ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2022, **14** (2), 354–363
doi:10.15330/cmp.14.2.354-363

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2022, Т.14, №2, С.354–363

Some new classes of degenerated generalized Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials

Ramírez W.[1], Cesarano C.[2,✉]

The aim of this paper is to study new classes of degenerated generalized Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials of order α and level m in the variable x. Here the degenerate polynomials are a natural extension of the classic polynomials. In more detail, we derive their explicit expressions, recurrence relations and some identities involving those polynomials and numbers. Most of the results are proved by using generating function methods.

Key words and phrases: Apostol-type polynomials, degenerated Apostol-type polynomials.

[1] University of the Coast, Calle 58 # 55 – 66, Barranquilla, Colombia
[2] International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186 Rome, Italy
✉ Corresponding author
E-mail: wramirez4@cuc.edu.co (Ramírez W.), c.cesarano@uninettunouniversity.net (Cesarano C.)

Introduction

New classes of generalized Apostol-Bernoulli polynomials, Apostol-Euler polynomials and Apostol-Genocchi polynomials with parameters $a, c \in \mathbb{R}^+$ by means of the following generating functions, defined in a suitable neighborhood of $t = 0$ (see [13] and the references therein):

$$t^{m\alpha}[A(\lambda, a; t)]^\alpha c^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n^{[m-1,\alpha]}(x; a, c; \lambda)\frac{t^n}{n!}, \tag{1}$$

$$2^{m\alpha}[B(\lambda, a; t)]^\alpha c^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n^{[m-1,\alpha]}(x; a, c; \lambda)\frac{t^n}{n!}, \tag{2}$$

$$(2t)^{m\alpha}[B(\lambda, a; t)]^\alpha c^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n^{[m-1,\alpha]}(x; a, c; \lambda)\frac{t^n}{n!}, \tag{3}$$

where

$$A(\lambda, a; t) = \left(\lambda a^t - \sum_{l=0}^{m-1} \frac{(t\log a)^l}{l!}\right)^{-1}, \qquad B(\lambda, a; t) = \left(\lambda a^t + \sum_{l=0}^{m-1} \frac{(t\log a)^l}{l!}\right)^{-1}.$$

The fundamental objective underlying previous and recent studies on generalized Apostol-type polynomials of level m attempts to make appropriate modifications to the generating

УДК 517.589
2020 *Mathematics Subject Classification:* 33E20, 11B83, 11B68, 30H50.

This research was funded by the International Telematic University Uninettuno (Italia) and the Universidad de la Costa (Colombia) for all the support provided supported by the project whit order code SAP E11P1070121C

functions associated with the classical classes of Apostol, Bernoulli, Euler and Genocchi polynomials, respectively, and obtain algebraic and/or similar polynomials, differential properties of these polynomials. On the subject of the Appell-type polynomials and their various extensions, a remarkably large number of investigations have appeared in the literature (see, for example, [1,3–7,13,14,16,19–21]).

On the other hand, for any nonzero real number a, the degenerate exponentials are given (see [9]):

$$e_a^x(t) = (1+at)^{x/a} \quad \text{and} \quad e_a(t) = (1+at)^{1/a}. \tag{4}$$

By (4), we get

$$(1+at)^{x/a} = \sum_{n=0}^{\infty} (x|a)_n \frac{t^n}{n!},$$

where $(x|a)_0 = 1$, $(x|a)_n = x(x-a)\dots(x-(n-1)a)$, $n \geq 1$.

Next, we recall the definitions of the degenerate Bernoulli polynomials $\mathcal{B}_n(x;a)$, the degenerate Euler polynomials $\mathcal{E}_n(x;a)$ and the degenerate Genocchi polynomials $\mathcal{G}_n(x;a)$ (see [2]) with parameter $a \in \mathbb{R}$ in the variable x and in a suitable neighborhood of $t=0$, by means of the corresponding generating functions.

For the degenerate Bernoulli polynomials we have

$$\frac{t}{(1+at)^{1/a}-1}(1+at)^{x/a} = \sum_{n=0}^{\infty} \mathcal{B}_n(x;a)\frac{t^n}{n!}. \tag{5}$$

If $x=0$, then $\mathcal{B}_n(a) := \mathcal{B}_n(0;a)$ are the corresponding degenerate Bernoulli numbers. It is to be noted from equation (5) that

$$\lim_{a\to 0} \mathcal{B}_n(x;a) = B_n(x), \quad n \geq 0,$$

where $B_n(x)$ are the nth order Bernoulli polynomials [15].

Now, for the degenerate Euler polynomials we obtain

$$\frac{2}{(1+at)^{1/a}+1}(1+at)^{x/a} = \sum_{n=0}^{\infty} \mathcal{E}_n(x;a)\frac{t^n}{n!}. \tag{6}$$

If $x=0$, then $\mathcal{E}_n(a) := \mathcal{E}_n(0;a)$ are the corresponding degenerate Euler numbers. It follows from equation (6) that

$$\lim_{a\to 0} \mathcal{E}_n(x;a) = E_n(x), \quad n \geq 0,$$

where $E_n(x)$ are the nth order ordinary Euler polynomials [15].

At last, for the degenerate Genocchi polynomials we have

$$\frac{2t}{(1+at)^{1/a}+1}(1+at)^{x/a} = \sum_{n=0}^{\infty} \mathcal{G}_n(x;a)\frac{t^n}{n!}. \tag{7}$$

If $x=0$, then $\mathcal{G}_n(a) := \mathcal{G}_n(0;a)$ are the corresponding degenerate Genocchi numbers. Consequently from equation (7), we have

$$\lim_{a\to 0} \mathcal{G}_n(x;a) = G_n(x), \quad n \geq 0,$$

where $G_n(x)$ are the nth order ordinary Genocchi polynomials [18].

In this paper, we present and develop some algebraic properties of the degenerated generalized new classes of Apostol-Bernouilli, Apostol-Eeuler and Apostol-Genocchi polynomials of level m. These results extend certain relations and identities of the related polynomials.

1 Degenerated new classes of generalized Apostol-Bernouilli, Apostol-Eeuler and Apostol-Genocchi polynomials of order α and level m

In this section, taking into account (1)–(4), we introduce the degenerated new classes of generalized Apostol-Bernouilli, Apostol-Eeuler and Apostol-Genocchi polynomials of order α and level m.

Definition 1. *For arbitrary parameters α, $b \in \mathbb{R}^+$ and for $a \in \mathbb{Z}$, the degenerated generalized the Apostol-Bernoulli polynomials $\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)$, the degenerated generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)$, and the degenerated generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)$, $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$, of order α and level m in the variable x, are defined, in a suitable neighborhood of $t = 0$, by means of the generating function:*

$$t^{m\alpha}[\sigma(\lambda;a,b;t)]^{\alpha}(1+at)^{x/a} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!}, \tag{8}$$

$$2^{m\alpha}[\psi(\lambda;a,b;t)]^{\alpha}(1+at)^{x/a} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!}, \tag{9}$$

$$(2t)^{m\alpha}[\psi(\lambda;a,b;t)]^{\alpha}(1+at)^{x/a} = \sum_{n=0}^{\infty} \mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!}, \tag{10}$$

respectively, where

$$\sigma(\lambda;a,b;t) = \left(\lambda(1+at)^{1/a} - \sum_{l=0}^{m-1}\frac{(t\log b)^l}{l!}\right)^{-1},$$

$$\psi(\lambda;a,b;t) = \left(\lambda(1+at)^{1/a} + \sum_{l=0}^{m-1}\frac{(t\log b)^l}{l!}\right)^{-1}.$$

Note that for $b = e$ in (8), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\lim_{a\to 0}\mathfrak{B}_n^{[m-1,\alpha]}(x;a,e;\lambda)\frac{t^n}{n!} &= \lim_{a\to 0}\frac{t^{m\alpha}(1+at)^{x/a}}{\left(\lambda(1+at)^{1/a} - \sum_{l=0}^{m-1}(t\log b)^l/l!\right)^{\alpha}}\\ &= \frac{t^{m\alpha}e^{xt}}{\left(\lambda e^t - \sum_{l=0}^{m-1}t^l/l!\right)^{\alpha}} = \sum_{n=0}^{\infty}\mathfrak{B}_n^{[m-1,\alpha]}(x)\frac{t^n}{n!},\end{aligned}$$

where $\mathfrak{B}_n^{[m-1,\alpha]}(x)$ are called the generalized Bernoulli polynomials of order α (see [11, Definition 2.3]). Also, for $b = e$ in (9), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\lim_{a\to 0}\mathfrak{E}_n^{[m-1,\alpha]}(x;a,e;\lambda)\frac{t^n}{n!} &= \lim_{a\to 0}\frac{2^{m\alpha}(1+at)^{x/a}}{\left(\lambda(1+at)^{1/a} + \sum_{l=0}^{m-1}(t\log b)^l/l!\right)^{\alpha}}\\ &= \frac{2^{m\alpha}e^{xt}}{\left(\lambda e^t + \sum_{l=0}^{m-1}t^l/l!\right)^{\alpha}} = \sum_{n=0}^{\infty}\mathfrak{E}_n^{[m-1,\alpha]}(x)\frac{t^n}{n!},\end{aligned}$$

where $\mathfrak{E}_n^{[m-1,\alpha]}(x)$ are called the generalized Euler polynomials of order α (see [10, equation (1.9)]). Similarly, for $b = e$ in (10), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\lim_{a\to 0}\mathfrak{G}_n^{[m-1,\alpha]}(x;a,e;\lambda)\frac{t^n}{n!} &= \lim_{a\to 0}\frac{(2t)^{m\alpha}(1+at)^{x/a}}{(\lambda(1+at)^{1/a} + \sum_{l=0}^{m-1}(t\log b)^l/l!)^{\alpha}}\\ &= \frac{(2t)^{m\alpha}e^{xt}}{(\lambda e^t + \sum_{l=0}^{m-1}(t)^l/l!)^{\alpha}} = \sum_{n=0}^{\infty}\mathfrak{G}_n^{[m-1,\alpha]}(x)\frac{t^n}{n!},\end{aligned}$$

where $\mathfrak{G}_n^{[m-1,\alpha]}(x)$ are called the generalized Genocchi polynomials of order α (see [10, equation (1.11)]).

In the table below, we introduce the standard notation for several subclasses the degenerated generalized the Apostol-Bernoulli polynomials $\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)$, the degenerated generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)$ and the degenerated generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)$, $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$ of order α and level m in the variable x (cf. [2,8,12,17]).

nth the degenerate Bernoulli polynomials	$B_n(x;a) := \mathfrak{B}_n^{[0,1]}(x;a,b,1)$
nth the degenerate Euler polynomials	$E_n(x;a) := \mathfrak{E}_n^{[0,1]}(x;a,b,1)$
nth the degenerate Genocchi polynomials	$G_n(x;a) := \mathfrak{G}_n^{[0,1]}(x;a,b,1)$
nth the degenerate Apostol-Bernoulli polynomials	$B_n^{(\alpha)}(x;a;\lambda) := \mathfrak{B}_n^{[0,\alpha]}(x;a,b,\lambda)$
nth the degenerate Apostol-Euler polynomials	$E_n^{(\alpha)}(x;a;\lambda) := \mathfrak{E}_n^{[0,\alpha]}(x;a,b,\lambda)$
nth the degenerate Apostol-Genocchi polynomials	$G_n^{(\alpha)}(x;a;\lambda) := \mathfrak{G}_n^{[0,\alpha]}(x;a,b,\lambda)$

If $x = 0$ in Definition 1, we obtain the degenerated generalized Apostol-Bernoulli numbers, degenerated generalized Apostol-Euler numbers and degenerated generalized Apostol-Genocchi numbers of order α and level m

$$t^{m\alpha}[\sigma(\lambda;a,b;t)]^{\alpha} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!}, \qquad 2^{m\alpha}[\psi(\lambda;a,b;t)]^{\alpha} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!},$$
$$(2t)^{m\alpha}[\psi(\lambda;a,b;t)]^{\alpha} = \sum_{n=0}^{\infty} \mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!}.$$

Example 1. *For $\lambda = 1$, the first few degenerate the Apostol-Bernoulli polynomials of level $m = 1$ are given as:*

$$\begin{aligned}
\mathfrak{B}_0^{[0,1]}(x;a,e;1) &= 1,\\
\mathfrak{B}_1^{[0,1]}(x;a,e;1) &= x^2 + \frac{a-1}{2}x + 1,\\
\mathfrak{B}_2^{[0,1]}(x;a,e;1) &= \frac{x^2}{2} - \frac{x^3}{2} + \frac{13-a^2}{12}x^2 + \frac{a-1}{2}x + 1,\\
\mathfrak{B}_3^{[0,1]}(x;a,e;1) &= \frac{x^6}{6} - \frac{a+1}{4}x^5 + \frac{7+3a}{12}x^4 + \frac{a^3-a-12}{24}x^3 + \frac{13-a^2}{12}x^2 - \frac{1}{2}x + 1.
\end{aligned}$$

Example 2. *For any $\lambda = 1$, the first few degenerated generalized the Apostol-Euler polynomial polynomials of level $m = 1$ are given as:*

$$\begin{aligned}
\mathfrak{E}_0^{[0,1]}(x;a,e;1) &= 1,\\
\mathfrak{E}_1^{[0,1]}(x;a,e;1) &= x^2 + \frac{-1}{2}x + 1,\\
\mathfrak{E}_2^{[0,1]}(x;a,e;1) &= \frac{x^4}{2} - \frac{a+1}{2}x^3 + \frac{6-a}{4}x^2 + \frac{-1}{2}x + 1,\\
\mathfrak{E}_3^{[0,1]}(x;a,e;1) &= \frac{x^6}{6} - \frac{2a+1}{4}x^5 + \frac{2a^2+3a+3}{6}x^4 + \left(a^2 - \frac{a}{12} - \frac{22}{43}\right)x^3 + \frac{24-a}{4}x^2 - \frac{1}{2}x + 1.
\end{aligned}$$

Similarly, for any $\lambda = 1$, the first few degenerated generalized the Apostol-Genocchi polynomials of level $m = 1$ are given as:

$$\mathfrak{G}_0^{[0,1]}(x;a,e;1) = 0, \quad \mathfrak{G}_1^{[0,1]}(x;a,e;1) = x, \quad \mathfrak{G}_2^{[0,1]}(x;a,e;1) = x^3 - \frac{1}{2}x,$$
$$\mathfrak{G}_3^{[0,1]}(x;a,e;1) = \frac{1}{2}x^5 - \frac{a+1}{2}x^4 + \frac{4+a}{4}x^3 - \frac{1}{2}x^2 + x.$$

2 Some properties for the polynomials $\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)$, $\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)$ and $\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)$

In this section, we state some properties for the new classes of degenerated generalized Apostol-type polynomials of order α and level m in the variable x defined in Section 1.

Theorem 1. *Let $\{\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)\}_{n\geq 0}$, $\{\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)\}_{n\geq 0}$ and $\{\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)\}_{n\geq 0}$ be the sequence of degenerated generalized Apostol-type polynomials of order $\alpha \in \mathbb{C}$ and level m in the variable x, where $a, b \in \mathbb{R}^+$. Then, for a fixed $m \in \mathbb{N}$, the following statemets hold.*

(A) Special values.

For $n \in \mathbb{N}_0$,

$$\mathfrak{B}_n^{[m-1,0]}(x;a,b;\lambda) = (x|a)_n, \quad \mathfrak{E}_n^{[m-1,0]}(x;a,b;\lambda) = (x|a)_n, \quad \mathfrak{G}_n^{[m-1,0]}(x;a,b;\lambda) = (x|a)_n.$$

(B) Summation formulas.

For $n \in \mathbb{N}_0$,

$$\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{B}_{n-k}^{[m-1,\alpha]}(a,b;\lambda)(x|a)_k,$$
$$\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_{n-k}^{[m-1,\alpha]}(a,b;\lambda)(x|a)_k,$$
$$\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{G}_{n-k}^{[m-1,\alpha]}(a,b;\lambda)(x|a)_k.$$

(C) Addition theorem of the argument.

For $\beta \in \mathbb{C}$ and $n \in \mathbb{N}_0$,

$$\begin{aligned}
\mathfrak{B}_n^{[m-1,\alpha+\beta]}(x+y;a,b;\lambda) &= \sum_{k=0}^{n}\binom{n}{k}\mathfrak{B}_k^{[m-1,\alpha]}(x;a,b;\lambda)\mathfrak{B}_{n-k}^{[m-1,\beta]}(y;a,b;\lambda),\\
\mathfrak{B}_n^{[m-1,\alpha]}(x+y;a,b;\lambda) &= \sum_{k=0}^{n}\binom{n}{k}\mathfrak{B}_{n-k}^{[m-1,\alpha]}(y;a,b;\lambda)(x|a)_k, \qquad (11)\\
\mathfrak{E}_n^{[m-1,\alpha+\beta]}(x+y;a,b;\lambda) &= \sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k^{[m-1,\alpha]}(x;a,b;\lambda)\mathfrak{E}_{n-k}^{[m-1,\beta]}(y;a,b;\lambda),\\
\mathfrak{E}_n^{[m-1,\alpha]}(x+y;a,b;\lambda) &= \sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_{n-k}^{[m-1,\alpha]}(y;a,b;\lambda)(x|a)_k,\\
\mathfrak{G}_n^{[m-1,\alpha+\beta]}(x+y;a,b;\lambda) &= \sum_{k=0}^{n}\binom{n}{k}\mathfrak{G}_k^{[m-1,\alpha]}(x;a,b;\lambda)\mathfrak{G}_{n-k}^{[m-1,\beta]}(y;a,b;\lambda),\\
\mathfrak{G}_n^{[m-1,\alpha]}(x+y;a,b;\lambda) &= \sum_{k=0}^{n}\binom{n}{k}\mathfrak{G}_{n-k}^{[m-1,\alpha]}(y;a,b;\lambda)(x|a)_k.
\end{aligned}$$

Proof. Statements *(A)* and *(B)* are obvious. Let us prove (11). From generating function of the degenerated generalized the Apostol-Bernoulli polynomials order α and level m in the variable x, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_n^{[m-1,\alpha]}(x+y;a,b;\lambda)\frac{t^n}{n!} &= t^{m\alpha}[\sigma(\lambda;a,b;t)]^{\alpha}(1+at)^{(x+y)/a}\\ &= t^{m\alpha}[\sigma(\lambda;a,b;t)]^{\alpha}(1+at)^{y/a}(1+at)^{x/a}\\ &= \sum_{n=0}^{\infty}\mathfrak{B}_n^{[m-1,\alpha]}(y;a,b;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}(x\mid a)_n\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{B}_{n-k}^{[m-1,\alpha]}(y;a,b;\lambda)(x\mid a)_k.\end{aligned}$$

Comparing the coefficients of $t^n/n!$ on the both sides of the above equation, we obtain the identity (11) at once.

The proof of others equalities from statement *(C)* proceeds analogously. □

Proposition. *The degenerated generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)$ of order α and level m satisfy the following relation*

$$\lambda\mathfrak{E}_n^{[m-1,\alpha]}(x+1;a,b;\lambda)+\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)=2\sum_{k=0}^{n}\mathfrak{E}_k^{[m-1,\alpha]}(x;a,b;\lambda)\mathfrak{E}_{n-k}^{[-1]}(0;a,e;1). \tag{12}$$

Proof. By (7) and (9), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}[\lambda\mathfrak{E}_n^{[m-1,\alpha]}(x+1;a,b;\lambda)+\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)]\frac{t^n}{n!}\\ &\qquad= 2^{m\alpha}[\psi(\lambda;a,b;t)]^{\alpha}(1+at)^{(x+1)/a}+\frac{t^{m\alpha}(1+at)^{x/a}}{(\lambda(1+az)^{1/a}-\sum_{l=0}^{m-1}(t\ln b)^l/l!)^{\alpha}}\\ &\qquad= 2^{m\alpha}[\psi(\lambda;a,b;t)]^{\alpha}(1+at)^{x/a}(1+\lambda(1+at)^{1/a})\\ &\qquad= 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k^{[m-1,\alpha]}(x;a,b;\lambda)\mathfrak{E}_{n-k}^{[-1]}(0;a,e;1)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $t^n/n!$ on the both sides of the above equation, we obtain the identity (12) at once. □

Theorem 2. *Let $\{\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)\}_{n\geq 0}$, $\{\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)\}_{n\geq 0}$ and $\{\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)\}_{n\geq 0}$ be the sequence of degenerated generalized Apostol-type polynomials of order $\alpha\in\mathbb{C}$ and level m in the variable x, where $a,b\in\mathbb{R}^+$. Then, for a fixed $m\in\mathbb{N}$,*

$$\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)=\mathfrak{B}_n^{[m-1,\alpha]}(x+a;a,b;\lambda)-an\mathfrak{B}_{n-1}^{[m-1,\alpha]}(x;a,b;\lambda), \tag{13}$$

$$\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)=\mathfrak{E}_n^{[m-1,\alpha]}(x+a;a,b;\lambda)-an\mathfrak{E}_{n-1}^{[m-1,\alpha]}(x;a,b;\lambda), \tag{14}$$

$$\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)=\mathfrak{G}_n^{[m-1,\alpha]}(x+a;a,b;\lambda)-an\mathfrak{G}_{n-1}^{[m-1,\alpha]}(x;a,b;\lambda). \tag{15}$$

Proof. Let us prove (13). From generating function of the degenerated generalized the Apostol-Bernoulli polynomials order α and level m in the variable x, we have

$$\sum_{n=0}^{\infty}\mathfrak{B}_n^{[m-1,\alpha]}(x+a;a,b;\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!}+\sum_{n=0}^{\infty}\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{at^{n+1}}{n!},$$

therefore,

$$\sum_{n=0}^{\infty}\mathfrak{B}_n^{[m-1,\alpha]}(x+a;a,b;\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!}+\sum_{n=0}^{\infty}n\,\mathfrak{B}_{n-1}^{[m-1,\alpha]}(x;a,b;\lambda)\frac{at^n}{n!}.$$

Thus, we have

$$\sum_{n=0}^{\infty}\mathfrak{B}_n^{[m-1,\alpha]}(x+a;a,b;\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}[\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)+an\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)]\frac{t^n}{n!}.$$

Comparing the coefficients of t^n in both sides of the equation, the result is

$$\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)=\mathfrak{B}_n^{[m-1,\alpha]}(x+a;a,b;\lambda)-an\mathfrak{B}_{n-1}^{[m-1,\alpha]}(x+a;a,b;\lambda).$$

The proofs of (14) and (15) are similar to that of (13). □

Theorem 3. *Let* $\{\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)\}_{n\geq 0}$, $\{\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)\}_{n\geq 0}$ *and* $\{\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)\}_{n\geq 0}$ *be the sequence of degenerated generalized Apostol-type polynomials of order* $\alpha\in\mathbb{C}$ *and level* m *in the variable* x*, where* $a,b\in\mathbb{R}^+$*. Then, for a fixed* $m\in\mathbb{N}$*,*

$$\frac{\partial\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)}{\partial x}=\sum_{k=0}^{n-1}n(-1)^k a^k\frac{k!}{k+1}\binom{n-1}{k}\mathfrak{B}_{n-1-k}^{[m-1,\alpha]}(x;a,b;\lambda), \tag{16}$$

$$\frac{\partial\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)}{\partial x}=\sum_{k=0}^{n-1}n(-1)^k a^k\frac{k!}{k+1}\binom{n-1}{k}\mathfrak{E}_{n-1-k}^{[m-1,\alpha]}(x;a,b;\lambda), \tag{17}$$

$$\frac{\partial\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)}{\partial x}=\sum_{k=0}^{n-1}n(-1)^k a^k\frac{k!}{k+1}\binom{n-1}{k}\mathfrak{G}_{n-1-k}^{[m-1,\alpha]}(x;a,b;\lambda). \tag{18}$$

Proof. Let us prove (18). Partially differentiating the generating function of the degenerated generalized the Apostol-Genocchi polynomials of order α and level m in the variable x, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!}&=(2t)^{m\alpha}[\psi(\lambda;a,b;t)]^{\alpha}(1+at)^{x/a}\ln(1+at)\frac{1}{a}\\&=\left(\sum_{n=0}^{\infty}\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}a^{n+1}t^{n+1}\frac{1}{a}\right)\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\mathfrak{G}_{n-k}^{[m-1,\alpha]}(x;a,b;\lambda)(-1)^k a^k\binom{n}{k}\frac{k!}{k+1}\frac{t^{n+1}}{n!}.\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\mathfrak{G}_{n-1-k}^{[m-1,\alpha]}(x;a,b;\lambda)(-1)^k a^k n\binom{n-1}{k}\frac{k!}{k+1}\frac{t^n}{n!}.$$

Comparing the coefficients of t^n in both sides of the equation, the result is

$$\frac{\partial\mathfrak{G}_n^{(\alpha)}(x;a,b;\lambda)}{\partial x}=\sum_{k=0}^{n-1}n(-1)^k a^k\frac{k!}{k+1}\binom{n-1}{k}\mathfrak{G}_{n-1-k}^{[m-1,\alpha]}(x;a,b;\lambda).$$

The proofs of (16) and (17) are similar to that of (18). □

Theorem 4. *Let* $\{\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)\}_{n\geq 0}$, $\{\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)\}_{n\geq 0}$ *and* $\{\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)\}_{n\geq 0}$ *be the sequence of degenerated generalized Apostol-type polynomials of order* $\alpha \in \mathbb{C}$ *and level* m *in the variable* x*, where* $a, b \in \mathbb{R}^+$*. Then, for a fixed* $m \in \mathbb{N}$,

$$\sum_{k=0}^{n} \mathfrak{B}_{n-k}^{[m-1,\alpha]}(x;a,b;\lambda)\mathfrak{B}_{k}^{[m-1,\alpha]}(x;a,b;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{B}_{k}^{[m-1,\alpha]}(2x;a,b;\lambda)\mathfrak{B}_{n-k}^{[m-1,\alpha]}(a,b;\lambda), \quad (19)$$

$$\sum_{k=0}^{n} \mathfrak{E}_{n-k}^{[m-1,\alpha]}(x;a,b;\lambda)\mathfrak{E}_{k}^{[m-1,\alpha]}(x;a,b;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{E}_{k}^{[m-1,\alpha]}(2x;a,b;\lambda)\mathfrak{E}_{n-k}^{[m-1,\alpha]}(a,b;\lambda), \quad (20)$$

$$\sum_{k=0}^{n} \mathfrak{G}_{n-k}^{[m-1,\alpha]}(x;a,b;\lambda)\mathfrak{G}_{k}^{[m-1,\alpha]}(x;a,b;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{G}_{k}^{[m-1,\alpha]}(2x;a,b;\lambda)\mathfrak{G}_{n-k}^{[m-1,\alpha]}(a,b;\lambda). \quad (21)$$

Proof. Let us prove (19). Consider the following expressions:

$$t^{m\alpha}[\sigma(\lambda;a,b;t)]^{\alpha}(1+at)^{x/a} = \sum_{n=0}^{\infty} \mathfrak{B}_{n}^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!}, \quad (22)$$

$$t^{m\alpha}[\sigma(\lambda;a,b;t)]^{\alpha}(1+at)^{x/a} = \sum_{k=0}^{\infty} \mathfrak{B}_{k}^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^k}{k!}. \quad (23)$$

From (22) and (23), we have

$$\sum_{n=0}^{\infty} \mathfrak{B}_{n}^{[m-1,\alpha]}(a,b;\lambda)\frac{t^n}{n!} \sum_{n=0}^{\infty} \mathfrak{B}_{n}^{[m-1,\alpha]}(2x;a,b;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathfrak{B}_{n}^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!} \times \sum_{k=0}^{\infty} \mathfrak{B}_{k}^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^k}{k!},$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} \mathfrak{B}_{n-k}^{[m-1,\alpha]}(a,b;\lambda)\mathfrak{B}_{n}^{[m-1,\alpha]}(2x;a,b;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} \mathfrak{B}_{n-k}^{[m-1,\alpha]}(x;a,b;\lambda) \times \mathfrak{B}_{k}^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!}.$$

Hence, we get assertion (19). The proofs of (20) and (21) are similar. □

Theorem 5. *Let* $\{\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)\}_{n\geq 0}$ *and* $\{\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)\}_{n\geq 0}$ *be the sequence of degenerated generalized Apostol-type polynomials of order* $\alpha \in \mathbb{C}$ *and level* m *in the variable* x*, where* $a, b \in \mathbb{R}^+$*. Then, for a fixed* $m \in \mathbb{N}$,

$$\mathfrak{B}_{n}^{[m-1,\alpha]}(x;a,b;-\lambda) = \frac{(-1)^{\alpha} n!}{(2)^{m\alpha}(n-m\alpha)!}\mathfrak{E}_{n-m\alpha}^{[m-1,\alpha]}(x;a,b;\lambda), \quad (24)$$

$$\mathfrak{E}_{n}^{[m-1,\alpha]}(x;a,b;-\lambda) = \frac{(-2)^{m\alpha} n!}{(n+m\alpha)!}\mathfrak{B}_{n+m\alpha}^{[m-1,\alpha]}(x;a,b;\lambda). \quad (25)$$

Proof. Let us prove (24). Considering the generating function (8)

$$t^{m\alpha}[\sigma(-\lambda;a,b;t)]^{\alpha}(1+at)^{x/a} = \sum_{n=0}^{\infty} \mathfrak{B}_{n}^{[m-1,\alpha]}(x;a,b;-\lambda)\frac{t^n}{n!}$$

or

$$\frac{(-1)^{\alpha}2^{m\alpha}}{2^{m\alpha}} t^{m\alpha}[\psi(\lambda;a,b;t)]^{\alpha}(1+at)^{x/a} = \sum_{n=0}^{\infty} \mathfrak{B}_{n}^{[m-1,\alpha]}(x;a,b;-\lambda)\frac{t^n}{n!},$$

we have
$$\sum_{n=0}^{\infty}\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;-\lambda)\frac{t^n}{n!}=\frac{(-1)^\alpha}{2^{m\alpha}}\sum_{n=0}^{\infty}\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^{n+m\alpha}}{n!}$$
or
$$\sum_{n=0}^{\infty}\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;-\lambda)\frac{t^n}{n!}=\frac{(-1)^\alpha}{2^{m\alpha}}\sum_{n=0}^{\infty}\mathfrak{E}_{n-m\alpha}^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{(n-m\alpha)!}.$$
Comparing the coefficients of t^n in both sides of the equation, the result is obtained.

The proof of (25) is similar to that of (24), considering the generating function (9). □

Theorem 6. *Let* $\{\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)\}_{n\geq 0}$, $\{\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)\}_{n\geq 0}$ *and* $\{\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)\}_{n\geq 0}$ *be the sequence of degenerated generalized Apostol-type polynomials of order* $\alpha\in\mathbb{C}$ *and level* m *in the variable* x*, where* $a,b\in\mathbb{R}^+$*. Then, for a fixed* $m\in\mathbb{N}$ *and* $n>m\alpha$*,*
$$\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;-\lambda)=(-2^m)^\alpha\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda), \tag{26}$$
$$\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)=\frac{n!}{(n-m\alpha)!}\mathfrak{E}_{n-m\alpha}^{[m-1,\alpha]}(x;a,b;\lambda). \tag{27}$$

Proof. Let us prove (26). Considering the generating function (8), we have
$$t^{m\alpha}[\sigma(\lambda;a,b;t)]^\alpha(1+at)^{x/a}=\sum_{n=0}^{\infty}\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!},$$
$$2^{m\alpha}t^{m\alpha}[\psi(-\lambda;a,b;t)]^\alpha(1+at)^{x/a}=(-2^m)^\alpha\sum_{n=0}^{\infty}\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!}. \tag{28}$$
Therefore from (10) and (28), we obtain
$$\sum_{n=0}^{\infty}\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;-\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}(-2)^{m\alpha}\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!}.$$
Comparing the coefficients of $t^n/n!$ on both sides, we get the desired result (26).

The proof of (27) is similar to that of (26), considering the generating function (10). □

3 Conclusions

The article aims to present the study of new degenerated generalized classes of Apotol-Bernouilli, Apostol-Eeuler and Apostol-Genocchi polynomials of order α and level m in the variable x, which play an important role in several diverse fields of physics, applied mathematics and engineering. Certain expressions, representations and sums of these polynomials are derived in terms of well-known classical special functions. The results we have considered in this paper indicate the usefulness of the series rearrangement technique used to deal with the theory of special functions.

References

[1] Bedoya D., Ortega M., Ramírez W., Urieles A. *New biparametric families of Apostol-Frobenius-Euler polynomials of level m.* Mat. Stud. 2021, **55** (1), 10–23. doi:10.30970/ms.55.1.10-23

[2] Carlitz L. *A degenerate Staudt-Clausen theorem.* Arch. Math. 1956, **7**, 28–33. doi:10.1007/BF01900520

[3] Cesarano C. *A note on generalized Hermite polynomial.* Int. J. Appl. Math. Inform. 2014, **8**, 1–6.

[4] Cesarano C., Ramírez W., Khan S. *A new class of degenerate Apostol-type Hermite polynomials and applications.* Dolomites Res. Notes Approx. 2022, **15** (1), 1–10. doi:10.14658/pupj-drna-2022-1-1

[5] Dattoli G., Cesarano C. *On a new family of Hermite polynomials associated to parabolic cylinder functions.* Appl. Math. Comput. 2003, **141** (1), 143–149. doi:10.1016/S0096-3003(02)00328-4

[6] Dattoli G., Sacchetti D., Cesarano C. *A note on Chebyshev polynomials.* Ann. Univ. Ferrara 2001, **7** (47), 107–115. doi:10.1007/BF02838178

[7] Khan W.A. *A note on degenerate Hermite poly-Bernoulli numbers and polynomials.* J. Class. Anal. 2016, **8** (1), 65–76. doi:10.7153/jca-08-06

[8] Kim T., Kim D.S. *Identities involving degenerate Euler numbers and polynomials arising from nonlinear differential equations.* J. Nonlinear Sci. Appl. 2016, **9**, 2086–2098.

[9] Kim T. *A note on degenerate Stirling polynomials of the second kind.* Proc. Jangjeon Math. Soc. 2017, **20** (3), 319–331. doi:10.17777/pjms2017.20.3.319

[10] Kurt B. *Some relationships between the generalized Apostol-Bernoulli and Apostol-Euler polynomials.* Turk. J. Anal. Number Theory 2013, **1** (1), 54–58.

[11] Kurt B. *A further generalization of the Bernoulli polynomials and on the 2D-Bernoulli polynomials* $B_n^2(x, y)$. App. Math. Sci. 2010, **4** (47), 2315–2322.

[12] Lim D. *Some identities of degenerate Genocchi polynomials.* Bull. Korean Math. Soc. 2016, **53** (2), 569–579. doi:10.4134/BKMS.2016.53.2.569

[13] Pathan M.A., Khan W.A. *Some new classes of generalized Hermite-based Apostol-Euler and Apostol-Genocchi polynomials.* Fasc. Math. 2015, **55**, 153–170. doi:10.1515/fascmath-2015-0020

[14] Natalini P., Bernardini A. *A generalization of the Bernoulli polynomials.* J. Appl. Math. 2003, **3**, 155–163. doi:10.1155/S1110757X03204101

[15] Rainville E.D. Special Functions. Chelsea Publishig Co., Bronx, New York, 1971.

[16] Ramírez W., Ortega M., Urieles A. *New generalized Apostol-Frobenius-Euler polynomials and their matrix approach.* Kragujev. J. Math. 2021, **45** (3), 393–407. doi:10.46793/KGJMAT2103.393O

[17] Subuhi K., Tabinda N., Mumtaz R. *On degenerate Apostol-type polynomials and applications.* Bol. Soc. Mat. Mex. 2019, **25**, 509–528. doi:10.1007/s40590-018-0220-z

[18] Srivastava H.M., Choi J. Series associated with the Zeta and related functions. Springer, Dordrecht, 2001.

[19] Tremblay R., Gaboury S., Fugére B.-J. *Some new classes of generalized Apostol-Euler and Apostol-Genocchi polynomials.* Int. J. Math. Math. Sci. 2012, **2012**, 182785. doi:10.1155/2012/182785

[20] Tremblay R., Gaboury S., Fugère B.-J. *A further generalization of Apostol-Bernoulli polynomials and related polynomials.* Honam Math. J. 2012, **34** (3), 311–326. doi:10.5831/HMJ.2012.34.3.311

[21] Urieles A., Ortega M., Ramirez W., Vega S. *New results on the q-generalized Bernoulli polynomials of level m.* Demonstr. Math. 2019, **52** (1), 511–522. doi:10.1515/dema-2019-0039

Received 30.06.2022
Revised 11.07.2022

Рамірез В., Чезарано К. *Деякі нові класи вироджених узагальнених поліномів Апостола-Бернуллі, Апостола-Ойлера та Апостола-Дженоккі* // Карпатські матем. публ. — 2022. — Т.14, №2. — С. 354–363.

Метою даної роботи є дослідження нових класів вироджених узагальнених поліномів Апостола-Бернуллі, Апостола-Ойлера та Апостола-Дженоккі порядку α та рівня m за змінною x. Тут вироджені поліноми є природним розширенням класичних поліномів. Докладніше, ми отримуємо їхні явні вирази, рекурентні співвідношення та деякі тотожності, що включають ці поліноми та числа. Більшість результатів доведено за допомогою методів твірних функцій.

Ключові слова і фрази: поліноми типу Апостола, вироджені поліноми типу Апостола.