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Non-inverse signed graph of a group

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Let G be a group with binary operation $*$. The non-inverse graph (in short, i^*-graph) of G, denoted by Γ, is a simple graph with vertex set consisting of elements of G and two vertices $x, y \in \Gamma$ are adjacent if x and y are not inverses of each other. That is, $x - y$ if and only if $x * y \neq i_G \neq y * x$, where i_G is the identity element of G. In this paper, we extend the study of i^*-graphs to signed graphs by defining i^*-signed graphs. We characterize the graphs for which the i^*-signed graphs and negated i^*-signed graphs are balanced, sign-compatible, consistent and k-clusterable. We also obtain the frustration index of the i^*-signed graph. Further, we characterize the homogeneous non-inverse signed graphs and study the properties like net-regularity and switching equivalence.

Key words and phrases: algebraic graph, non-inverse graph, non-inverse signed graph.

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1 Introduction

Many researchers have constructed graphs using the concepts of algebra such as intersection graphs (see [2]), zero divisor graphs (see [6]), cyclic groups (see [13]), non-cyclic graphs (see [1]), power graphs (see [9]) etc. The total graph of a commutative ring R was introduced in [5] and is defined as a graph with all elements of R as vertices and for distinct vertices $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in Z(R)$, where $Z(R)$ is the set of zero divisors.

The notion of center graph $\Gamma_Z(G)$ of a group G was introduced in [7]. It is a graph whose vertices are the elements of the group G and two distinct vertices $a, b \in \Gamma_Z(G)$ are adjacent if $ab \in Z(G)$, where $Z(G)$ is the center of G. The structure for $\Gamma_Z(G)$ and conditions under which $\Gamma_Z(G)$ is regular or biregular were also determined.

A new graph construction called inverse graph associated with finite group was introduced in [3], motivated by which, the non-inverse graph of a group was introduced and certain properties of the non-inverse graphs were investigated in [4].

Signed graphs have been extensively studied in the literature. A signed graph is a graph whose edges receive either positive or negative signs. The concept of signed graphs was introduced in [11]. In [16], a graph G was defined to be p-signed (l-singed) if every point (line) is signed either positive or negative. The paper gives the characterization of both p-balanced and pl-balanced signed graphs.

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Further research has been done by extending the concept of algebraic graphs to signed graphs. In [14], two signed graphs namely order prime signed graph and general order prime signed graph of a finite group are introduced. The notion of distance coprime signed graph was introduced in [12] and structural characterisations and switching equivalent characterisations were obtained.

Motivated by the above mentioned literature, we extend the concept of non-inverse graphs to signed graphs called non-inverse signed graphs which is discussed in the following sections.

For the terms and definitions in graph theory, we refer to [19] and for those in group theory, refer to [10]. Throughout the paper, n denotes the order of the group G.

2 Non-inverse signed graph

Definition 1 ([4]). *Let G be a group with binary operation $*$. The non-inverse graph (in short, i^*-graph) of G, denoted by Γ, is a simple graph with vertex set consisting of elements of G and two vertices $x, y \in \Gamma$ are adjacent if x and y are not inverses of each other. That is, $x - y \iff x * y \neq i_G \neq y * x$, where i_G is the identity element of G.*

Definition 2. *A non-inverse signed graph associated with a group G is an ordered pair $\Gamma_\Sigma(G) = (K_n, \sigma)$, where K_n is a complete graph of order n and for an edge ab of K_n the signature function σ is defined as*

$$\sigma(ab) = \begin{cases} +, & \text{if } ab \in E\big(\Gamma(G)\big), \\ -, & \text{otherwise.} \end{cases}$$

Theorem 1. *Let $\Gamma_\Sigma(G)$ be a non-inverse signed graph associated with a group G. Then, for any vertex v of $\Gamma_\Sigma(G)$, we have*

$$d^+(v) = \begin{cases} n-1, & \text{if } v \text{ is self-inverse element in } G, \\ n-2, & \text{otherwise,} \end{cases}$$

and

$$d^-(v) = \begin{cases} 0, & \text{if } v \text{ is self-inverse element in } G, \\ 1, & \text{otherwise.} \end{cases}$$

Proof. Let $\Gamma_\Sigma(G)$ be a non-inverse signed graph associated with a group G. The total degree $d(u)$ of a vertex u in $\Gamma_\Sigma(G)$ must be $n-1$ since the underlying graph is a complete graph.

By Definition 1, any vertex v in $\Gamma(G)$, which is a self-inverse element in G, has degree $n-1$ and therefore by Definition 2, $d^+(v)$ must be $n-1$ in $\Gamma_\Sigma(G)$ and $d^-(v)$ must be 0 since $d^+(v) + d^-(v) = n-1$.

By Definition 1, any vertex w in $\Gamma(G)$, which is not a self-inverse element in G, has degree $n-2$ and hence by Definition 2, $d^+(G)$ must be $n-2$ and $d^-(G)$ must be 1 as total degree of w in $\Gamma_\Sigma(G)$ is $n-1$. Hence the result. □

Theorem 2 ([4]). *The non-inverse signed graph associated with any group $(G, *)$ is complete multipartite graph with $\frac{n+l}{2}$ partitions, where l is the number of self-inverse elements in G.*

Theorem 3. *The negation $-\Gamma_\Sigma(G)$ of non-inverse signed graph associated with a group G is balanced if and only if $n \leq 3$, and if $n = 3$, then i_G is the only self-inverse element in G.*

Proof. Let $-\Gamma_\Sigma(G)$ be the negation of the non-inverse signed graph associated with a group G. First, assume that $-\Gamma_\Sigma(G)$ is balanced.

If $n > 3$, then by Theorem 2, there are at least three parts in the vertex partition of $-\Gamma_\Sigma(G)$ and a cycle, containing vertices from each of these three parts, has all negative edges (see Definition 1 and Definition 2). Therefore, there exists at least one cycle with odd number of negative edges, which is a contradiction to the hypothesis that $-\Gamma_\Sigma(G)$ is balanced. Therefore, $n \leq 3$.

Now, let $n \leq 3$. When $n = 1, 2$, the signed graph $-\Gamma_\Sigma(G)$ is trivially balanced as it is acyclic. Hence, assume that $n = 3$ and $-\Gamma_\Sigma(G)$ is balanced. If G has all self inverses say $G = \{i_G, a, b\}$, then by Definition 1 and Definition 2, $-\Gamma_\Sigma(G)$ is a cycle with odd number of negative edges, which is a contradiction to the hypothesis that $-\Gamma_\Sigma(G)$ is balanced. Therefore, i_G should be the only self-inverse element in G.

Conversely, let $n \leq 3$ and if $n = 3$, let i_G be the only self-inverse element in G. When $n = 1, 2$, $-\Gamma_\Sigma(G)$ is a acyclic signed graph. Hence, it is balanced.

If $n = 3$ and i_G is the only self-inverse element in G, then $-\Gamma_\Sigma(G)$ is given by Figure 1.

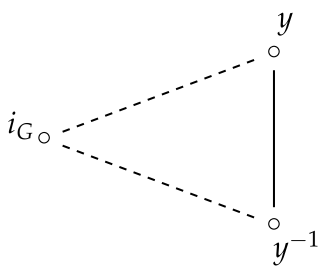


Figure 1. $-\Gamma_\Sigma(G)$ with one self inverse element

Clearly, $-\Gamma_\Sigma(G)$ shown in Figure 1 is a balanced graph. Hence the result. □

Theorem 4. *Let G a group with identity element i_G and $\Gamma_\Sigma(G)$ be its non-inverse signed graph. Then the signed graph $-\Gamma_\Sigma(G) - i_G$ is balanced if and only if $n \leq 3$ or G is either of the form $G = \{i_G, a, b, b^{-1}\}$ or $G = \{i_G, a, a^{-1}, b, b^{-1}\}$, where $a, b \in G$.*

Proof. Given a group G with identity element i_G and the negation of non-inverse signed graph $-\Gamma_\Sigma(G)$, let $-\Gamma_\Sigma(G) - i_G$ be balanced.

Assume that $n > 5$. Then, by Theorem 2, there are at least three parts of the vertex partition in $-\Gamma_\Sigma(G) - i_G$ and a cycle, containing vertices from each of these three parts, has all negative edges (see Definition 1 and Definition 2). Therefore, there exists at least one cycle with odd number of negative edges, which is a contradiction to the hypothesis that $-\Gamma_\Sigma(G) - i_G$ is balanced. Therefore, $n \leq 5$.

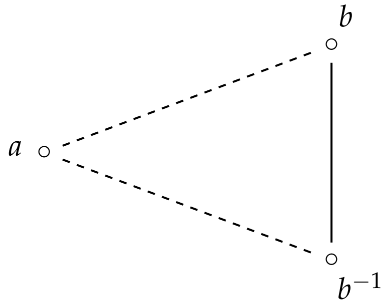
Now, let $n \leq 5$. When $n = 2, 3$, $-\Gamma_\Sigma(G) - i_G$ is trivially balanced since $-\Gamma_\Sigma(G) - i_G$ is acyclic.

Next, let $n = 4$ and let $-\Gamma_\Sigma(G) - i_G$ be balanced. Assume that there are only self-inverse elements in G. Then, by Theorem 2, there are exactly three parts of the vertex partition in $-\Gamma_\Sigma(G) - i_G$ each containing one vertex. Therefore, $-\Gamma_\Sigma(G) - i_G$ is a cycle of length 3 with all negative edges, which is a contradiction that $-\Gamma_\Sigma(G) - i_G$ is balanced. Therefore, there must be two non-self-inverse elements in G. That is, G is of the form $G = \{i_G, a, b, b^{-1}\}$.

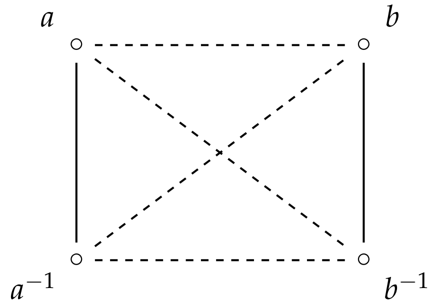
Now, let $n = 5$ and let $-\Gamma_\Sigma(G) - i_G$ be balanced. Assume that there is a self-inverse element in G other than i_G. Then, by Theorem 2, there are at least three parts of the vertex partition in $-\Gamma_{\Sigma}(G) - i_G$ and a cycle, containing vertices from each of these three parts, has all negative edges (see Definition 1 and Definition 2). Therefore, there exists at least one cycle with odd number of negative edges, which is a contradiction to the hypothesis that $-\Gamma_{\Sigma}(G) - i_G$ is balanced. Therefore, i_G is the only self-inverse element in G. That is, G is of the form $G = \{i_G, a, a^{-1}, b, b^{-1}\}$.

Conversely, let $n \leq 3$ or G be either of the form $G = \{i_G, a, b, b^{-1}\}$ or $G = \{i_G, a, a^{-1}, b, b^{-1}\}$. If $n = 2, 3$, the signed graph $-\Gamma_{\Sigma}(G) - i_G$ is acyclic. Hence, it is balanced.

If G is either of the form $G = \{i_G, a, b, b^{-1}\}$ or $G = \{i_G, a, a^{-1}, b, b^{-1}\}$, then $-\Gamma_{\Sigma}(G) - i_G$ is given by either Figure 2 (a) or Figure 2 (b), respectively.



(a) $-\Gamma_{\Sigma}(G) - i_G$ associated with $G = \{i_G, a, b, b^{-1}\}$



(b) $-\Gamma_{\Sigma}(G) - i_G$ associated with $G = \{i_G, a, a^{-1}, b, b^{-1}\}$

Figure 2. Graph $-\Gamma_{\Sigma}(G) - i_G$

The graphs in both Figure 2 (a) and Figure 2 (b) are balanced. Hence the result. □

Theorem 5. *The negation of the non-inverse signed graph $\Gamma_{\Sigma}(G)$ of a group G is k-clusterable, where $k = \frac{n+l}{2}$ and l is the number of self-inverse elements in G.*

Proof. By Theorem 2, the non-inverse signed graph associated with any group $(G, *)$ is complete multipartite graph with $\frac{n+l}{2}$ parts, where l is the number of self-inverse elements in G. Therefore, by Definition 2 and Theorem 2, the vertex set of $-\Gamma_{\Sigma}(G)$ can be partitioned into l singleton parts consisting of self-inverse element each and $\frac{n-l}{2}$ parts each consisting of a non-self-inverse element and its inverse such that the negative edges lie across the parts and positive edges lie within the part as shown in the following Figure 3.

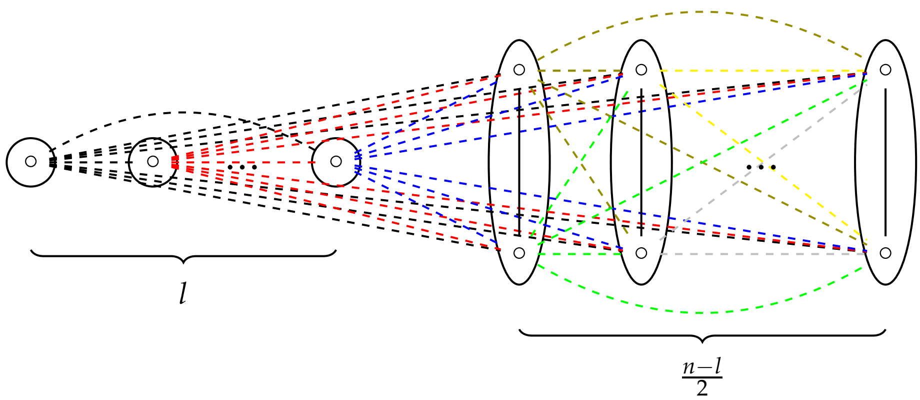


Figure 3. $-\Gamma_{\Sigma}(G)$ with $\frac{n+l}{2}$ parts

Hence, $-\Gamma_{\Sigma}(G)$ associated with a group G is k-clusterable, where $k = \frac{n+l}{2}$. □

Theorem 6. *The non-inverse signed graph $\Gamma_\Sigma(G)$ associated with a group G is positive homogeneous if and only if all the elements in G are self-inverses.*

Proof. Let the non-inverse signed graph $\Gamma_\Sigma(G)$ associated with a group G be positive homogeneous. Then, By Definition 2 and Definition 1, the associated non-inverse signed graph $\Gamma(G)$ must be a complete graph which implies that all the elements in G are self-inverses by Definition 1.

Conversely, let all the elements in G be self-inverses. Then, by Definition 1, the non-inverse signed graph associated with G is a complete graph. Hence, by Definition 2, all the edges in $\Gamma_\Sigma(G)$ are positive. Therefore, $\Gamma_\Sigma(G)$ is a positive homogeneous signed graph. $\square$

Note, that the non-inverse signed graph $\Gamma_\Sigma(G)$ associated with a group G can never be negative homogeneous since there exists an identity element in G which is adjacent to all the other vertices in $\Gamma(G)$.

Theorem 7. *Given a group G, its non-inverse signed graph is $(n-1)$-net-regular if and only if all the elements in G are self-inverses.*

Proof. Let G be a group and let $\Gamma_\Sigma(G)$ be $(n-1)$-net-regular. Assume that there is at least one non-self-inverse element in G, say x. Then there should be another non-self-inverse element in G which is inverse of x, say y. By Theorem 1, $d^+(x) = n-2$ and $d^-(x) = 1$. Therefore, $d^+(x) - d^-(x) = n-3$. Similarly, $d^+(y) - d^-(y) = n-3$, which is a contradiction that $\Gamma_\Sigma(G)$ is $(n-1)$-net-regular. Therefore, all the elements in G must be self-inverses.

Conversely, let all the elements in G be self-inverses. Then, by Theorem 1, the difference between the positive degree and negative degree of any vertex in $\Gamma_\Sigma(G)$ is $n-1$. Therefore $\Gamma_\Sigma(G)$ is $(n-1)$-net-regular. $\square$

Proposition 1. *Let G be a group with exactly one self-inverse element i_G, then $\Gamma_\Sigma(G) - i_G$ is $(n-4)$-net-regular, where i_G is the identity element of G.*

Proof. Let G be a group and let the identity element i_G be the only self-inverse element in G. Then the signed graph $\Gamma_\Sigma(G) - i_G$ has elements with positive degree $n-3$ and negative degree 1 by Theorem 1. Therefore, the difference between the positive degree and the negative degree of every vertex in $\Gamma_\Sigma(G) - i_G$ is $n-4$. Hence, it is $(n-4)$-net-regular. $\square$

Definition 3 ([12])**.** *Given a signed graph S, for any vertex $v \in V(S)$, where $V(S)$ is the vertex set of S, the marking μ of S, defined by*

$$\mu(v) = \prod_{uv \in E(S)} \sigma(uv),$$

is called a canonical marking of S.

Definition 4 ([18])**.** *A signed graph S is sign-compatible if there exists a marking μ of its vertices such that the end vertices of every negative edge receive '$-$' sign in μ and no positive edge in S has both of its ends '$-$' sign in μ.*

Definition 5 ([17])**.** *A marked signed graph is consistent if every cycle in the signed graph possesses an even number of negative vertices.*

Theorem 8. $\Gamma_\Sigma(G)$ *is sign-compatible and consistent under canonical marking if and only if there are at most two non-self-inverse elements in the group* G.

Proof. Let G be a group and $\Gamma_\Sigma(G)$ be its non-inverse signed graph. Consider the canonical marking of the vertices in $\Gamma_\Sigma(G)$ and therefore, by Definition 2, all the vertices corresponding to self-inverse elements receive '$+$' sign and all the remaining vertices '$-$' sign.

Now, let $\Gamma_\Sigma(G)$ be sign-compatible. Assume that there are more than two elements in G that are not self-inverses say, x, x^{-1}, y and y^{-1}. Then the vertices corresponding to these elements receive '$-$' sign under canonical marking. By Definition 2, there are positive edges between x and y, x and y^{-1}, y and x^{-1}, y^{-1} and x^{-1}, which is a contradiction to the fact that $\Gamma_\Sigma(G)$ is sign-compatible. Also, the cycle formed by the vertices x, x^{-1} and y has odd number of negative vertices, which is a contradiction that $\Gamma_\Sigma(G)$ is consistent. Hence, there are at most two non-self-inverse elements in G.

Conversely, let there be at most two non-self-inverse elements in G say x and x^{-1}. Hence, the vertices corresponding to these two elements receive '$-$' sign. Therefore, the negative edge between the vertices corresponding to x and x^{-1} has both its end vertices '$-$' sign and no positive edge in $\Gamma_\Sigma(G)$ has both of its end vertices '$-$' sign. Therefore, $\Gamma_\Sigma(G)$ is sign-compatible under canonical marking. Further, every cycle containing x and x^{-1} have even number of negative vertices. Therefore, $\Gamma_\Sigma(G)$ is consistent under canonical marking. □

Theorem 9. *The negation of non-inverse signed graph* $\Gamma_\Sigma(G)$ *associated with a group* G *is consistent under canonical marking if and only if*

(i) all the elements in G *are self-inverses, when* n *is odd,*

and

(ii) all the elements in G *are non-self-inverses, when* n *is even.*

Proof. Let $-\Gamma_\Sigma(G)$ be the negation of non-inverse signed graph associated with a group G. Consider the canonical marking of the vertices of $-\Gamma_\Sigma(G)$. We know that, for the negation of non-inverse signed graph, by Theorem 1, the negative degree of a vertex corresponding to a self-inverse element is $n-1$ and that of a non-self-inverse element is $n-2$. The following two cases arise.

Case 1. Let n be odd and let $-\Gamma_\Sigma(G)$ be consistent. Assume that there are two non-self-inverse elements in G say, x, x^{-1}. Since n is odd, under canonical marking, the vertices corresponding to self-inverse elements receive '$+$' sign and the vertices corresponding to non-self-inverse elements receive '$-$' sign. Therefore, x and x^{-1} receive '$-$' sign and any cycle containing exactly one of these two vertices has odd number of negative vertices, which is a contradiction that $-\Gamma_\Sigma(G)$ is consistent. Hence, all the elements in G are self-inverses.

Conversely, let there be only self-inverse elements in G. Since n is odd, the vertices corresponding to all the self-inverses receive '$+$' sign. Since, there are no vertices with '$-$' sign, there are no cycles with odd number of negative vertices. Hence, $-\Gamma_\Sigma(G)$ is consistent.

Case 2. Let n be even and let $-\Gamma_\Sigma(G)$ be consistent. Assume that there is one self-inverse element in G, say x. Since n is even, under canonical marking, the vertices corresponding to self-inverse elements receive '$-$' sign and the vertices corresponding to non-self-inverse elements receive '$+$' sign. Therefore, x receives '$-$' sign and any cycle containing x has odd number of negative vertices, which is a contradiction that $-\Gamma_\Sigma(G)$ is consistent. Hence, all the elements in G are non-self-inverses.

Conversely, let there be no self-inverse element in G. Since n is even, the vertices corresponding to all the non-self-inverses receive '$+$' sign. Since, there are no vertices with '$-$' sign, there are no cycles with odd number of negative vertices. Hence, $-\Gamma_\Sigma(G)$ is consistent. □

Theorem 10. *$-\Gamma_\Sigma(G)$ associated with a group G is sign-compatible under canonical marking if and only if n is even and G has all self-inverse elements.*

Proof. Let $-\Gamma_\Sigma(G)$ be the negation of the non-inverse signed graph associated with the group G. Let $-\Gamma_\Sigma(G)$ be sign-compatible under canonical marking. Assume that n is odd. We know that, for a negated non-inverse signed graph, by Theorem 1, the negative degree of a vertex corresponding to a self-inverse element is $n-1$ and that of a non-self inverse element is $n-2$. Therefore, by canonical marking, the vertices corresponding to self-inverse elements receive '$+$' sign and the vertices corresponding to non-self-inverse elements receive '$-$' sign. Hence, there will be at least two negative edges whose end vertices are not both negative, which is a contradiction that $-\Gamma_\Sigma(G)$ is sign-compatible. Hence, n must be even.

Now, let n be even and $-\Gamma_\Sigma(G)$ be sign-compatible under canonical marking. Assume that there are at least two non-self-inverse elements in G. Since n is even, the vertices corresponding to non-self-inverse elements in $-\Gamma_\Sigma(G)$ receive '$+$' sign and all the other vertices corresponding to self-inverse elements receive '$-$' sign, by Theorem 1. Therefore, there will be at least one negative edge whose end vertices are not both negative, which is a contradiction that $-\Gamma_\Sigma(G)$ is sign-compatible. Hence, all the elements in G must be self-inverses.

Hence, $-\Gamma_\Sigma(G)$ is sign-compatible under canonical marking if n is even and G has all self-inverse elements.

Conversely, let n be even and G has all self-inverse elements. Then, by Theorem 1, all the vertices in $-\Gamma_\Sigma(G)$ receive '$-$' sign. Therefore, all the negative edges have both its end vertices '$-$' sign. Hence, $-\Gamma_\Sigma(G)$ is sign-compatible. □

Definition 6 ([12]). *Switching a signed graph S with respect to a marking μ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The resulting signed graph $S_\mu(S)$ is said to be a switched signed graph.*

Definition 7 ([12]). *Two signed graphs $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ are said to be weakly isomorphic or cycle isomorphic if there exists an isomorphism $\phi : G_1 \to G_2$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2.*

Theorem 11 ([15]). *Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic.*

Theorem 12. *$-\Gamma_\Sigma(G)$ and its switched graph with respect to canonical marking are switching equivalent.*

Proof. Let $-\Gamma_\Sigma(G)$ be the negation of non-inverse signed graph associated with a group G and let S' be its switched graph under canonical marking. We know that, for the negation of non-inverse signed graph, by Theorem 1, the negative degree of a vertex corresponding to a self-inverse element is $n-1$ and that of a non-self inverse element is $n-2$. We also know that, by Theorem 2, $-\Gamma_\Sigma(G)$ can be partitioned into 2-partite sets, each containing a vertex corresponding to an element of G and the vertex corresponding to its inverse and into singleton sets, each containing a vertex corresponding to self-inverse element. The following cases arise.

Case 1. Let n be odd. Then, under canonical marking, all the vertices of $-\Gamma_\Sigma(G)$ corresponding to self-inverse elements receive '$+$' sign and the vertices corresponding to non-self-inverse elements receive '$-$' sign. Now, the following subcases arise.

Subcase 1.1. G has only self-inverses. Clearly, all the vertices in $-\Gamma_\Sigma(G)$ have '$+$' sign and therefore, $-\Gamma_\Sigma(G)$ switches to itself. Hence, $-\Gamma_\Sigma(G)$ and S' are switching equivalent.

Subcase 1.2. Let G have both self-inverse and non-self inverse elements. Then the edges between any two 2-partite sets remain the same after switching under canonical marking and the negative edges between any 2-partite set and a singleton set switch to positive edges. It is evident that the sign of each cycle in $-\Gamma_\Sigma(G)$ is equal to the sign of the corresponding cycle in S'. Therefore, $-\Gamma_\Sigma(G)$ and S' are switching equivalent under canonical marking by Theorem 11.

Case 2. Let n be even. Then, under canonical marking, all the vertices of $-\Gamma_\Sigma(G)$ corresponding to self-inverse elements receive '$-$' sign and the vertices corresponding to non-self-inverse elements receive '$+$' sign. Now, the following subcases arise.

Subcase 2.1. G has only self-inverses. Clearly, all the vertices in $-\Gamma_\Sigma(G)$ have '$-$' sign and therefore $-\Gamma_\Sigma(G)$ switches to itself. Hence, $-\Gamma_\Sigma(G)$ and S' are switching equivalent.

Subcase 2.2. G has both self-inverse and non-self-inverse elements. Same argument as in Subcase 1.2.

Subcase 2.3. Let G has only non-self-inverse elements. Since all the vertices in $-\Gamma_\Sigma(G)$ have '$+$' sign, $-\Gamma_\Sigma(G)$ switches to itself. Hence, $-\Gamma_\Sigma(G)$ and S' are switching equivalent. This completes the proof. □

Definition 8 ([8])**.** *Frustration index is the least number of edges whose deletion makes the signed graph balanced.*

Theorem 13. *The upper bound for the frustration index $l(\Sigma)$ of the signed graph $-\Gamma_\Sigma(G)$ associated with a group G is given by the following three cases.*

(i) When all the elements in G are self-inverse elements, then

$$l(\Sigma) \leq \left\lceil \frac{n-3}{2} \right\rceil + \left\lceil \left(\frac{n-2}{2} \right)^2 \right\rceil.$$

(ii) When all the elements except the identity element are self-inverses, then

$$l(\Sigma) \leq \frac{n^2-4n+3}{4}.$$

(iii) When n is odd and there are l self-inverse elements in G, then

$$l(\Sigma) \leq \left(\frac{n-3}{2} \right)^2 + \left(\frac{n-3}{2} \right) + \left(\frac{l-1}{2} \right).$$

Proof. Let $-\Gamma_{\Sigma}(G)$ be the negation of non-inverse signed graph associated with a group G. $-\Gamma_{\Sigma}(G)$ is a complete multi-partite graph with singleton sets containing self-inverse elements and 2-element sets each containing an element and its inverse. We define a marking of the vertices with '$+$' and '$-$' signs as follows. Arrange all the singleton partitions in the beginning followed by the 2-element partitions so that we can alternatively assign the signs to the partitions starting with either '$+$' or '$-$' sign. The vertices inside the 2-element partition receive the same sign. To obtain the frustration index $l(\Sigma)$, we remove one edge each from the odd cycle consisting of all negative edges. This can be done by removing the edges joining two vertices which have the same sign under the given marking. We obtain $l(\Sigma)$ for the following cases.

Case 1. Let all the elements in G be self-inverse elements. For $n = 3, 4, 5, 6, 7, \ldots$, the frustration index $l(\Sigma) \leq 1, 2, 4, 6, 9, \ldots$, respectively. Therefore, in general, $l(\Sigma) \leq \left\lceil \frac{n-3}{2} \right\rceil + \left\lceil \left(\frac{n-2}{2}\right)^2 \right\rceil$ when all the elements in G are self-inverse elements.

Case 2. Let all the elements except the identity element be self-inverses. Then n must be odd as the identity element is the only self-inverse element. For $n = 5, 7, 9, 11, 13, \ldots$, the frustration index $l(\Sigma) \leq 2, 6, 12, 20, 30, \ldots$, respectively. Therefore, in general, $l(\Sigma) \leq \frac{n^2-4n+3}{4}$.

Case 3. Let n be odd and there are l self-inverse elements in G. Then, for $n = 5$, $l = 3$, we get $l(\Sigma) \leq 3$, for $n = 7$, $l = 3$, we get $l(\Sigma) \leq 7$ and $l = 5$ gives $l(\Sigma) \leq 8$, for $n = 9$, $l = 3$ we get $l(\Sigma) \leq 13$, $l = 5$ gives $l(\Sigma) \leq 14$ and $l = 7$ gives $l(\Sigma) \leq 15$ and so on. Therefore, after generalisation and simplification, $l(\Sigma) \leq \left(\frac{n-3}{2}\right)^2 + \left(\frac{n-3}{2}\right) + \left(\frac{l-1}{2}\right)$, when n is odd and there are l self-inverse elements in G. □

It is difficult to generalise frustration index for the case when the order of the group n is even and there are l self-inverse elements in the group G.

3 Conclusion

In this paper, we define non-inverse signed graph of a group and characterise the graphs for which non-inverse signed graphs and the negation of non-inverse signed graphs are balanced, sign-compatible, consistent and k-clusterable. We also obtaine the frustration index of the i^*-signed graph. Further, we characterize the homogeneous non-inverse signed graphs and study the properties like net-regularity and switching equivalence of non-inverse signed graphs.

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Нехай G — група з бінарною операцією $*$. Граф без інверсій (скорочено, i^*-граф) групи G, який ми позначимо Γ, є простим графом з множиною вершин, що складається з елементів G, де дві вершини $x, y \in \Gamma$ є сусідніми, якщо x та y не є інверсіями одна одної. Тобто, $x - y$ тоді і тільки тоді, коли $x * y \neq i_G \neq y * x$, де i_G — нейтральний елемент групи G. У цій статті ми розширюємо вивчення i^*-графів на випадок знакових графів, вводячи i^*-знакові графи. Ми характеризуємо графи, для яких i^*-знакові графи та неговані i^*-знакові графи є збалансованими, знаково-сумісними, послідовними та k-кластеризованими. Крім того, ми визначаємо індекс фрустрації i^*-знакового графа. Додатково ми характеризуємо однорідні знакові графи без інверсій та досліджуємо такі їх властивості, як мережева регулярність і еквівалентність відносно перестановок знаків.

Ключові слова і фрази: алгебраїчний граф, граф без інверсій, знаковий граф без інверсій.