*. Ukrain. Mat. Zh. 2022, **74** (6), 857–868. (in Ukrainian)

[34] Stasyuk S.A., Fedunyk O.V. *Approximation characteristics of the classes $B^{\Omega}_{p,\theta}$ of periodic functions of many variables*. Ukrainian Math. J. 2006, **58** (5), 779–793. doi:10.1007/s11253-006-0101-x (translation of Ukrain. Mat. Zh. 2006, **58** (5), 692–704. (in Ukrainian))

[35] Stechkin S.B. *On the order of the best approximations of continuous functions*. Izv. Ross. Akad. Nauk Ser. Mat. 1951, **15** (3), 219–242. (in Russian)

[36] Temlyakov V.N. Approximation of Periodic Functions. Nova Science Publishers Inc., New York, 1993.

[37] Temlyakov V.N. *Approximation of functions with bounded mixed derivative*. Proc. Steklov Inst. Math. 1989, **178**, 1–121. (translation of Tr. Mat. Inst. Steklova 1986, **178**, 3–113. (in Russian))

[38] Temlyakov V.N. *Diameters of some classes of functions of several variables*. Dokl. Akad. Nauk 1982, **267** (2), 314–317. (in Russian)

[39] Temlyakov V.N. *Estimates of the asymptotic characteristics of classes of functions with bounded mixed derivative or difference*. Proc. Steklov Inst. Math. 1990, **189**, 161–197. (translation of Tr. Mat. Inst. Steklova 1989, **189**, 138–168. (in Russian))

[40] Yongsheng S., Heping W. *Representation and approximation of multivariate periodic functions with bounded mixed moduli of smoothness*. Tr. Mat. Inst. Steklova 1997, **219**, 356–377.

[41] Zaderey P.V., Hembars'ka S.B. *Best orthogonal trigonometric approximations of the Nikol'skii-Besov-type classes of periodic functions in the space $B_{\infty,1}$*. Ukrain. Mat. Zh. 2022, **74** (6), 784–795. (in Ukrainian)

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Федуник-Яремчук О.В., Гембарська С.Б. *Найкращі ортогональні тригонометричні наближення класів типу Нікольського-Бєсова періодичних функцій однієї та багатьох змінних* // Карпатські матем. публ. — 2022. — Т.14, №1. — С. 171–184.

Встановлено точні за порядком оцінки найкращих ортогональних тригонометричних наближень періодичних функцій однієї та багатьох змінних з класів типу Нікольського-Бєсова $B^{\omega}_{1,\theta}$ ($B^{\Omega}_{1,\theta}$ у багатовимірному випадку $d \geq 2$) у просторі $B_{\infty,1}$. Виявлено, що в багатовимірному випадку порядки згаданих апроксимаційних характеристик класів функцій $B^{\Omega}_{1,\theta}$ реалізуються за наближення їх східчасто-гіперболічними сумами Фур'є, які містять необхідну кількість гармонік. У одновимірному випадку оптимальними, з точки зору порядкових оцінок найкращих ортогональних тригонометричних наближень відповідних класів функцій, є звичайні частинні суми їх рядів Фур'є. В якості наслідків з одержаних результатів встановлено також точні за порядком оцінки ортопоперечників класів $B^{\omega}_{1,\theta}$ ($B^{\Omega}_{1,\theta}$ при $d \geq 2$) у просторі $B_{\infty,1}$. Слід зазначити, що в одновимірному випадку, на противагу багатовимірному, оцінки розглянутих апроксимаційних характеристик не залежать від параметра θ.

Ключові слова і фрази: клас типу Нікольського-Бєсова, східчасто-гіперболічна сума Фур'є, найкраще ортогональне тригонометричне наближення, ортопоперечник.