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Analytical investigation of a boundary value problem for functional differential inclusion

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This paper investigates the existence of mild solutions for an initial value problem involving fractional-order differential inclusions with nonlocal boundary conditions, specifically infinite-point or Riemann-Stieltjes integral conditions. We establish sufficient conditions for the uniqueness of the solution and examine its continuous dependence on the given data. To demonstrate the practical applicability of our findings, we conclude with two illustrative examples.

Key words and phrases: functional integro-differential inclusion, fixed point theorem, Riemann-Stieltjes integral boundary condition, infinite-point boundary condition.

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1 Introduction

Differential and integral equation models have been instrumental in modeling various phenomena across physical sciences and applied mathematics, particularly in boundary value problems (BVPs) involving fractional differential equations (see [1, 12]). Fractional differential equations have gained significant attention in recent years due to their ability to model memory and hereditary properties in diverse systems, which classical integer-order differential equations cannot effectively describe [14, 15].

Traditionally, boundary conditions in such problems are imposed locally. However, nonlocal conditions are increasingly being utilized because of their precision and applicability in real-world problems. Nonlocal conditions often provide more accurate measurements compared to local ones, as they incorporate global information about the state of the system across the domain rather than at discrete points [2, 5]. This shift has led to extensive research on fractional boundary value problems with nonlocal and resonant boundary conditions.

One key area of interest is the study of fractional BVPs at resonance, where fruitful results have emerged [4, 6]. In [7], it is examined a class of fractional differential equations with m-point boundary conditions, and similar work was extended in [8], where fractional-order three-point BVPs in resonant cases were investigated. Despite these advances, there remains a scarcity of studies concerning fractional boundary value problems involving Riemann-Stieltjes integrals, which are important for modeling more complex, real-world systems, where integration with respect to non-differentiable functions is required. H.M. Srivastava et. al. [16] explored the Riemann-Stieltjes integral in the context of multi-point boundary conditions, further demonstrating its versatility in mathematical modeling.

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Additionally, some authors have examined boundary value problems involving nonlocal, integral, and infinite-point boundary conditions, which expand the scope of applicability for fractional differential equations [13, 17]. For example, H.M. Srivastava et. al. [16] addressed nonlinear BVPs for arbitrary fractional-order differential equations with Riemann-Stieltjes functional integral and infinite-point boundary conditions, providing a robust framework for solving such problems.

In this paper, we extend these developments by studying the existence of mild solutions for a functional integro-differential inclusion of the form

$$ {}^{c}D^{\eta}\mu(\mathfrak{r}) \in \Phi_1\big(\mathfrak{r}, I^{\sigma}\phi_2(\mathfrak{r}, \mu(\varphi(\mathfrak{r})))\big), \quad \eta \in (0,1),\ \mathfrak{r} \in (0,T], \tag{1} $$

equipped with Riemann-Stieltjes integro boundary conditions

$$ \mu(0) + \int_0^T \mu(\varsigma)dh(\varsigma) = \mu_\circ, \quad h : [0,T] \to \mathbb{R} \text{ is a nondecreasing function,} \tag{2} $$

or infinite-point boundary conditions given by

$$ \mu(0) + \sum_{k=1}^{\infty} a_k \mu(\tau_k) = \mu_\circ, \quad a_k > 0,\ \tau_k \in (0,T]. \tag{3} $$

Here, ${}^{c}D^{\sigma}$ is the Caputo fractional derivative, and $\Phi_1 : [0,T] \times \mathbb{R}^+ \to P(\mathbb{R})$ is a set-valued mapping, where $P(\mathbb{R})$ denotes the family of nonempty subsets of $\mathbb{R}$. The function $h(\varsigma)$ plays a critical role in defining the nonlocal boundary conditions via the Riemann-Stieltjes integral.

Our approach focuses on reformulating the functional integro-differential inclusion into a coupled system under several appropriate assumptions for the set-valued function Φ_1. We will first address the continuous solution of problem (1) with the m-point boundary conditions

$$ \mu(0) + \sum_{k=1}^{m} a_k \mu(\tau_k) = \mu_\circ, \quad a_k > 0,\ \tau_k \in [0,1]. \tag{4} $$

We then explore the solutions for BVPs involving the Riemann-Stieltjes integral given by (1) and (2), as well as BVPs with infinite-point boundary conditions provided by (1) and (3). Using Schauder's fixed point theorem, we establish sufficient conditions for the existence of at least one continuous solution. Notably, our work extends previous results by applying fixed point theorems in the context of nonlinear fractional differential equations, which have been widely used in the literature for proving existence and uniqueness of solutions (see [3, 9–11]).

The structure of this paper is as follows. In Section 2, we present our main findings for the problem (1)–(4), followed by an investigation of BVPs given by (1)–(2) and (1)–(3). We demonstrate sufficient conditions for the existence of solutions under both Riemann-Stieltjes functional integral boundary conditions and infinite-point boundary conditions. In Section 3, we discuss the continuous dependence and uniqueness of solutions. Several examples are provided in Section 4 to illustrate the application of our findings. Finally, Section 5 concludes the paper with a summary of our contributions and potential directions for future work.

2 Existence of a continuous solution to (1) using the m-point boundary conditions (4)

Take into account the following assumptions.

(i) The set-valued map $\Phi_1 : I \times \mathbb{R} \to 2^{\mathbb{R}}$ is continuous and Lipschitzian set-valued map with a nonempty compact convex subset of $2^{\mathbb{R}^+}$, with a Lipschitz constant $k > 0$, such that

$$H_d\big(\Phi_1(\mathfrak{r}, \mu) - \Phi_1(\mathfrak{r}, \nu)\big) \le k|\mu - \nu|.$$

Note. The set of Lipschitz selections for Φ_1 is not empty and by [1, Section 9, Chapter 1, Theorem 1] there exists $\phi_1 \in \Phi_1$ such that

$$|\phi_1(\mathfrak{r}, \mu) - \phi_1(\mathfrak{r}, \nu)| \le k\,|\mu - \nu|.$$

(ii) The function $\varphi : I \to I$ is continuous.

(iii) The Caratheodory requirement is satisfied for function $\phi_2 : I \times \mathbb{R} \to \mathbb{R}$, i.e. ϕ_2 is measurable in t for any $\mu \in \mathbb{R}$ and continuous in μ for almost all $t \in I$. There exists a function $a(t)$ that is measurable, bounded and there is a positive constant $b > 0$ such that

$$|\phi_2(\mathfrak{r}, \mu)| \le a(\mathfrak{r}) + b|\mu| \quad \forall\, \mathfrak{r} \in I \ \text{ and } \ x \in \mathbb{R}.$$

(iv) The following statement is true

$$\Big[a\Big(|\mu_\circ| + \sum_{k=1}^{m} |a_k|\Big) + 1\Big]kT < 1, \qquad \frac{bT^\sigma}{\Gamma(\sigma+1)} < 1, \quad \text{and} \quad I_c^\gamma a(\cdot) \le M \quad \forall\, \gamma \le \sigma,\ \ c \ge 0.$$

Note. Using hypothesis (i), we can infer that the set S_{Φ_1} is nonempty and that there is a Lipschitzian function $\phi_1 \in \Phi_1$ such that $|\phi_1(\mathfrak{r}, \mu) - \phi_1(\mathfrak{r}, \nu)| \le k\,|\mu - \nu|$, which implies

$$D^\eta \mu(\mathfrak{r}) = \phi_1\big(\mathfrak{r}, I^\sigma \phi_2\big(\mathfrak{r}, \mu(\varphi(\mathfrak{r}))\big)\big), \quad \sigma \in (0,1),\ \mathfrak{r} \in I. \tag{5}$$

Then the solution of the single-valued problem (5)–(4) is a solution of the inclusion problem (1)–(4).

Lemma 1. *The integral representation of the mild solution of the single-valued problem* (5) *with the non-local condition* (4) *is given by*

$$\begin{aligned} \mu(\mathfrak{r}) =& \frac{1}{1 + \sum_{k=1}^{m} a_k}\Big(\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2\big(\varsigma, \mu(\varphi(\varsigma))\big)\big)\, d\varsigma\Big) \\ &+ \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2\big(\varsigma, \mu(\varphi(\varsigma))\big)\big)\, d\varsigma. \end{aligned} \tag{6}$$

Proof. We begin by considering the problem (5) with m-point boundary conditions (4). Integrating both sides of (5), we obtain

$$\mu(\mathfrak{r}) = \mu(0) + I^\eta \phi_1\big(\mathfrak{r}, I^\sigma \phi_2\big(\mathfrak{r}, \mu(\varphi(\mathfrak{r}))\big)\big).$$

When we substitute the value of $\mu(0)$ from (4), we get

$$\mu(\mathfrak{r}) = \mu_\circ - \sum_{k=1}^{m} a_k \mu(\tau_k) + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2\big(\varsigma, \mu(\varphi(\varsigma))\big)\big)\, d\varsigma. \tag{7}$$

In fact, when we set $t = \tau_k \in [0, T]$ in (7), we obtain

$$\mu(\tau_k) = \mu_\circ - \sum_{k=1}^{m} a_k \mu(\tau_k) + \int_0^{\tau_k} \frac{(\tau_k - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2\big(\varsigma, \mu(\varphi(\varsigma))\big)\big)\, d\varsigma. \tag{8}$$

So, we have

$$\begin{aligned} \mu(\tau_k) = \mu(\mathfrak{r}) &+ \int_0^{\tau_k} \frac{(\tau_k - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2\big(\varsigma, \mu(\varphi(\varsigma))\big)\big)\, d\varsigma \\ &- \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2\big(\varsigma, \mu(\varphi(\varsigma))\big)\big)\, d\varsigma. \end{aligned} \tag{9}$$

Substituting (9) in (7), we get

$$\begin{aligned} \mu(\mathfrak{r}) = \mu_\circ - \sum_{k=1}^{m} a_k \Bigg(\mu(\mathfrak{r}) &+ \int_0^{\tau_k} \frac{(\tau_k - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2\big(\varsigma, \mu(\varphi(\varsigma))\big)\big)\, ds \\ &- \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2\big(\varsigma, \mu(\varphi(\varsigma))\big)\big)\, ds \Bigg) \\ + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\eta-1}}{\Gamma(\eta)} &\phi_1\big(\varsigma, I^\sigma \phi_2\big(\varsigma, \mu(\varphi(\varsigma))\big)\big)\, d\varsigma. \end{aligned}$$

As a result, we obtain

$$\begin{aligned} \Big(1 + \sum_{k=1}^{m} a_k\Big) \mu(\mathfrak{r}) = \mu_\circ &- \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2\big(\varsigma, \mu(\varphi(\varsigma))\big)\big)\, d\varsigma \\ &+ \Big(1 + \sum_{k=1}^{m} a_k\Big) \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2\big(\varsigma, \mu(\varphi(\varsigma))\big)\big)\, d\varsigma, \end{aligned}$$

which is equivalent to (6). □

It is obvious from assumption (i), that the mild solution of the single-valued integral equation (6), where $\phi_1 \in S_{F_1}$, is a solution to the inclusion (1) with $\mu(0) + \sum_{k=1}^{m} a_k x(\tau_k) = \mu_\circ$.

It should be noticed that ϕ_1 met the Lipschitz selection $|\phi_1(\mathfrak{r}, \mu) - \phi_1(\mathfrak{r}, \nu)| \le k|\mu - \nu|$. Note that $|\phi_1(\mathfrak{r}, \mu)| \le k|\mu| + \phi_1^*$, where $\phi_1^* = \sup_{\mathfrak{r} \in [0,T]} |\phi_1(\mathfrak{r}, 0)|$.

Let us go on to the next step. Let $a = (1 + \sum_{k=1}^{m} a_k)^{-1}$ and

$$\nu(\mathfrak{r}) = I^\sigma \phi_2\big(\mathfrak{r}, \mu(\varphi(\mathfrak{r}))\big), \quad \mathfrak{r} \in I. \tag{10}$$

Thus, the nonlinear functional integral equation (6) can be expressed as

$$\mu(\mathfrak{r}) = a \Bigg(\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, \nu(\varsigma)\big)\, d\varsigma \Bigg) + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, \nu(\varsigma)\big)\, d\varsigma \tag{11}$$

for $\mathfrak{r} \in I$. As a result, the coupled system (10) and (11) and the functional integral equation (6) are equal.

Now, we investigate the existence of a continuous solution of (6), by obtaining the continuous solution of the coupled system (10) and (11).

Now for the existence of at least one solution, $u = (\mu, \nu)$, $\mu, \nu \in C(I)$, of the coupled system (10), (11) we have the following theorem.

Theorem 1. *Assume that assumptions* (i)–(iv) *hold. Then the coupled system* (10), (11) *has at least one continuous solution* $u = (\mu, \nu)$, $\mu, \nu \in C(I)$.

Proof. Let the set Q_r be defined by

$$Q_r = \{u = (\mu, \nu) \in R^2, \|u\| \le r\},$$

where

$$r = r_1 + r_2 = \frac{a|\mu_\circ| + \left[a\sum_{k=1}^m |a_k| + 1\right] f_1^* T}{1 - \left[a\sum_{k=1}^m |a_k| + 1\right] kT} + \left(1 - \frac{bT^\sigma}{\Gamma(\sigma+1)}\right)^{-1} \frac{MT^{\sigma-\gamma}}{\Gamma(\sigma-\gamma+1)}.$$

It is clear that the set Q_r is nonempty, bounded, closed and convex.

Afterwards, let us indicate by A the operator defined on the space $C(I)$ by

$$Au(\mathfrak{r}) = A(\mu, \nu)(\mathfrak{r}) = \big(A_1\nu(\mathfrak{r}), A_2\mu(\mathfrak{r})\big),$$

where

$$A_1\nu(\mathfrak{r}) = a\left(\mu_\circ - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, \nu(\varsigma)\big)\, d\varsigma\right) + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, \nu(\varsigma)\big)\, d\varsigma, \quad \mathfrak{r} \in I,$$

$$A_2\mu(\mathfrak{r}) = \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\sigma-1}}{\Gamma(\sigma)} \phi_2\big(\varsigma, \mu(\varphi(\varsigma))\big)\, d\varsigma, \quad \mathfrak{r} \in I.$$

Hence, according to $u = (\mu, \nu) \in Q_r$, we have

$$\begin{aligned}
|A_1\nu(\mathfrak{r})| &= \left| a\left(\mu_\circ - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, \nu(\varsigma)\big)\, d\varsigma\right) + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, \nu(\varsigma)\big)\, d\varsigma \right| \\
&\le a|\mu_\circ| + a\sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k - \varsigma)^{\eta-1}}{\Gamma(\eta)} |\phi_1\big(\varsigma, \nu(\varsigma)\big)|\, d\varsigma + \int_0^t \frac{(t - \varsigma)^{\eta-1}}{\Gamma(\eta)} |\phi_1\big(\varsigma, \nu(\varsigma)\big)|\, ds \\
&\le a|\mu_\circ| + \left[a\sum_{k=1}^m |a_k| + 1\right] \frac{(k|\nu| + \phi_1^*)T^\eta}{\Gamma(\eta+1)}.
\end{aligned}$$

Therefore

$$\|A_1\nu\| \le a|\mu_\circ| + \left[a\sum_{k=1}^m |a_k| + 1\right] \frac{(k|\nu| + \phi_1^*)T^\eta}{\Gamma(\eta+1)} = r_1,$$

$$r_1 = \frac{a|\mu_\circ| + \left[a\sum_{k=1}^m |a_k| + 1\right] \frac{\phi_1^* T^\eta}{\Gamma(\eta+1)}}{1 - \left[a\sum_{k=1}^m |a_k| + 1\right] \frac{kT^\eta}{\Gamma(\eta+1)}}.$$

Also

$$\begin{aligned}
|A_2\mu(\mathfrak{r})| &= \left| \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\sigma-1}}{\Gamma(\sigma)} \phi_2\big(\varsigma, \mu(\varphi(\varsigma))\big)\, d\varsigma \right| \\
&\le \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\sigma-1}}{\Gamma(\sigma)} |\phi_2\big(\varsigma, \mu(\varphi(\varsigma))\big)|\, d\varsigma \le \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\sigma-1}}{\Gamma(\sigma)} \big[a(\varsigma) + b\big|\mu(\varphi(\varsigma))\big|\big]\, d\varsigma.
\end{aligned}$$

Taking supremum over $\mathfrak{r} \in I$, we obtain

$$\begin{aligned}
\|A_2\mu\| &\le \int_0^{\mathfrak{r}} a(\varsigma) \frac{(\mathfrak{r} - \varsigma)^{\sigma-1}}{\Gamma(\sigma)}\, d\varsigma + \int_0^{\mathfrak{r}} b|\mu(\varphi(\varsigma))| \frac{(\mathfrak{r} - \varsigma)^{\sigma-1}}{\Gamma(\sigma)}\, d\varsigma \\
&\le I^\sigma a(\mathfrak{r}) + b\, r_2 \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\sigma-1}}{\Gamma(\sigma)}\, d\varsigma \le I^{\sigma-\gamma} I^\gamma a(\mathfrak{r}) + b\, r_2 I^\sigma(\mathfrak{r}) \\
&\le M \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\sigma-\gamma-1}}{\Gamma(\sigma-\gamma)}\, d\varsigma + b\, r_2 \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \varsigma)^{\sigma-1}}{\Gamma(\sigma)}\, d\varsigma \le \frac{M\mathfrak{r}^{\sigma-\gamma}}{\Gamma(\sigma-\gamma+1)} + b\, r_2 \frac{\mathfrak{r}^\sigma}{\Gamma(\sigma+1)} \\
&\le \frac{MT^{\sigma-\gamma}}{\Gamma(\sigma-\gamma+1)} + \frac{b\, r_2 T^\sigma}{\Gamma(\sigma+1)} = r_2, \qquad r_2 = \left(1 - \frac{bT^\sigma}{\Gamma(\sigma+1)}\right)^{-1} \frac{MT^{\sigma-\gamma}}{\Gamma(\sigma-\gamma+1)}.
\end{aligned}$$

Now

$$\|Au\|_X = \|A_1\nu\|_C + \|A_2\mu\|_C \le r_1 + r_2$$
$$\le \frac{a|\mu_\circ| + \left[a\sum_{k=1}^m |a_k| + 1\right]\frac{\phi_1^*}{\Gamma(\beta+1)}}{1 - \left[a\sum_{k=1}^m |a_k| + 1\right]\frac{k}{\Gamma(\beta+1)}} + \left(1 - \frac{bT^\sigma}{\Gamma(\sigma+1)}\right)^{-1}\frac{MT^{\sigma-\gamma}}{\Gamma(\sigma-\gamma+1)} = r.$$

Hence, the class $\{Au\}$, $u \in Q_r$, is uniformly bounded for $AQ_r \subset Q_r$.

Currently, for $u = (\mu,\nu) \in Q_r$, for all $\delta > 0$ and for each $\mathfrak{r}_1, \mathfrak{r}_2 \in [0,T]$, $\mathfrak{r}_1 < \mathfrak{r}_2$, such that $|\mathfrak{r}_2 - \mathfrak{r}_1| < \delta$, we get

$$\begin{aligned}
|A_1\nu(\mathfrak{r}_2) - A_1\nu(\mathfrak{r}_1)| \\
&= \Bigg| a\left(\mu_\circ - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma,\nu(\varsigma))\,d\varsigma\right) + \int_0^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\varsigma)^{\eta-1}}{\Gamma(\eta)} f_1(\varsigma,\nu(\varsigma))\,d\varsigma \\
&\quad - a\left(\mu_\circ - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma,\nu(\varsigma))\,d\varsigma\right) + \int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_1-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\varsigma,\nu(\varsigma))\,d\varsigma \Bigg| \\
&\le \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\varsigma)^{\eta-1}}{\Gamma(\eta)} |\phi_1(\varsigma,\nu(\varsigma))|\,d\varsigma + \int_0^{\mathfrak{r}_1}\left[\frac{(\mathfrak{r}_2-\varsigma)^{\eta-1}}{\Gamma(\eta)} - \frac{(\mathfrak{r}_1-\varsigma)^{\eta-1}}{\Gamma(\eta)}\right]|\phi_1(\varsigma,\nu(\varsigma))|\,d\varsigma \\
&\le (k|\nu| + \phi_1^*)\frac{(\mathfrak{r}_2-\mathfrak{r}_1)^\eta}{\Gamma(\eta+1)} + (k|\nu| + \phi_1^*)\left(\frac{-(\mathfrak{r}_2-\mathfrak{r}_1)^\eta}{\Gamma(\eta+1)} + \frac{\mathfrak{r}_2^\sigma}{\Gamma(\eta+1)} - \frac{\mathfrak{r}_1^\eta}{\Gamma(\beta+1)}\right) \\
&\le (k|\nu| + \phi_1^*)\left(\frac{\mathfrak{r}_2^\eta - \mathfrak{r}_1^\eta}{\Gamma(\eta+1)}\right)
\end{aligned}$$

and

$$\begin{aligned}
|A_2\mu(\mathfrak{r}_2) - A_2\mu(\mathfrak{r}_1)| \\
&\le \left|\int_0^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\varsigma,\mu(\varphi(\varsigma)))\,d\varsigma - \int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_1-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\varsigma,\mu(\varphi(\varsigma)))\,d\varsigma\right| \\
&\le \left|\int_0^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\varsigma,\mu(\varphi(\varsigma)))\,d\varsigma - \int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\varsigma,\mu(\varphi(\varsigma)))\,d\varsigma\right| \\
&\quad + \left|\int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\varsigma,\mu(\varphi(\varsigma)))\,d\varsigma - \int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_1-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\varsigma,\mu(\varphi(\varsigma)))\,d\varsigma\right| \\
&\le \left|\int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1}}{\Gamma(\alpha)}\phi_2(\varsigma,\mu(\varphi(\varsigma)))\,d\varsigma\right| + \left|\int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\varsigma,\mu(\varphi(\varsigma)))\,d\varsigma\right| \\
&\quad - \left|\int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_1-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\varsigma,\mu(\varphi(\varsigma)))\,d\varsigma\right| \\
&\le \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}|\phi_2(\varsigma,\mu(\varphi(\varsigma)))|\,d\varsigma + \int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1} - (\mathfrak{r}_1-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}|\phi_2(\varsigma,\mu(\varphi(\varsigma)))|\,d\varsigma \\
&\le \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} [a + b|\mu(\varphi(\varsigma))|]\frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\,d\varsigma + \int_0^{\mathfrak{r}_1}[a + b|\mu(\varphi(s))|]\frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1} - (\mathfrak{r}_1-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\,d\varsigma \\
&\le (a + br_2)\int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\,ds + (a + br_2)\int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_2-\varsigma)^{\sigma-1} - (\mathfrak{r}_1-\varsigma)^{\sigma-1}}{\Gamma(\sigma)}\,d\varsigma \\
&\le (a + br_2)\frac{(\mathfrak{r}_2-\mathfrak{r}_1)^\sigma}{\Gamma(\sigma+1)} + (a + br_2)\left(\frac{-(\mathfrak{r}_2-\mathfrak{r}_1)^\sigma}{\Gamma(\sigma+1)} + \frac{\mathfrak{r}_2^\sigma}{\Gamma(\sigma+1)} - \frac{\mathfrak{r}_1^\sigma}{\Gamma(\sigma+1)}\right) \\
&\le (a + br_2)\frac{(\mathfrak{r}_2^\sigma - \mathfrak{r}_1^\sigma)}{\Gamma(\sigma+1)}.
\end{aligned}$$

For the operator A and $u \in Q_r$, we have

$$\begin{aligned} Au(\mathfrak{r}_2) - Au(\mathfrak{r}_1) &= A(\mu,\nu)(\mathfrak{r}_2) - A(\mu,\nu)(\mathfrak{r}_1) \\ &= (A_2\mu(\mathfrak{r}_2), A_1\nu(\mathfrak{r}_2)) - (A_2\mu(\mathfrak{r}_1), A_1\nu(\mathfrak{r}_1)) \\ &= (A_2\mu(\mathfrak{r}_2) - A_2\mu(\mathfrak{r}_1), A_1\nu(\mathfrak{r}_2) - A_1\nu(\mathfrak{r}_1)), \end{aligned}$$

then

$$\begin{aligned} |Au(\mathfrak{r}_2) - Au(\mathfrak{r}_1)|_X &= |A(\mu,y)(\mathfrak{r}_2) - A(\mu,y)(\mathfrak{r}_1)|_X \\ &= |A_1\nu(\mathfrak{r}_2) - A_1\nu(\mathfrak{r}_2)|_C + |A_2\mu(\mathfrak{r}_2) - A_2\mu(\mathfrak{r}_1)|_C \\ &= (k|\nu| + \phi_1^*)\frac{(\mathfrak{r}_2^\eta - \mathfrak{r}_1^\eta)}{\Gamma(\eta+1)} + (a + br_2)\frac{(\mathfrak{r}_2^\sigma - \mathfrak{r}_1^\sigma)}{\Gamma(\sigma+1)}. \end{aligned}$$

As a result, the class of functions $\{Au\}$ is equi-continuous on Qr. The operator A is compact via the Arzela-Ascoli Theorem [5]. The continuity of $A : Qr \to Qr$ still needs to be proven. Let $u_n = (\mu_n, \nu_n)$ be a sequence in Qr with $x_n \to \mu$ and $y_n \to y$. Since $\phi_2(\mathfrak{r}, \mu(\mathfrak{r}))$ is continuous in $C(I) \times R$, we obtain that $\phi_2(\mathfrak{r}, \mu_n(\mathfrak{r}))$ converges to $\phi_2(\mathfrak{r}, \mu(\mathfrak{r}))$, thus $\phi_2(\mathfrak{r}, \mu_n(\varphi(\mathfrak{r})))$ converges to $\phi_2(\mathfrak{r}, \mu(\varphi(\mathfrak{r})))$. Using assumptions (iii)–(iv) and applying Lebesgue Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\sigma-1}}{\Gamma(\sigma)} \phi_2\big(\varsigma, \mu_n(\varphi(\varsigma))\big)\, d\varsigma = \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\sigma-1}}{\Gamma(\sigma)} \phi_2\big(\varsigma, \mu(\varphi(\varsigma))\big)\, d\varsigma,$$

then

$$\begin{aligned} \lim_{n\to\infty} A_2\mu_n(\mathfrak{r}) &= \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\sigma-1}}{\Gamma(\sigma)} \lim_{n\to\infty} \phi_2\big(\varsigma, \mu_n(\varphi(\varsigma))\big)\, d\varsigma \\ &= \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\sigma-1}}{\Gamma(\sigma)} \phi_2\big(\varsigma, \mu(\varphi(\varsigma))\big)\, ds = A_2\mu(\mathfrak{r}), \\ \lim_{n\to\infty} A_1\nu_n(\mathfrak{r}) &= a\Big(\mu_\circ - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \lim_{n\to\infty} \phi_1\big(\varsigma, \nu_n(\varsigma)\big)\, d\varsigma\Big) \\ &\quad + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)} \lim_{n\to\infty} \phi_1\big(\varsigma, \nu_n(\varsigma)\big)\, d\varsigma \\ &= a\Big(x_\circ - \sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, \nu(\varsigma)\big)\, d\varsigma\Big) + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, \nu(\varsigma)\big)\, d\varsigma \\ &= A_1\nu(\mathfrak{r}). \end{aligned}$$

So,

$$\begin{aligned} \lim_{n\to\infty} Au_n(\mathfrak{r}) &= \lim_{n\to\infty} (A_1\nu_n(\mathfrak{r}), A_2\mu_n(\mathfrak{r})) \\ &= (\lim_{n\to\infty} A_1\nu_n(\mathfrak{r}), \lim_{n\to\infty} A_2\mu_n(\mathfrak{r})) = (A_1\nu(\mathfrak{r}), A_2\mu(\mathfrak{r})) = Au(\mathfrak{r}). \end{aligned}$$

Therefore, $Au_n \to Au$ as $n \to 1$. The operator A is continuous as a result. While all criteria of the Schauder fixed-point theorem are achieved, we get that A has a fixed point $u \in Q_r$. Hence, the system (11), (10) has at least one continuous solutions $u = (\mu, \nu) \in Q_r$, $\mu; \nu \in C(I)$.

Therefore, there is at least one solution $\mu \in C(I)$ to the functional integral equation (1). □

2.1 Riemann-Stieltjes integral boundary conditions (2)

Let $\mu \in C(I)$ represent the mild solution to the non-local problem (1)–(4). Let us denote $a_k = h(\mathfrak{r}_k) - h(\mathfrak{r}_{k-1})$. Note, that the function h is nondecreasing and $\tau_k \in (\mathfrak{r}_{k-1}, \mathfrak{r}_k)$ for

$0 = \mathfrak{r}_0 < \mathfrak{r}_1 < \mathfrak{r}_2 < \cdots < T$. Then the nonlocal condition (4) take the following form

$$\mu(0) + \sum_{k=1}^{m} \mu(\tau_k)\big(h(\mathfrak{r}_j) - h(\mathfrak{r}_{j-1})\big) = \mu_\circ.$$

We derive from [15] the continuation of the solution of the nonlocal problem (1)–(4), i.e.

$$\lim_{m\to\infty} \sum_{k=1}^{m} \mu(\tau_k)\big(h(\mathfrak{r}_j) - h(\mathfrak{r}_{j-1})\big) = \int_0^T \mu(\varsigma)\, dh(\varsigma).$$

That is, the non-local conditions (4) is changed to Riemann-Steltjes integral condition

$$\mu(0) + \lim_{m\to\infty} \sum_{k=1}^{m} \mu(\tau_k)\big(h(\mathfrak{r}_j) - h(\mathfrak{r}_{j-1})\big) = \mu(0) + \int_0^T \mu(\varsigma)\, dh(\varsigma) = \mu_\circ.$$

Theorem 2. *Assume that assumptions* (i)–(iv) *hold and* $h : [0,T] \to [0,T]$ *is an increasing function. Then the Riemann-Stieltjes functional integral condition* (2) *and the non-local problem* (1) *have a mild solution* $\mu \in C(I)$ *that is represented by*

$$\begin{aligned}\mu(\mathfrak{r}) = \frac{1}{1 + h(T) - h(0)}\bigg(\mu_\circ - \int_0^T \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\beta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma)))\big)\, d\varsigma\, dh(\varsigma)\bigg) \\ + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma)))\big)\, ds.\end{aligned} \tag{12}$$

Proof. The non-local problem (1)–(4) will have the following solution as $m \to \infty$:

$$\begin{aligned}\mu(\mathfrak{r}) &= \lim_{m\to\infty} \frac{1}{1 + \sum_{k=1}^{m} a_k}\bigg(\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\beta-1}}{\Gamma(\beta)} \phi_1\big(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma)))\big)\, d\varsigma\bigg) \\ &\quad + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma)))\big)\, d\varsigma \\ &= \frac{1}{1 + h(T) - h(0)}\bigg(\mu_\circ - \lim_{m\to\infty} \sum_{k=1}^{m} \big(h(\mathfrak{r}_k) - h(\mathfrak{r}_{k-1})\big) \\ &\quad \times \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\beta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma)))\big)\, d\varsigma\bigg) \\ &\quad + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma)))\big)\, d\varsigma \\ &= \frac{1}{1 + h(T) - h(0)}\bigg(\mu_\circ - \int_0^T \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\beta)} \phi_1\big(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma)))\big)\, d\varsigma\, dh(\varsigma)\bigg) \\ &\quad + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2(\varsigma, \mu(\varphi(\varsigma)))\big)\, d\varsigma.\end{aligned}$$

□

As a result, the mild solution $\mu \in C(I)$ of the first-order nonlinear differential equation (1) with the Riemann-Stieltjes integral condition (2) is represented by (12).

Consequently, there exist at least one mild solution $\mu \in C(I)$ of the nonlocal problem of functional differential inclusion (1)–(2).

2.2 Infinite-point boundary condition (3)

Take into account that $\mu \in C(I)$ is the mild solution to the non-local problem presented by (1) and (3).

Theorem 3. *Assume that assumptions* (i)–(iv) *hold and let* $B_m^{-1} = 1 + \sum_{k=1}^{m} a_k$ *be convergent sequence. Then the non-local problem* (1)–(3) *represented by the integral equation*

$$\begin{aligned}\mu(\mathfrak{r}) = B_m\mu_\circ - B_m\sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1\big(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma)))\big)\,d\varsigma \\ + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1\big(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma)))\big)\,d\varsigma\end{aligned} \tag{13}$$

has at least one mild solution $\mu \in C(I)$.

Proof. Let $\mu \in C(I)$ be a solution of the infinite point BVP (1) and (3) provided by (6). Let

$$\begin{aligned}\mu_m(\mathfrak{r}) = \frac{1}{1+\sum_{k=1}^{m} a_k}\left(x_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\beta-1}}{\Gamma(\beta)}\phi_1\big(\varsigma, I^\sigma f_2(\varsigma,\mu(\varphi(\varsigma)))\big)\,d\varsigma\right) \\ + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\beta-1}}{\Gamma(\beta)}\phi_1\big(\varsigma, I^\sigma\phi_2(\varsigma,\mu_m(\varphi(\varsigma)))\big)\,d\varsigma.\end{aligned} \tag{14}$$

Consider the limit in (14) as $m \to \infty$, we obtain

$$\begin{aligned}&\lim_{m\to\infty}\mu_m(\mathfrak{r}) \\ &= \lim_{m\to\infty}\left[\frac{1}{1+\sum_{k=1}^{m} a_k}\left(\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\beta-1}}{\Gamma(\eta)}\phi_1\big(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma)))\big)\,d\varsigma\right)\right. \\ &\quad \left. + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1\big(\varsigma, I^\sigma\phi_2(\varsigma,\mu_m(\varphi(\varsigma)))\big)\,d\varsigma\right] \\ &= \lim_{m\to\infty}\frac{1}{1+\sum_{k=1}^{m} a_k}\left[\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1\big(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma)))\big)\,ds\right] \\ &\quad + \lim_{m\to\infty}\int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1\big(\varsigma, I^\sigma\phi_2(\varsigma,\mu_m(\varphi(\varsigma)))\big)\,d\varsigma.\end{aligned} \tag{15}$$

Now, we have $|a_k\mu(\tau_k)| \le |a_k|\|\mu\|$, so, using a comparison test, we get that

$$\sum_{k=1}^{m} a_k\mu(\tau_k)$$

is convergent.

Also

$$\begin{aligned}&\left|\int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1\big(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma)))\big)\,d\varsigma\right| \\ &\quad \le \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\beta-1}}{\Gamma(\eta)}\big(k\big|I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma)))\big| + \phi_1^*\big)\,ds \\ &\quad \le k\int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\left[I^\sigma a(\varsigma) + b\int_0^{\varsigma}\frac{(\varsigma-\theta)^{\sigma-1}}{\Gamma(\sigma)}|\mu(\varphi(\theta))|\,d\theta\right]d\varsigma + k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1^*\,ds \\ &\quad \le k\int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\beta-1}}{\Gamma(\eta)}\left[I^{\sigma-\gamma}I^\gamma a(\varsigma) + br_2\int_0^{\varsigma}\frac{(\varsigma-\theta)^{\sigma-1}}{\Gamma(\sigma)}\,d\theta\right]d\varsigma + k\,\phi_1^*\frac{T^\beta}{\Gamma(\beta+1)}\end{aligned}$$

$$\leq kM\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\beta-1}}{\Gamma(\beta)}\int_0^{\varsigma}\frac{(\varsigma-\theta)^{\sigma-\gamma-1}}{\Gamma(\sigma-\gamma)}\,d\theta\,ds$$
$$+\,k\,br_2\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\beta-1}}{\Gamma(\beta)}\int_0^{\varsigma}\frac{(\varsigma-\theta)^{\sigma-1}}{\Gamma(\sigma)}\,d\theta\,d\varsigma+\frac{k\phi_1^*T^{\eta}}{\Gamma(\eta+1)}$$
$$\leq\frac{T^{\beta}}{\Gamma(\beta+1)}\left[\frac{kMT^{\sigma-\gamma+1}}{\Gamma(\sigma-\gamma+1)}+\frac{k\,br_2T^{\sigma+1}}{\Gamma(\sigma+1)}+k\,\phi_1^*\right]\leq N,$$

therefore

$$|a_k|\left|\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\,\phi_1\big(\varsigma,I^{\sigma}\phi_2(\varsigma,\mu(\varphi(\varsigma)))\big)\,d\varsigma\right|\leq|a_k|\,N,$$

and by the comparison test

$$\sum_{k=1}^{m}a_k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\beta-1}}{\Gamma(\beta)}\,\phi_1\big(\varsigma,I^{\sigma}\phi_2(\varsigma,\mu(\varphi(\varsigma)))\big)\,d\varsigma$$

is convergent.

Using assumptions (i)–(iii) and applying Lebesgue Dominated Convergence Theorem [12], from (15) we obtain (13). Furthermore, from (13), we have

$$\Big(1+\sum_{k=1}^{m}a_k\Big)\mu(\tau_k)=B_m^{-1}B_mx_\circ-B_m^{-1}B_m\sum_{k=1}^{m}a_k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\beta-1}}{\Gamma(\eta)}\phi_1\big(\varsigma,I^{\sigma}\phi_2(\varsigma,\mu(\varphi(\varsigma)))\big)\,d\varsigma$$
$$+\Big(1+\sum_{k=1}^{m}a_k\Big)\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\beta-1}}{\Gamma(\eta)}\phi_1\big(\varsigma,I^{\sigma}\phi_2(\varsigma,\mu(\varphi(\varsigma)))\big)\,d\varsigma,$$

$$\mu(\tau_k)+\sum_{k=1}^{m}a_k\mu(\tau_k)=x_\circ-\sum_{k=1}^{m}a_k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\beta-1}}{\Gamma(\eta)}\phi_1\big(\varsigma,I^{\sigma}\phi_2(\varsigma,\mu(\varphi(\varsigma)))\big)\,d\varsigma$$
$$+\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1\big(\varsigma,I^{\sigma}\phi_2(\varsigma,\mu(\varphi(\varsigma)))\big)\,d\varsigma$$
$$+\sum_{k=1}^{m}a_k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1\big(\varsigma,I^{\sigma}\phi_2(\varsigma,\mu(\varphi(\varsigma)))\big)\,d\varsigma,$$

$$\sum_{k=1}^{m}a_k\mu(\tau_k)=\mu_\circ-\mu(\tau_k)+\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1\big(\varsigma,I^{\sigma}\phi_2(\varsigma,\mu(\varphi(\varsigma)))\big)\,d\varsigma. \tag{16}$$

From (6), we obtain

$$\mu(0)=a\bigg(\mu_\circ-\sum_{k=1}^{m}a_k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1\big(\varsigma,I^{\sigma}\phi_2(\varsigma,\mu(\varphi(\varsigma)))\big)\,ds\bigg)$$

and

$$\mu(\tau_k)=a\bigg(x_\circ-\sum_{k=1}^{m}a_k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\beta-1}}{\Gamma(\beta)}\phi_1\big(\varsigma,I^{\sigma}\phi_2(\varsigma,\mu(\varphi(\varsigma)))\big)\,d\varsigma\bigg)$$
$$+\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\beta-1}}{\Gamma(\eta)}\phi_1\big(\varsigma,I^{\sigma}\phi_2(\varsigma,\mu(\varphi(\varsigma)))\big)\,d\varsigma.$$

So

$$\mu(0)=\mu(\tau_k)-\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1\big(\varsigma,I^{\sigma}\phi_2(s,\mu(\varphi(\varsigma)))\big)\,d\varsigma.$$

Going back to (16), we obtain infinite-point boundary condition (3). Consequently, the nonlocal problem of functional differential inclusion (1)–(3) has at least one mild solution $\mu\in C(I)$. $\square$

3 Existence of unique solutions to (1) with the m-point boundary conditions (4)

In this section, the necessary condition for the uniqueness result for non-local problem (1)–(4) is provided. Assume the following assumption.

$(iii)^*$ Suppose that $\phi_2 : I \times R \to R$ is a continuous function that satisfies the Lipschitz condition $|\phi_2(\mathfrak{r},\mu) - \phi_2(\mathfrak{r},\nu)| \le c\,|\mu - \nu|$.

Theorem 4. *Assume that the conditions of Theorem 1 hold, with condition (iii) replaced by $(iii)^*$. If*

$$\frac{kc\left(a\sum_{k=1}^{m} a_k + 1\right) T^{\eta+\sigma+2}}{\Gamma(\eta+1)\Gamma(\sigma+1)} < 1,$$

then the non-local problem (1)–(4) has a unique solution $x \in C(I)$.

Proof. Let $\mu_1(\mathfrak{r})$ and $\mu_2(\mathfrak{r})$ be two solutions of the functional integral equation (6). Then

$$\begin{aligned}
\mu_1(\mathfrak{r}) - \mu_2(\mathfrak{r}) &= a\left(\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2(\varsigma, \mu_1(\varphi(\varsigma)))\big)\, d\varsigma\right) \\
&\quad + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2(\varsigma, \mu_1(\varphi(\varsigma)))\big)\, d\varsigma \\
&\quad - a\left(\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\beta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2(\varsigma, \mu_2(\varphi(\varsigma)))\big)\, d\varsigma\right) \\
&\quad - \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2(\varsigma, \mu_2(\varphi(\varsigma)))\big)\, d\varsigma, \\
|\mu_1(\mathfrak{r}) - x_2(\mathfrak{r})| &\le a \left| \sum_{k=1}^{m} a_k \left(\int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2(\varsigma, \mu_2(\varphi(\varsigma)))\big)\, d\varsigma \right.\right. \\
&\quad \left.\left. - \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \phi_1\big(\varsigma, I^\sigma \phi_2(\varsigma, \mu_1(\varphi(\varsigma)))\big)\, d\varsigma \right) \right| \\
&\quad + \left| \int_0^{t} \frac{(t-\varsigma)^{\eta-1}}{\Gamma(\eta)} \left[\phi_1\big(\varsigma, I^\sigma \phi_2(\varsigma, \mu_1(\varphi(\varsigma)))\big) - \phi_1\big(\varsigma, I^\sigma \phi_2(\varsigma, \mu_2(\varphi(\varsigma)))\big)\right] d\varsigma \right| \\
&\le a \sum_{k=1}^{m} a_k \left(\int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\beta-1}}{\Gamma(\eta)} \left|\phi_1\big(\varsigma, I^\sigma \phi_2(\varsigma, \mu_2(\varphi(\varsigma)))\big) - \phi_1\big(\varsigma, I^\sigma \phi_2(\varsigma, \mu_1(\varphi(\varsigma)))\big)\right| d\varsigma \right) \\
&\quad + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\beta-1}}{\Gamma(\beta)} \left|\phi_1\big(\varsigma, I^\sigma \phi_2(\varsigma, \mu_1(\varphi(\varsigma)))\big) - \phi_1\big(\varsigma, I^\sigma \phi_2(\varsigma, \mu_2(\varphi(\varsigma)))\big)\right| d\varsigma.
\end{aligned}$$

Lipschitz condition for ϕ_1 allows us to obtain

$$\begin{aligned}
|\mu_1(\mathfrak{r}) - \mu_2(\mathfrak{r})| &\le a \sum_{k=1}^{m} a_k k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\beta-1}}{\Gamma(\eta)} \left| I^\sigma \phi_2(\varsigma, \mu_1(\varphi(\varsigma))) - I^\sigma \phi_2(\varsigma, \mu_2(\varphi(\varsigma)))\right| d\varsigma \\
&\quad + k \int_0^{t} \frac{(t-\varsigma)^{\beta-1}}{\Gamma(\beta)} \left| I^\sigma \phi_2(\varsigma, \mu_1(\varphi(\varsigma))) - I^\sigma \phi_2(\varsigma, \mu_2(\varphi(\varsigma)))\right| d\varsigma \\
&\le a \sum_{k=1}^{m} a_k k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)} \int_0^{\varsigma} \frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)} \left|\phi_2(\tau, \mu_1(\varphi(\tau))) - \phi_2(\tau, \mu_2(\varphi(\tau)))\right| d\tau\, d\varsigma \\
&\quad + k \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\beta-1}}{\Gamma(\beta)} \int_0^{\varsigma} \frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)} \left|\phi_2(\tau, \mu_1(\varphi(\tau))) - \phi_2(\tau, \mu_2(\varphi(\tau)))\right| d\tau\, d\varsigma.
\end{aligned}$$

Lipschitz condition for ϕ_2 allows us to obtain

$$\begin{aligned}|\mu_1(\mathfrak{r})-\mu_2(\mathfrak{r})| &\leq a\sum_{k=1}^{m}a_k kc\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\big|\mu_1(\varphi(\tau))-\mu_2(\varphi(\tau))\big|\,d\tau\,d\varsigma\\ &\quad+kc\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\big|\mu_1(\varphi(\tau))-\mu_2(\varphi(\tau))\big|\,d\tau\,d\varsigma\\ &\leq a\sum_{k=1}^{m}a_k kc\|\mu_1-\mu_2\|\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\,d\tau\,d\varsigma\\ &\quad+kc\|\mu_1-\mu_2\|\int_0^{t}\frac{(t-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\,d\tau\,d\varsigma,\\ \|\mu_1-\mu_2\| &\leq \frac{(a\sum_{k=1}^{m}a_k+1)T^{\eta+1}T^{\sigma+1}kc}{\Gamma(\eta+1)\Gamma(\sigma+1)}\|\mu_1-\mu_2\|.\end{aligned}$$

Hence

$$\left(1-\frac{kc(a\sum_{k=1}^{m}a_k+1)T^{\eta+\sigma+2}}{\Gamma(\eta+1)\Gamma(\sigma+1)}\right)\|\mu_1-\mu_2\|\leq 0.$$

Since $\frac{kc(a\sum_{k=1}^{m}a_k+1)T^{\eta+\sigma+2}}{\Gamma(\eta+1)\Gamma(\sigma+1)}<1$, we get $\mu_1(\mathfrak{r})=\mu_2(\mathfrak{r})$ and the solution of the integral equation (6) is unique. As a result, this establishes the existence of unique solution to the non-local problem (1)–(4). □

Corollary 1. *Assume that assumptions of Theorem 4 hold. Then the solution of the non-local problem* (1)–(4) *is continuously dependent on the set S_{Φ_1} of all Lipschitzian selections of Φ_1.*

Proof. Let $\phi_1(\mathfrak{r},\mu(\mathfrak{r}))$ and $\phi_1^*(\mathfrak{r},\mu(\mathfrak{r}))$ be two separate Lipschitzian selections of $\Phi_1(\mathfrak{r},\mu(\mathfrak{r}))$, so that

$$|\phi_1(\mathfrak{r},\mu(\mathfrak{r}))-\phi_1^*(\mathfrak{r},\mu(\mathfrak{r})|<\epsilon,\quad \epsilon>0,\ \ \mathfrak{r}\in I.$$

Then for two related solutions $\mu_{\phi_1}(\mathfrak{r})$ and $\mu_{\phi_1^*}(\mathfrak{r})$ of (6) we have the following

$$\begin{aligned}&|\mu_{\phi_1}(\mathfrak{r})\ -\mu^*_{\phi_1}(\mathfrak{r})|\\ &\quad=\bigg|a\sum_{k=1}^{m}a_k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\big[\phi_1\big(\varsigma,I^{\sigma}\phi_2\big(\varsigma,\mu(\varphi(\varsigma))\big)\big)-\phi_1^*\big(\mathfrak{r},I^{\sigma}\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big)\big]\,ds\\ &\qquad+\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\beta)}\big[\phi_1\big(\varsigma,I^{\sigma}\phi_2\big(\varsigma,\mu(\varphi(\varsigma))\big)\big)-\phi_1^*\big(\mathfrak{r},I^{\sigma}\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big)\big]\,ds\bigg|\\ &\quad\leq a\sum_{k=1}^{m}a_k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\beta)}\big|\phi_1\big(\varsigma,I^{\sigma}\phi_2\big(\varsigma,\mu(\varphi(\varsigma))\big)\big)-\phi_1^*\big(\mathfrak{r},I^{\sigma}\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big)\big|\,d\varsigma\\ &\qquad+\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\big|\phi_1\big(\varsigma,I^{\sigma}\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big)-\phi_1^*\big(\varsigma,I^{\sigma}\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big)\big|\,d\varsigma\\ &\quad\leq a\sum_{k=1}^{m}a_k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\big|\phi_1\big(\varsigma,I^{\sigma}\phi_2\big(\varsigma,\mu(\varphi(\varsigma))\big)\big)-\phi_1\big(\varsigma,I^{\sigma}\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big)\big|\,d\varsigma\\ &\qquad+a\sum_{k=1}^{m}a_k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\big|\phi_1\big(\varsigma,I^{\sigma}\phi_1\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big)-\phi_1^*\big(\varsigma,I^{\sigma}\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big)\big|\,d\varsigma\\ &\qquad+\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\big|\phi_1\big(\varsigma,I^{\sigma}\phi_2\big(\varsigma,\mu(\varphi(\varsigma))\big)\big)-\phi_1\big(\varsigma,I^{\sigma}\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big)\big|\,ds\\ &\qquad+\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\big|\phi_1\big(\varsigma,I^{\sigma}\phi_1\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big)-\phi_1^*\big(\varsigma,I^{\sigma}\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big)\big|\,d\varsigma\end{aligned}$$

$$\leq a\sum_{k=1}^{m}a_k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\big(\big|\phi_1\big(\varsigma,I^\sigma\phi_2\big(\varsigma,\mu(\varphi(\varsigma))\big)\big)-\phi_1\big(\varsigma,I^\sigma\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big)\big|+\delta\big)\,d\varsigma$$

$$+\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\big|\phi_1\big(\varsigma,I^\sigma\phi_2\big(\varsigma,\mu(\varphi(\varsigma))\big)\big)-\phi_1\big(\varsigma,I^\sigma\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big)\big|\,d\varsigma+\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\delta\,d\varsigma$$

$$\leq a\sum_{k=1}^{m}a_kk\Big(\int_0^{\mathfrak{r}_k}\big|I^\sigma\phi_2\big(\mathfrak{r},\mu(\varphi(\mathfrak{r}))\big)-I^\sigma\phi_2\big(\mathfrak{r},\mu^*(\varphi(\mathfrak{r}))\big)\big|\,ds+\delta\frac{T^\eta}{\Gamma(\eta+1)}\Big)$$

$$+k\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\big|I^\sigma\phi_2\big(\mathfrak{r},\mu(\varphi(\mathfrak{r}))\big)-I^\sigma\phi_2\big(\mathfrak{r},\mu^*(\varphi(\mathfrak{r}))\big)\big|\,ds+\frac{\delta T^\eta}{\Gamma(\eta+1)}$$

$$\leq a\sum_{k=1}^{m}a_kk\Big(\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\big|\phi_2\big(\tau,\mu(\varphi(\tau))\big)-\phi_2\big(\tau,\mu^*(\varphi(\tau))\big)\big|\,d\tau\,d\varsigma+\frac{\delta T^\eta}{\Gamma(\eta+1)}\Big)$$

$$+k\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\beta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\big|\phi_2\big(\tau,\mu(\varphi(\tau))\big)-\phi_2\big(\tau,\mu^*(\varphi(\tau))\big)\big|\,d\tau\,d\varsigma+\frac{\delta T^\eta}{\Gamma(\eta+1)}$$

$$\leq a\sum_{k=1}^{m}a_kkc\Big(\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\big|\mu(\varphi(\varsigma))-\mu^*(\varphi(\varsigma))\big|\,d\tau\,d\varsigma+\frac{\delta T^\eta}{\Gamma(\eta+1)}\Big)$$

$$+kc\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\big|\mu(\varphi(\varsigma))-\mu^*(\varphi(\varsigma))\big|\,d\tau\,d\varsigma+\frac{\delta T^\eta}{\Gamma(\eta+1)}$$

$$\leq\|\mu-\mu^*\|\Big(a\sum_{k=1}^{m}a_kkc\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\,d\tau\,d\varsigma$$

$$+kc\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\,d\tau\,d\varsigma\Big)+(a\sum_{k=1}^{m}a_kkc+1)\frac{\delta T^\eta}{\Gamma(\eta+1)},$$

$$\|\mu_{\phi_1}-\mu_{\phi_1^*}\|\leq\frac{(a\sum_{k=1}^{m}a_k+1)kcT^{\sigma+\eta}}{\Gamma(\eta+1)\Gamma(\sigma+1)}\|\mu_{\phi_1}-\mu_{\phi_1^*}\|+(a\sum_{k=1}^{m}a_kkc+1)\frac{\delta T^\eta}{\Gamma(\eta+1)},$$

$$\|\mu_{\phi_1}-\mu_{\phi_1^*}\|\leq\Big(1-\frac{\big(a\sum_{k=1}^{m}a_k+1\big)kcT^{\sigma+\eta}}{\Gamma(\eta+1)\Gamma(\sigma+1)}\Big)^{-1}\Big(a\sum_{k=1}^{m}a_kkc+1\Big)\frac{\delta T^\eta}{\Gamma(\eta+1)}=\epsilon.$$

Hence, $\|\mu_{\phi_1}-\mu_{\phi_1^*}\|\leq\epsilon$. It demonstrates that the solution on the set S_{Φ_1} of all Lipschitzian selection of Φ_1 is continuous dependence. □

Corollary 2. *Assume that assumptions of Theorem 4 hold. Then the solution of the non-local problem* (1)–(4) *depends continuously on the Lipschitz function* ϕ_2.

Proof. Let $\phi_2(\mathfrak{r},\mu(\mathfrak{r}))$ and $\phi_2^*(\mathfrak{r},\mu(\mathfrak{r}))$ be two different Lipschitz functions such that

$$|\phi_2(\mathfrak{r},\mu(\mathfrak{r}))-\phi_2^*(\mathfrak{r},\mu(\mathfrak{r})|<\delta,\quad\delta>0,\ \mathfrak{r}\in I.$$

Then for two corresponding solutions μ and μ^* of (6), we have

$$|\mu(\mathfrak{r})\ -\mu^*(\mathfrak{r})|$$

$$\leq a\sum_{k=1}^{m}a_k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\big|\phi_1\big(\varsigma,I^\sigma\phi_2\big(\varsigma,\mu_2(\varphi(\varsigma))\big)\big)-\phi_1\big(\varsigma,I^\sigma\phi_2^*\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big)\big|\,ds$$

$$+\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\big|\phi_1\big(\varsigma,I^\sigma\phi_2\big(\varsigma,\mu_1(\varphi(\varsigma))\big)\big)-\phi_1\big(\varsigma,I^\sigma\phi_2^*\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big)\big|\,d\varsigma$$

$$\leq a\sum_{k=1}^{m}a_kk\int_0^{\mathfrak{r}_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\big|I^\sigma\phi_2\big(\varsigma,\mu(\varphi(\varsigma))\big)-I^\sigma\phi_2^*\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big|\,d\varsigma$$

$$+k\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\big|I^\sigma\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)-I^\sigma\phi_2^*\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big|\,d\varsigma$$

$$
\begin{aligned}
&\le a\sum_{k=1}^{m} a_k k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\big|I^{\sigma}\phi_2\big(\varsigma,\mu(\varphi(\varsigma))\big)-I^{\sigma}\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big|\,d\varsigma\\
&\quad+a\sum_{k=1}^{m} a_k k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\big|I^{\sigma}\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)-I^{\sigma}\phi_2^*\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big|\,d\varsigma\\
&\quad+k\int_0^{\mathfrak{t}}\frac{(\mathfrak{t}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\big|I^{\sigma}\phi_2\big(\varsigma,\mu(\varphi(\varsigma))\big)-I^{\sigma}\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big|\,d\varsigma\\
&\quad+k\int_0^{\mathfrak{t}}\frac{(\mathfrak{t}-\varsigma)^{\eta-1}}{\Gamma(\beta)}\big|I^{\sigma}\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)-I^{\sigma}\phi_2^*\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big|\,d\varsigma\\
&\le a\sum_{k=1}^{m} a_k\Big[k\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\big|\phi_2\big(\tau,\mu(\varphi(\tau))\big)-\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big|\,d\tau\,d\varsigma\\
&\quad+\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\big|\phi_2\big(\tau,\mu^*(\varphi(\tau))\big)-\phi_2^*\big(\tau,\mu^*(\varphi(\tau))\big)\big|\,d\tau\,d\varsigma\Big]\\
&\quad+k\Big[\int_0^{\mathfrak{t}}\frac{(\mathfrak{t}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\big|\phi_2\big(\tau,\mu(\varphi(\tau))\big)-\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big|\,d\tau\,d\varsigma\\
&\quad+\int_0^{\mathfrak{t}}\frac{(\mathfrak{t}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\big|\phi_2\big(\tau,\mu^*(\varphi(\tau))\big)-\phi_2^*\big(\tau,\mu^*(\varphi(\tau))\big)\big|\,d\tau\,d\varsigma\Big]\\
&\le a\sum_{k=1}^{m} a_k kc\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\big[|\mu(\varphi(\tau))-\mu^*(\varphi(\tau))|+k\delta\big]\,d\tau\,d\varsigma\\
&\quad+kc\int_0^{\mathfrak{t}}\frac{(\mathfrak{t}-\varsigma)^{\beta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\big[|\mu(\varphi(\tau))-\mu^*(\varphi(\tau))|+k\delta\big]\,d\tau\,d\varsigma\\
&\le a\sum_{k=1}^{m} a_k\big[kc\|\mu-\mu^*\|+k\delta\big]\int_0^{\tau_k}\frac{(\tau_k-\varsigma)^{\beta-1}}{\Gamma(\beta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\,d\tau\,d\varsigma\\
&\quad+\big[kc\|\mu-\mu^*\|+k\delta\big]\int_0^{\mathfrak{t}}\frac{(\mathfrak{t}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\,d\tau\,d\varsigma\\
&\le a\sum_{k=1}^{m} a_k k\big[c\|\mu-\mu^*\|+\delta\big]\frac{T^{\sigma+\eta}}{\Gamma(\sigma+1)\Gamma(\eta+1)}+k\big[c\|\mu-\mu^*\|+\delta\big]\frac{T^{\sigma+\eta}}{\Gamma(\sigma+1)\Gamma(\eta+1)}.
\end{aligned}
$$

Taking supremum over $t\in I$, we get

$$
\|\mu-\mu^*\|\le\frac{(a\sum_{k=1}^{m}a_k+1)kcT^{\sigma+\eta}}{\Gamma(\sigma+1)\Gamma(\eta+1)}\|\mu-\mu^*\|+\frac{(a\sum_{k=1}^{m}a_k+1)kcT^{\sigma+\eta}}{\Gamma(\sigma+1)\Gamma(\beta+1)},
$$

$$
\|\mu-\mu^*\|\le\left(1-\frac{\big(a\sum_{k=1}^{m}a_k+1\big)kcT^{\sigma+\eta}}{\Gamma(\sigma+1)\Gamma(\eta+1)}\right)^{-1}\frac{\big(a\sum_{k=1}^{m}a_k+1\big)kcT^{\sigma+\eta}}{\Gamma(\sigma+1)\Gamma(\eta+1)}=\epsilon.
$$

Hence, $\|\mu-\mu^*\|\le\epsilon$, which proves the continuous dependence of the solution on the Lipschitz function ϕ_2. □

Corollary 3. *Let the assumptions of Theorem 4 be satisfied. Then the solution of the non-local problem* (1)–(4) *depends continuously on initial data* $\mu_\circ$.

Proof. The solution of the integral inclusion (6) depends continuously on initial data $\mu_\circ$, if for all $\epsilon>0$ there exists $\delta(\epsilon)$ such that $|\mu_\circ-\mu_\circ^*|<\delta$ implies $\|\mu-\mu^*\|<\epsilon$.

Then for two corresponding solutions $\mu(\mathfrak{t})$ and $\mu^*(\mathfrak{t})$ of the integral equation (6) we obtain

$$
\begin{aligned}
&|\mu(\mathfrak{r}) - \mu^*(\mathfrak{r})| \\
&= \Bigg| a\Bigg(\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1\big(\varsigma, I^\sigma\phi_2\big(\varsigma,\mu_2(\varphi(\varsigma))\big)\big)\,d\varsigma\Bigg) \\
&\quad + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\beta)}\phi_1\big(\varsigma, I^\sigma\phi_2\big(\varsigma,\mu(\varphi(\varsigma))\big)\big)\,d\varsigma \\
&\quad - a\Bigg(\mu_\circ^* - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\beta)}\phi_1\big(\varsigma, I^\sigma\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big)\,d\varsigma\Bigg) \\
&\quad - \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\phi_1\big(\varsigma, I^\sigma\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big)\,d\varsigma\Bigg| \\
&\le a|\mu_\circ-\mu_\circ^*| + a\sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\big|\phi_1\big(\varsigma, I^\sigma\phi_2(\varsigma,\mu(\varphi(\varsigma)))\big) - \phi_1\big(\varsigma, I^\sigma\phi_2(\varsigma,\mu^*(\varphi(\varsigma)))\big)\big|\,d\varsigma \\
&\quad + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\big|\phi_1\big(\varsigma, I^\sigma\phi_2\big(\varsigma,\mu(\varphi(\varsigma))\big)\big) - \phi_1\big(\varsigma, I^\sigma\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big)\big|\,d\varsigma \\
&\le a\delta + a\sum_{k=1}^{m} a_k k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\big|I^\sigma\phi_2\big(\varsigma,\mu(\varphi(\varsigma))\big) - I^\sigma\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big|\,d\varsigma \\
&\quad + k\int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\beta-1}}{\Gamma(\eta)}\big|I^\sigma\phi_2\big(\varsigma,\mu(\varphi(\varsigma))\big) - I^\sigma\phi_2\big(\varsigma,\mu^*(\varphi(\varsigma))\big)\big|\,d\varsigma \\
&\le a\delta + a\sum_{k=1}^{m} a_k k \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\big|\phi_2\big(\tau,\mu(\varphi(\tau))\big) - \phi_2\big(\tau,\mu^*(\varphi(\tau))\big)\big|\,d\tau\,d\varsigma \\
&\quad + k\int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\big|\phi_2\big(\tau,\mu(\varphi(\tau))\big) - \phi_2\big(\tau,\mu^*(\varphi(\tau))\big)\big|\,d\tau\,d\varsigma \\
&\le a\delta + a\sum_{k=1}^{m} a_k kc \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}|\mu(\varphi(\tau)) - \mu^*(\varphi(\tau))|\,d\tau\,d\varsigma \\
&\quad + kc\int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}|\mu(\varphi(\tau)) - \mu^*(\varphi(\tau))|\,d\tau\,d\varsigma \\
&\qquad \le a\delta + a\sum_{k=1}^{m} a_k kc\|\mu-\mu^*\| \int_0^{\tau_k} \frac{(\tau_k-\varsigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\,d\tau\,d\varsigma \\
&\qquad\quad + kc\|\mu-\mu^*\|\int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{\beta-1}}{\Gamma(\eta)}\int_0^{\varsigma}\frac{(\varsigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\,d\tau\,d\varsigma.
\end{aligned}
$$

Taking supermum over $\mathfrak{r} \in I$, we get

$$
\begin{aligned}
\|\mu-\mu^*\| &\le a\delta + \frac{\big[a\sum_{k=1}^{m} a_k + 1\big]kcT^{\sigma+\eta}}{\Gamma(\sigma+1)\Gamma(\eta+1)}\|\mu-\mu^*\|, \\
\|\mu-\mu^*\| &\le \left(1 - \frac{\big[a\sum_{k=1}^{m} a_k + 1\big]kcT^{\sigma+\eta}}{\Gamma(\sigma+1)\Gamma(\eta+1)}\right)^{-1} a\delta = \epsilon.
\end{aligned}
$$

Hence, $\|\mu-\mu^*\| \le \epsilon$. Thus, the integral equation (6) has a continuous dependence on $\mu_\circ$. So for the the non-local problem (1)–(4) its solution depends continuously on initial data $\mu_\circ$. □

4 Illustrative examples

We provide some examples in this section to demonstrate our findings.

Example 1. *Consider the following nonlinear integro-differential inclusion*

$$ {}^{c}D^{\eta}\mu(\mathfrak{r}) \in \Phi_1(\mathfrak{r}, I^{\sigma}\phi_2(\mathfrak{r}, \mu(\varphi(\mathfrak{r})))), \quad \mathfrak{r} \in [0,1], \ \eta, \sigma \in (0,1), \tag{17} $$

with infinite point boundary condition

$$ \mu(0) + \sum_{k=1}^{\infty} \frac{1}{k^2}\mu\Big(\frac{k-1}{k}\Big) = \mu_{\circ}. \tag{18} $$

We choose $\Phi_1 : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}^+}$ *as*

$$ \Phi_1\big(\mathfrak{r}, I^{\frac{1}{4}}\phi_2(\mathfrak{r}, \mu(\mathfrak{r}))\big) = \Big[0, \, \mathfrak{r}^2 + \mathfrak{r} + 1 + \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{-\frac{3}{4}}}{2\Gamma(\frac{3}{4})}\Big(\sin\mu(\varsigma) + \frac{\mu(\varsigma)}{e^{\varsigma}}\Big)\, ds\Big]. $$

Set

$$ \phi_2(\mathfrak{r}, \mu(\mathfrak{r})) = \frac{1}{2}\Big(\sin\mu(\varsigma) + \frac{\mu(\varsigma)}{e^{\varsigma}}\Big). $$

Define a continuous map $\phi_1 : [0,1] \times \mathbb{R} \to \mathbb{R}$. *Note that* $\phi_1 \in S_{\Phi_1}$. *Then we have*

$$ \big|\phi_1\big(\mathfrak{r}, I^{\frac{1}{4}}\phi_2\big(\mathfrak{r}, \mu(\varphi(\mathfrak{r}))\big)\big) - \phi_1\big(\mathfrak{r}, I^{\frac{1}{4}}\phi_2\big(\mathfrak{r}, \nu(\varphi(\mathfrak{r}))\big)\big)\big| \le \frac{1+e}{2\, e\, \Gamma(\frac{1}{4})}|\mu - \nu| $$

and

$$ \big|\phi_2(\mathfrak{r}, \mu(\mathfrak{r}))\big| \le \frac{1}{2}|\cos(\mu(\mathfrak{r}) + 1)| + \frac{|\mu(\mathfrak{r})|}{2e}. $$

So, conditions (i) *and* (iii) *hold with* $k = \frac{1+e}{2e\Gamma(\frac{1}{4})} \approx 0.1889 < 1$, $a(\mathfrak{r}) = \frac{1}{2}\cos(\mu(t)+1) \in L^1[0,1]$, $b = \frac{1}{2e}$ *and the series* $\sum_{k=1}^{\infty} \frac{1}{k^4}$ *is convergent. Also,* $\big[a(|\mu_\circ| + \sum_{k=1}^{m}|a_k|) + 1\big]kT \approx 0.6136 < 1$ *and* $\frac{bT^{\sigma}}{\Gamma(\sigma+1)} \approx 0.2029 < 1$. *We deduce from Theorem 1 that the nonlocal problem* (17)–(18) *has at least one continuous solution.*

Example 2. *Consider the following nonlinear integro-differential inclusion*

$$ {}^{c}D^{\eta}\mu(\mathfrak{r}) \in \Phi_1(\mathfrak{r}, I^{\sigma}\phi_2(\mathfrak{r}, \mu(\varphi(\mathfrak{r})))), \quad \mathfrak{r} \in [0,1], \ \eta, \sigma \in (0,1), \tag{19} $$

with infinite point boundary condition

$$ \mu(0) + \sum_{k=1}^{\infty} \frac{1}{k^4}\mu\Big(\frac{k^2+k-1}{k^2+k}\Big) = \mu_{\circ}. \tag{20} $$

We choose $\Phi_1 : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}^+}$ *as follows*

$$ \Phi_1\big(\mathfrak{r}, I^{\frac{1}{2}}\phi_2(\mathfrak{r}, \mu(\mathfrak{r}))\big) = \Big[0, \, \frac{e^{-\mathfrak{r}}}{e^{\mathfrak{r}} + 5} + \int_0^{\mathfrak{r}} \frac{\sqrt{(\mathfrak{r}-\varsigma)}}{2\sqrt{\pi}e^{\varsigma+1}}\Big(\frac{2 + |\sin\mu(\varsigma)|}{1 + |\sin\mu(\varsigma)|}\Big)\, d\varsigma\Big]. $$

Set

$$ \phi_2(\mathfrak{r}, \mu(\mathfrak{r})) = \frac{1}{2e^{\mathfrak{r}+1}}\Big(\frac{2 + |\sin\mu(\mathfrak{r})|}{1 + |\sin\mu(\mathfrak{r})|}\Big). $$

Define a continuous map $\phi_1 : [0,1] \times \mathbb{R} \to \mathbb{R}$. *Note that* $\phi_1 \in S_{\Phi_1}$. *Then we have*

$$ \big|\phi_1\big(\mathfrak{r}, I^{\frac{1}{2}}\phi_2\big(\mathfrak{r}, \mu(\varphi(\mathfrak{r}))\big)\big) - \phi_1\big(\mathfrak{r}, I^{1/2}\phi_2\big(\mathfrak{r}, \nu(\varphi(\mathfrak{r}))\big)\big)\big| \le \frac{1}{2e^2\sqrt{\pi}}|\mu - \nu| $$

and

$$ \big|\phi_2(\mathfrak{r}, \mu(\mathfrak{r})) - \phi_2(\mathfrak{r}, \nu(\mathfrak{r}))\big| \le \frac{1}{2e^2}|\mu - \nu|. $$

Thus conditions (i)–$(iii)^*$ *are satisfied with* $k = \frac{1}{2e^2\sqrt{\pi}}$ *and* $c = \frac{1}{2e^2}$. *The series* $\sum_{k=1}^{\infty} \frac{1}{k^3}$ *is convergent. Also* $\frac{kc(a\sum_{k=1}^{m} a_k + 1)T^{\eta+\sigma+2}}{\Gamma(\eta+1)\Gamma(\sigma+1)} = \frac{1}{4e^4\pi} < 1$. *We deduce from Theorem 1 that the nonlocal problem* (19)–(20) *has a unique continuous solution.*

5 Conclusion

In this study, we have investigated the existence of mild solutions for functional integro-differential inclusions involving fractional-order derivatives, under both Riemann-Stieltjes integral and infinite-point boundary conditions. By reformulating the original problem into a coupled system, we successfully applied fixed point theorems to establish the existence of at least one continuous solution. Furthermore, we extended the analysis by presenting sufficient conditions for the uniqueness of these solutions.

Additionally, we explored the continuous dependence of the solution on initial data, showing that small variations in initial conditions lead to controlled deviations in the solution. This highlights the robustness of the proposed model in practical applications.

Finally, we provided two illustrative examples to demonstrate the applicability of our theoretical results. These examples serve to underline the effectiveness of the proposed approach in handling complex boundary value problems, further expanding the scope of fractional differential inclusions in various fields of study.

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Аль-Ісса Ш.М., Каддура І.Г., Хамзае Г.М. *Аналітичне дослідження крайової задачі для функціонального диференціального включення* // Карпатські матем. публ. — 2025. — Т.17, №2. — С. 661–678.

У цій статті досліджується існування м'яких розв'язків задачі Коші, що містить диференціальні включення дробового порядку з нелокальними крайовими умовами, зокрема умовами з нескінченною кількістю точок або інтегральними умовами типу Рімана-Стілтьєса. Встановлено достатні умови єдиності розв'язку та досліджено його неперервну залежність від заданих даних. Для демонстрації практичної застосовності отриманих результатів роботу завершено двома ілюстративними прикладами.

Ключові слова і фрази: функціональне інтегро-диференціальне включення, теорема про нерухому точку, інтегральна крайова умова Рімана-Стілтьєса, крайова умова з нескінченною кількістю точок.