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Regular behavior of subharmonic in space functions of the zero kind

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Let u be a subharmonic in $\mathbb{R}^m$, $m \geq 3$, function of the zero kind with Riesz measure μ on negative axis Ox_1, $n(r,u) = \mu(\{x \in \mathbb{R}^m : |x| \leq r\})$, $N(r,u) = (m-2)\int_1^r n(t,u)/t^{m-1}dt$, $\rho(r)$ is a proximate order, $\rho(r) \to \rho$ as $r \to +\infty$, $0 < \rho < 1$. We found the asymptotic of $u(x)$ as $|x| \to +\infty$ by the condition $N(r,u) = (1+o(1))\, r^{\rho(r)}$, $r \to +\infty$. We also investigated the inverse relationship between a regular growth of u and a behavior of $N(r,u)$ as $r \to +\infty$.

Key words and phrases: regular growth, subharmonic function, Riesz measure, proximate order.

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Introduction

Let f be an entire transcendent (hereinafter entire) function of noninteger order ρ such that $f(0) = 1$. Denote by $n(r) = n(r, 0, f)$ a number of zeros of f in a circle $\{z : |z| \leq r\}$. Let $\ln f(z)$ be single-valued in the angle $\{z : |\arg z| < \pi\}$ branch of the multi-valued function $\operatorname{Ln} f(z) = \ln|f(z)| + i \operatorname{Arg} f(z)$ such that $\ln f(0) = 0$. Let $\rho(r)$ be a proximate order, that is

1) $\rho(r)$ is a continuously differentiable function on $[0, +\infty)$;
2) $\rho(r) \to \rho > 0$ as $r \to +\infty$;
3) $r\rho'(r) \ln r = o(1)$, $r \to +\infty$.

If zeros of f are negative and

$$n(r) \sim \Delta r^{\rho(r)}, \quad 0 < \Delta < +\infty, \ r \to +\infty, \tag{1}$$

then, as follows from the results of G. Valiron [15], we have

$$\ln\left|f\left(re^{i\theta}\right)\right| \sim \frac{\pi\Delta\cos\rho\theta}{\sin\pi\rho} r^{\rho(r)}, \quad |\theta| < \pi, \ r \to +\infty.$$

Conversely, if f has only negative zeros and

$$\ln|f(r)| \sim \frac{\pi\Delta}{\sin\pi\rho} r^{\rho(r)}, \quad r \to +\infty,$$

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then the relation (1) holds.

E. Titchmarsh proved the last proposition simpler in [13]. Because of that, the propositions on the connection between regular growth of $n(r)$ and $\ln\left|f(z)\right|$ are called Valiron-type and Valiron-Titchmarsh-type theorems.

Such theorems are proved in [9, 10] in terms of the two-term asymptotic, that is the case when

$$n(r) = \Delta r^{\rho(r)} + \Delta_1 r^{\rho_1(r)} + o\left(r^{\rho_1(r)}\right), \quad r \to +\infty,$$

and respectively

$$\ln f\left(re^{i\theta}\right) = \frac{\pi \Delta e^{i\rho\theta}}{\sin \pi\rho} r^{\rho(r)} + \frac{\pi \Delta_1 e^{i\rho_1\theta}}{\sin \pi\rho_1} r^{\rho_1(r)} + o\left(r^{\rho_1(r)}\right), \quad r \to +\infty,$$

where $\Delta_1 \in \mathbb{R}$, $\rho_1(r)$ is a proximate order, $\rho_1(r) \to \rho_1$ as $r \to +\infty$, $[\rho] < \rho_1 < \rho$.

The similar problems for entire functions of order zero were considered in [16]. In [1, 5, 8], relationships between regular growth of the counting function of the Riesz measure and behavior of a subharmonic in $\mathbb{R}^m$, $m \geq 3$, function were investigated. In particular, for a function $u\left(r\cos\theta, x_2, \ldots, x_m\right)$ of noninteger order ρ with Riesz measure on negative axis Ox_1 was found the representation of indicator $H(\theta; u)$ based on Gegenbauer polynomials $G_j^{(m-2)/2}(\cos\theta)$ given by the generating function $\left(1 - 2t\cos\theta + t^2\right)^{-(m-2)/2}$ (see [5]) or by the associated Legendre spherical functions of the first kind $P^{(3-m)/2}_{\rho+(m-3)/2}(\cos\theta)$ (see [8]).

In the paper, for such a function $u(r\cos\theta, x_2, \ldots, x_m)$ of zero kind the representation $H(\theta; u)$ is established using the Gauss hypergeometric function ${}_2F_1(a, b; c, z)$ (Theorem 2) and a new proof of Valiron-Titchmarsh-type theorem is given (Theorem 3).

1 Definitions and main results

Let u be a subharmonic function in $\mathbb{R}^m$, $m \geq 3$, and u is harmonic in a unit neighborhood of the point 0, $u(0) = 0$, μ be its Riesz measure. Denote $n(t) = n(t, u) = \mu\left(\{x \colon |x| \leq t\}\right)$, $d_m = m - 2$, $N(t) = N(t, u) = d_m \int_1^t n(\tau)\tau^{1-m} d\tau$, $u^+(x) = \max\left\{u(x); 0\right\}$.

Let $c_m = 2\pi^{m/2}/\Gamma(m/2)$ be a surface area of the unit sphere $\left\{x \in \mathbb{R}^m \colon |x| = 1\right\}$. Denote by $d\sigma(x)$ an element of the surface area of the sphere $S(0, r) = \left\{x \colon |x| = r\right\}$. Let

$$T(r, u) = \frac{1}{c_m r^{m-1}} \int_{S(0,r)} u^+(x) d\sigma(x)$$

be the Nevanlinna characteristic of the function u. Numbers

$$\rho[u] = \varlimsup_{r \to +\infty} \frac{\ln T(r, u)}{\ln r}, \quad \rho[N] = \varlimsup_{r \to +\infty} \frac{\ln N(r, u)}{\ln r}$$

are called orders of functions u and $N(r, u)$, respectively. We will say that u is of a zero kind if $T(r, u) = o(r)$ as $r \to +\infty$.

We denote by $SH_m(\rho)$ a class of subharmonic in $\mathbb{R}^m$, $m \geq 2$, functions of order ρ, by $SH_m^-(\rho)$ a subclass of functions u from $SH_m(\rho)$ such that u is harmonic outside the negative axis Ox_1.

Let $V(r) = r^{\rho(r)}$ and

$$\mathcal{P}(t,r,\theta;\alpha) = \left(t^2 + 2tr\cos\theta + r^2\right)^{-\alpha}, \quad \alpha > 0, \tag{2}$$

$$\mathcal{J}\big(r,\theta;\alpha,\psi(t)\big) = \int_0^{+\infty} \psi(t)\mathcal{P}(t,r,\theta;\alpha)dt, \tag{3}$$

$$\mathcal{J}\left(r,\theta;\alpha,\beta\right) = \mathcal{J}\left(r,\theta;\alpha,t^{\beta}\right), \quad 0 < \beta < 2\alpha - 1,$$

where ψ is a locally integrated on $[0,+\infty)$ function such that the integral in (3) is convergent.

Theorem 1. *Let u be a subharmonic in $\mathbb{R}^m$, $m \geq 3$, function of zero kind such that it is harmonic outside the negative axis Ox_1, $r = |x|$, $x = (r\cos\theta, x_2, \ldots, x_m)$, $|\theta| < \pi$. Then*

$$u(x) = mr^2\sin^2\theta \cdot \mathcal{J}\left(1,\theta;(m+2)/2, t^{m-1}N(rt)\right) + (m-1)r\cos\theta \cdot \mathcal{J}\left(1,\theta;m/2, t^{m-2}N(rt)\right).$$

Theorem 2. *Let $0 < \rho < 1$, $m \geq 3$, $u \in SH_m^-(\rho)$, $r = |x|$, $x = (r\cos\theta, x_2, \ldots, x_m)$, $0 < \delta < 1$. If*

$$N(t) = \big(1 + o(1)\big)V(t), \quad t \to +\infty, \tag{4}$$

then for $|\theta| < \pi$ there exists the limit

$$\lim_{r\to+\infty} u(x)/V(r) = H(\theta;u), \tag{5}$$

moreover equality (5) holds uniformly relative to θ on the set $\{\theta\colon |\theta| < \pi - \delta\}$, where

$$H(\theta;u) = m\sin^2\theta \cdot \mathcal{J}\big(1,\theta;(m+2)/2, m-1+\rho\big) + (m-1)\cos\theta \cdot \mathcal{J}\left(1,\theta;m/2, m-2+\rho\right). \tag{6}$$

Remark 1. *Integrals in (6) are convergent, since for subintegral expressions (see (2)) we have*

$$t^{m-1+\rho}\mathcal{P}\left(t,1,\theta;\frac{m+2}{2}\right) \sim \frac{1}{t^{3-\rho}}, \quad t^{m-2+\rho}\mathcal{P}\left(t,1,\theta;\frac{m}{2}\right) \sim \frac{1}{t^{2-\rho}} \quad \text{as } t \to +\infty.$$

Remark 2. *Owing to properties of the beta-function, we have*

$$\begin{aligned}\mathcal{J}\left(1,0;\frac{m}{2}, m-2+\rho\right) &= \int_0^{+\infty} \frac{t^{m-2+\rho}}{(t+1)^m}dt = B(m-1+\rho, 1-\rho)\\ &= \frac{\Gamma(m-1+\rho)\Gamma(1-\rho)}{\Gamma(m)} = \frac{\pi\rho(\rho+1)\cdot\ldots\cdot(m-2+\rho)}{(m-1)!\sin\pi\rho}.\end{aligned}$$

Remark 3. *Denote $n_1(t) = n_1(t,u) = \frac{d_m}{\rho}t^{2-m}n(t)$. It turns out that (4) in the case $\rho > 0$ is equivalent to*

$$n_1(t) = \big(1 + o(1)\big)V(t), \quad t \to +\infty. \tag{7}$$

In the case $m = 2$ it was independently proved by A. Baernstein [2] and A. Goldberg [4]. In the case $m \geq 3$ equivalence of (4) and (7) can be easily proved using methods of the paper [4].

Remark 4. *Under condition* (7) *with* $V(t) = t^\rho$ *for functions* u *of noninteger order, which satisfy condition* (5)*, in [5] and [8] it was proved, respectively,*

$$H(\theta; u) = (\rho + m - 2) \int_0^\infty s^{-\rho-1} h_m\big(s, \theta, [\rho]\big) ds,$$

where $h_m(s, \theta, [\rho]) = -\big(1 + 2s\cos\theta + s^2\big)^{(2-m)/2} + \sum_{j=0}^{[\rho]} (-s)^j G_j^{(m-2)/2}(\cos\theta)$*, and* $G_j^{(m-2)/2}(\cos\theta)$ *are the Gegenbauer polynomials given by the generating function* $\big(1 - 2t\cos\theta + t^2\big)^{(2-m)/2}$*;*

$$H(\theta; u) = \frac{\pi(\rho+1)\cdot\ldots\cdot(\rho+m-2)\Gamma\left(\frac{m-1}{2}\right)2^{(m-3)/2}}{(m-3)!\sin\pi\rho\sin^{(m-3)/2}\theta} P_{\rho+(m-3)/2}^{(3-m)/2}(\cos\theta),$$

where $P_\nu^\mu(\cos\theta)$ *are the associated Legendre spherical functions of the first kind.*

Remark 5. *Integrals in* (6) *can be rewritten as (see [11, p. 311; p. 788])*

$$\begin{aligned}\mathcal{J}(1, \theta; p, \alpha) &= \int_0^{+\infty} \frac{t^{\alpha-1} dt}{(t^2 + 2t\cos\theta + 1)^p} \\ &= \frac{1}{2} B\left(\frac{\alpha}{2}, p - \frac{\alpha}{2}\right) {}_2F_1\left(\frac{\alpha}{2}, p - \frac{\alpha}{2}; \frac{1}{2}, \cos^2\theta\right) \\ &\quad - \frac{1}{2} B\left(\frac{\alpha-1}{2}, p - \frac{\alpha-1}{2}\right) {}_2F_1\left(\frac{\alpha-1}{2}, p - \frac{\alpha-1}{2}; \frac{3}{2}, \cos^2\theta\right),\end{aligned}$$

where $0 < \alpha < 2p$*,* $0 < |\theta| < \pi$*,* ${}_2F_1(a, b; c, z)$ *is Gauss hypergeometric function.*

Theorem 3. *Let* u*,* ρ*,* m *be such as in Theorem 2,* $u(r) = u(r, 0, \ldots, 0)$*. If*

$$u(r) = \frac{\pi\rho(\rho+1)\cdot\ldots\cdot(\rho+m-2)}{(m-2)!\sin\pi\rho} V(r) + o\big(V(r)\big), \quad r \to +\infty, \tag{8}$$

then $N(t)$ *satisfies* (4).

For completeness let us formulate analogues of Valiron and Valiron-Titchmarsh theorems for the set $SH_2^-(\rho)$.

Proposition 1. *Let* $0 < \rho < 1$*,* $u \in SH_2^-(\rho)$*,* $x = (r\cos\theta, r\sin\theta)$*,* $r = |x|$*,* $u(r) = u(r, 0)$*.*

(A) If $n(t, u) = \big(1 + o(1)\big)V(t)$*,* $t \to +\infty$*, then for* $|\theta| < \pi$ *we have*

$$u(x) = \frac{\pi\cos\rho\theta}{\sin\pi\rho} V(r) + o\big(V(r)\big), \quad r \to +\infty,$$

moreover last equality holds uniformly relative to θ *on the set* $\{\theta\colon |\theta| < \pi - \delta\}$*.*

(B) If

$$u(r) = \frac{\pi}{\sin\pi\rho} V(r) + o\big(V(r)\big), \quad r \to +\infty,$$

then $n(t, u) = \big(1 + o(1)\big)V(t)$*,* $t \to +\infty$*.*

2 Additional results

Positive continuous on $[0, +\infty)$ function l is said to be slowly varying if for any $\lambda > 0$ asymptotical equality $l(\lambda x) \sim l(x)$, $x \to +\infty$, holds.

We will use next propositions to prove theorems.

Lemma 1 ([12, p. 63–64]). *Let l be a slowly varying function, φ be a locally integrated on $[0, +\infty)$ function and for some $\eta > 0$ the integral $\int_0^{+\infty} t^\eta \varphi(t)dt$ is convergent. Then*

$$\int_0^{+\infty} \varphi(t) l(xt) dt \sim l(x) \int_0^{+\infty} \varphi(t)dt, \quad x \to +\infty.$$

Lemma 2. *Let g be differentiated, g' be nonincreasing on $[0, +\infty)$ functions, $V(r) = r^{\rho(r)}$, $\rho > 0$, $0 < \Delta < +\infty$ and $g(r) = \Delta V(r) + o\big(V(r)\big)$, $r \to +\infty$. Then*

$$g'(r) = \Delta\rho V(r)/r + o\big(V(r)/r\big), \quad r \to +\infty.$$

The proof of Lemma 2 is similar to the proof of lemma from [14].

Lemma 3. *Let $V(r) = r^{\rho(r)}$, $\rho > 0$, $\varepsilon(r)$ be a locally integrated on $[1, +\infty)$ function such that $\varepsilon(r) \to 0$ as $r \to +\infty$. Then:*

1) for $\alpha < \rho + 1$,

$$\int_1^r \frac{\varepsilon(t)V(t)}{t^\alpha} dt = o\left(\frac{V(r)}{r^{\alpha-1}}\right), \quad r \to +\infty;$$

2) for $\alpha > \rho + 1$,

$$\int_r^{+\infty} \frac{\varepsilon(t)V(t)}{t^\alpha} dt = o\left(\frac{V(r)}{r^{\alpha-1}}\right), \quad r \to +\infty.$$

It is easy to see that this lemma is true if you use the L'opital rule.

Lemma 4 ([7]). *Let $0 < a < b < a+1$, $m \geq 1$, α and β are positive, nondecreasing on $[0, +\infty)$ functions, α is differentiated, $\alpha(x) \to +\infty$ as $x \to +\infty$, $a\alpha(x) < x\alpha'(x) < b\alpha(x)$ for $x \geq x_0 > 0$,*

$$F(x) = \int_0^{+\infty} \frac{t^{m-1} d\alpha(t)}{(t+x)^{m+k}}, \quad G(x) = \int_0^{+\infty} \frac{t^{m-1} d\beta(t)}{(t+x)^{m+k}}, \quad k = [b].$$

If $F(x) \sim G(x)$, then $\alpha(x) \sim \beta(x)$ as $x \to +\infty$.

Lemma 5. *Let $m \geq 1$, $V(t) = t^{\rho(t)}$, $0 < \rho < 1$. Then*

$$I(x) = \int_0^{+\infty} \frac{t^{m-1} dV(t)}{(t+x)^m} = \big(1 + o(1)\big)\rho B(m + \rho - 1, 1 - \rho) V(x)/x, \quad x \to +\infty.$$

Proof. By integrating by parts and replacing $t = \tau r$, we obtain

$$I(x) = \int_0^{+\infty} \frac{\big(t - (m-1)x\big)t^{m-2}}{(t+x)^{m+1}} V(t) dt = \frac{1}{x} \int_0^{+\infty} \frac{\big(\tau - (m-1)\big)\tau^{m-2}}{(\tau+1)^{m+1}} V(\tau x) d\tau.$$

Since (see, e.g., [3, p. 73]) $V(t) = t^{\rho} l(t)$, where $l(t) = t^{\rho(t)-\rho}$ is a slowly varying function on $[0, +\infty)$, then owing to Lemma 1 with $\eta = \dfrac{1-\rho}{2} > 0$ we get

$$\begin{aligned}
I(x) &= x^{\rho-1} \int_0^{\infty} l(tx) \frac{t^{m+\rho-2}\big(t-(m-1)\big)dt}{(t+1)^{m+1}} \\
&= \big(1+o(1)\big)\frac{r^{\rho} l(x)}{x} \int_0^{\infty} \left(\frac{t^{m+\rho-1}}{(t+1)^{m+1}} - (m-1)\frac{t^{m+\rho-2}}{(t+1)^{m+1}} \right) dt \\
&= \big(1+o(1)\big)\frac{V(x)}{x}\big(B(m+\rho, 1-\rho) - (m-1)B(m+\rho-1, 2-\rho)\big) \\
&= \big(1+o(1)\big)\rho B(m+\rho-1, 1-\rho)\frac{V(x)}{x}, \quad x \to +\infty.
\end{aligned}$$

This completes the proof. □

3 Proof of results

Proof of Theorem 1. Let u be a subharmonic in $\mathbb{R}^m$, $m \geq 3$, function of the zero kind. Then (see, e.g., [6, p.174])

$$u(x) = u\left(x_1, x_2, \ldots, x_m\right) = \int_{|\xi|<+\infty} K_0(x,\xi) d\mu_{\xi}, \quad \xi \in \mathbb{R}^m,$$

where μ is a Riesz measure of u, $K_0(x,\xi) = |\xi|^{2-m} - |x-\xi|^{2-m}$.

If $u \in SH_m^-(\rho)$, $|\xi| = t$, $|x| = r$, $x_1 = r\cos\theta$, $|\theta| < \pi$, $\bar{j} = (1, 0, \ldots, 0)$ is m-dimensional vector, then

$$\begin{aligned}
u(x) = u\left(r\cos\theta, x_2, \ldots, x_m\right) &= \int_0^{+\infty} \left(t^{2-m} - \left|x + \bar{j}t\right|^{2-m} \right) dn(t) \\
&= \int_0^{+\infty} \left(t^{2-m} - \left(t^2 + 2tr\cos\theta + r^2\right)^{(2-m)/2} \right) dn(t) \\
&= n(t) \left(t^{2-m} - \left(t^2 + 2tr\cos\theta + r^2\right)^{(2-m)/2} \right) \Bigg|_0^{+\infty} \\
&\quad + (m-2) \int_0^{+\infty} \left(t^{1-m} - \frac{t + r\cos\theta}{\left(t^2 + 2tr\cos\theta + r^2\right)^{m/2}} \right) n(t) dt \\
&= \int_0^{+\infty} \left(1 - \frac{t^{m-1}(t + r\cos\theta)}{\left(t^2 + 2tr\cos\theta + r^2\right)^{m/2}} \right) dN(t),
\end{aligned} \tag{9}$$

since $n(0) = 0$, $n(t) = \dfrac{t^{m-1}}{m-2}\dfrac{d}{dt}N(t)$,

$$\frac{\left(t^2 + 2tr\cos\theta + r^2\right)^{(m-2)/2} - t^{m-2}}{t^{m-2}\left(t^2 + 2tr\cos\theta + r^2\right)^{(m-2)/2}} = \big(1+o(1)\big)\frac{(m-2)r\cos\theta}{t^{m-1}},$$

and $n(t) = o\left(t^{m-1}\right)$, $t \to +\infty$.

Moreover, integrating by parts the integral (9), we obtain (see (2), (3))

$$\begin{aligned} u(x) &= \left(1 - \frac{t^{m-1}(t + r\cos\theta)}{(t^2 + 2tr\cos\theta + r^2)^{m/2}}\right) N(t)\Big|_0^{+\infty} \\ &\quad + mr^2 \sin^2\theta \int_0^{+\infty} \frac{t^{m-1}N(t)dt}{(t^2 + 2tr\cos\theta + r^2)^{(m+2)/2}} \\ &\quad + (m-1)r\cos\theta \int_0^{+\infty} \frac{t^{m-2}N(t)dt}{(t^2 + 2tr\cos\theta + r^2)^{m/2}} \\ &= mr^2 \sin^2\theta \mathcal{J}\left(r,\theta; \frac{m+2}{2}, t^{m-1}N(t)\right) + (m-1)r\cos\theta \mathcal{J}\left(r,\theta; \frac{m}{2}, t^{m-2}N(t)\right), \end{aligned} \tag{10}$$

because $N(0) = 0$, $1 - \dfrac{t^{m-1}(t + r\cos\theta)}{(t^2 + 2tr\cos\theta + r^2)^{m/2}} = (1 + o(1))\dfrac{(m-1)r\cos\theta}{t}$, $N(t) = o(t)$ as $t \to +\infty$. Substituting $t = \tau r$ into integrals $\mathcal{J}\left(r,\theta; \frac{m+2}{2}, t^{m-1}N(t)\right)$ and $\mathcal{J}\left(r,\theta; \frac{m}{2}, t^{m-2}N(t)\right)$, from (10) we obtain

$$u(x) = m\sin^2\theta \mathcal{J}\left(1,\theta; \frac{m+2}{2}, t^{m-1}N(rt)\right) + (m-1)\cos\theta \mathcal{J}\left(1,\theta; \frac{m}{2}, t^{m-2}N(rt)\right). \tag{11}$$

□

Remark 6. *At* $\theta = 0$ *from* (9) *and* (10) *we get*

$$u(r) = u(r,0,\dots,0) = \int_0^{+\infty} \left(1 - \frac{t^{m-1}}{(t+r)^{m-1}}\right) dN(t) = (m-1)r \int_0^{+\infty} \frac{t^{m-2}N(t)}{(t+r)^m} dt. \tag{12}$$

Remark 7. *In the case of* $m = 2$*, for convenience, instead of* $x = (x_1, x_2)$ *we write* $z = re^{i\theta}$*,* $r = |z|$*,* $-\pi \le \theta < \pi$*. If now* $u \in SH_2^-(\rho)$*,* $0 < \rho < 1$*, then*

$$u(z) = \int_0^{+\infty} \ln\left|1 + \frac{z}{t}\right| dn(t) = Re \int_0^{+\infty} \ln\left(1 + \frac{z}{t}\right) dn(t). \tag{13}$$

Proof of Theorem 2. Let a subharmonic function u satisfies the conditions of Theorem 2. Then by (10) and (11) we obtain

$$\begin{aligned} u(x) = m\sin^2\theta &\Bigg(\mathcal{J}\left(1,\theta; \frac{m+2}{2}, t^{m-1}V(tr)\right) + r^2 \mathcal{J}\left(r,\theta; \frac{m+2}{2}, t^{m-1}\varepsilon(t)V(t)\right) \Bigg) \\ &+ (m-1)\cos\theta \Bigg(\mathcal{J}\left(1,\theta; \frac{m}{2}, t^{m-2}V(tr)\right) + r\mathcal{J}\left(r,\theta; \frac{m}{2}, t^{m-2}\varepsilon(t)V(t)\right) \Bigg), \end{aligned} \tag{14}$$

where $\varepsilon(t) = \big(N(t) - V(t)\big)/V(t) \to 0$ as $t \to +\infty$. By [3, p. 73] the function $V(t) = t^\rho l(t)$, with $l(t) = t^{\rho(t)-\rho}$, is slowly varying on $[0, +\infty)$, therefore by Lemma 1 with $\eta = (1-\rho)/2 > 0$ we get

$$\begin{aligned} \mathcal{J}\left(1,\theta; \frac{m+2}{2}, t^{m-1}V(tr)\right) &= r^\rho \int_0^{+\infty} l(tr)t^{m+\rho-1}\mathcal{P}\left(t,1,\theta; \frac{m+2}{2}\right) dt \\ &= \big(1 + o(1)\big)V(r)\mathcal{J}\left(1,\theta; \frac{m+2}{2}, m+\rho-1\right), \quad r \to +\infty. \end{aligned} \tag{15}$$

Similarly,

$$\mathcal{J}\left(1,\theta;\frac{m}{2},t^{m-2}V(tr)\right)=\big(1+o(1)\big)V(r)\mathcal{J}\left(1,\theta;\frac{m}{2},m+\rho-2\right),\quad r\to+\infty. \tag{16}$$

Considering (see [3, p. 92]) $\left|t^2+2tr\cos\theta+r^2\right|=\left|t+re^{i\theta}\right|^2\geq(t+r)^2\sin^2\delta$ for $|\theta|\leq\pi-\delta$, $\delta>0$ by Lemma 3 and by estimate

$$r^2\sin^{m+2}\delta\left|\mathcal{J}\left(r,\theta;\frac{m+2}{2},t^{m-1}\varepsilon(t)V(t)\right)\right|\leq\frac{1}{r^m}\int_1^r\frac{\varepsilon(t)V(t)}{t^{1-m}}dt+r^2\int_r^{+\infty}\frac{\varepsilon(t)V(t)}{t^3}dt,$$

we obtain $mr^2\sin\theta\mathcal{J}\left(r,\theta;\frac{m+2}{2},t^{m-1}\varepsilon(t)V(t)\right)=o\big(V(r)\big)$ as $r\to+\infty$. Similarly, it can be shown that $(m-1)r\cos\theta\mathcal{J}\left(r,\theta;\frac{m}{2},t^{m-2}\varepsilon(t)V(t)\right)=o\big(V(r)\big)$, $r\to+\infty$. From the above and from (14)–(16) it follows that the relation (5) is fulfilled. □

Proof of Theorem 3. From (12) we have

$$u'(r)=(m-1)\int_0^{+\infty}\frac{t^{m-1}}{(t+r)^m}dN(t),$$

thus, u' is a nonincreasing function on $[0,+\infty)$. On the other hand, owing to Lemmas 2 and 5 by (8) (see Remark 2) we get

$$\begin{aligned}u'(r)&=(m-1)\rho B(m+\rho-1,1-\rho)\frac{V(r)}{r}+o\left(\frac{V(r)}{r}\right)\\&=\big(1+o(1)\big)(m-1)\int_0^{+\infty}\frac{t^{m-1}dV(t)}{(t+r)^m},\quad r\to+\infty.\end{aligned}$$

Taking into account that

$$\frac{\rho}{2}V(r)<rV'(r)=\big(\rho(r)+r\rho'(r)\ln r\big)V(r)<\frac{1+\rho}{2}V(r),\quad r\geq r_0,$$

owing to Lemma 4 with $k=\left[\frac{1+\rho}{2}\right]=0$, we obtain $N(t)=\big(1+o(1)\big)V(t)$ as $t\to+\infty$. □

Remark 8. *It is well known the asymptotics of the integral* $\int_0^{+\infty}\ln\left(1+\frac{z}{t}\right)dn(t)$*, when* $n(t)\sim V(t)$*,* $t\to+\infty$*. For* $-\pi<\theta<\pi$ *we have (see, e.g., [3, p. 92–94])*

$$\int_0^{+\infty}\ln\left(1+\frac{z}{t}\right)dV(t)=\frac{\pi}{\sin\pi\rho}e^{i\rho\theta}V(r)+o\big(V(r)\big),\quad r\to+\infty.$$

From the above and from (13) *statement (A) of Proposition 1 follows. The proof of statement (B) of Proposition 1 is similar to the proof of Theorem 3.*

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Нехай u — субгармонійна в $\mathbb{R}^m$, $m \geq 3$, функція нульового роду з мірою Рісса μ на від'ємній півосі Ox_1, $n(r,u) = \mu(\{x \in \mathbb{R}^m : |x| \leq r\})$, $N(r,u) = (m-2)\int_1^r n(t,u)/t^{m-1}dt$, $\rho(r)$ — уточнений порядок, $\rho(r) \to \rho$ при $r \to +\infty$, $0 < \rho < 1$. За умови $N(r,u) = (1+o(1))\, r^{\rho(r)}$, $r \to +\infty$, в цій статті знайдено асимптотику $u(x)$ при $|x| \to +\infty$. Досліджено також обернений зв'язок між регулярним зростанням u та поводженням $N(r,u)$ при $r \to +\infty$.

Ключові слова і фрази: регулярний ріст, субгармонійна функція, міра Рісса, уточнений порядок.